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STRONG CONVERGENCE THEOREMS FOR EQUILIBRIUM PROBLEMS AND FIXED POINTS OF ASYMPTOTICALLY NONEXPANSIVE MAPS

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ABSTRACT. Zhenhua He and Wei-Shih Du, *Fixed Point Theory and Applications* 2011, **2011**:33 introduced a new method of finding a common element in the intersection of the set of solutions of a finite family of equilibrium problems and the set of fixed points of a nonexpansive mapping in real Hilbert spaces. In this paper we modify the algorithm of He and Du and prove strong convergence results for finding a common element in the intersection of the set of solutions of a finite family of equilibrium problems and the set of fixed points of an asymptotically nonexpansive mapping in real Hilbert spaces.

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1. INTRODUCTION

Let H be a real Hilbert space with the inner product $\langle ., . \rangle$ and the induced norm $||.||$. Let C be a nonempty closed convex subset of H. A mapping $T : C \to C$ is said to be L-*Lipschitzian* if there exists $L > 0$ such that

$$||Tx - Ty|| \leq L||x - y||, \ \forall x, y \in C. \tag{1.1}$$

T is said to be a *contraction* if $L \in [0, 1)$ and T is said to be *nonexpansive* if $L = 1$. T is said to be *asymptotically nonexpansive* (see for example [1]) if there exists a sequence $\{k_n\}_{n=1}^{\infty} \subseteq [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$||T^n x - T^n y|| \leq k_n||x - y||, \ \forall x, y \in C. \tag{1.2}$$

It is well known (see for example [1]) that the class of nonexpansive mappings is a proper subclass of the class of asymptotically nonexpansive mappings. T is said to be uniformly L-Lipschitzian if there

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exists $L > 0$ such that

$$||T^n x - T^n y|| \leq L||x - y||, \ \forall x, y \in C. \tag{1.3}$$

T is said to be *asymptotically regular at a point* $x \in C$ (see for example [2-5]) if $\lim_{n\to\infty} ||T^{n+1}x - T^n x|| = 0$, and it is said to be *asymptotically regular on* C if it is asymptotically regular at all $x \in C$. T is said to be *uniformly asymptotically regular* (see for example [2-5]) if for any $\epsilon > 0$ there exists an N such that for all $x \in C$ and for all $n \geq N$, $||T^{n+1}x - T^n x|| < \epsilon$. It is known (see for example [5]) that if T is nonexpansive, then for all $\lambda \in (0, 1)$, $T_\lambda = \lambda I + (1-\lambda)T$ is asymptotically regular at x if $\{T_\lambda^n(x) : n \in \mathbb{N}\}$ is bounded, and if C is bounded; then T_λ is uniformly asymptotically regular. T is said to be *demiclosed* at p if whenever $\{x_n\}_{n=1}^\infty$ is a sequence in C which converges weakly to $x^* \in C$ and $\{Tx_n\}_{n=1}^\infty$ converges strongly to p, then $Tx^* = p$.

Let $P_C : H \to C$ denote the metric projection (the proximity map) which assigns to each point $x \in H$ the unique nearest point in C, denoted by $P_C(x)$. It is well known that $z = P_C(x)$ if and only if $\langle x - z, z - y\rangle \geq 0, \ \forall y \in C$, and that P_K is nonexpansive.

We recall that a *bi-function* $f : C \times C \to \Re$ is any function such that $f(x, x) = 0$ for all $x \in C$. A bi-function f is said to be monotone if for all $x, y \in C, f(x, y) + f(y, x) \leq 0$. If f is a bi-function, the classical equilibrium problem is to find $x \in C$ such that

$$f(x, y) \geq 0, \forall y \in C. \tag{1.4}$$

Let $EP(f)$ denote the set of all solutions of the problem (1.4). Since several problems in Physics, Optimization and Economics reduce to finding a solution of (1.4) (see for example [6-7]), some authors had proposed some methods to find the solution of the equilibrium problem (see for example [6-9]). If $F(T) := \{x \in C : Tx = x\} \neq \emptyset$, several authors have applied various iterative methods such as the composite iterative algorithm, the CQ algorithm, viscosity approximation methods, etc, to find a common element in $EP(f) \cap F(T)$ (see for example [10-17]). Let I denote an index set, for each $i \in I$, let f_i be a bi-function from $C \times C$ into $\Re$. The system of equilibrium problem is to find $x \in C$ such that

$$f_i(x, y) \geq 0, \ \forall y \in C \text{ and } \forall i \in I. \tag{1.5}$$

$\bigcap_{i\in I} EP(f_i)$ is the set of all solutions of the system of equilibrium problem (1.5). For each $i \in I$, if $f_i(x, y) = \langle A_i x, y - x\rangle$, where $A_i : C \to C$ is a nonlinear operator, then the problem (1.5) becomes

the following system of variational inequality problem:
Find an element $x \in C$ such that

$$\langle A_i x, y - x\rangle \geq 0, \ \forall y \in C. \tag{1.6}$$

As we already mentioned above, the equilibrium problem unifies a lot of different problems in Optimization and Nonlinear Analysis and for this reason the existence of solutions for equilibrium problems was studied in a large number of papers. Recently, there were many papers devoted to algorithms for finding such solutions (see for example [10-17]).

Recently, He and Du [17] proved the following strong convergence theorem for equilibrium problems and fixed points of nonexpansive mappings in real Hilbert spaces:

Theorem 1.1 ([17]). Let C be a nonempty closed convex subset of a real Hilbert space H and $I = \{1, 2, 3, \dots, k\}$ be a finite index set. For each $i \in I$, let f_i be a bi-function from $C \times C$ into $\Re$ which satisfies the following conditions:

(A1) $f_i(x, x) = 0$ for all $x \in C$,
(A2) f_i is monotone,
(A3) for each $x, y, z \in C$, $\lim_{t\downarrow 0} f_i(tz + (1-t)x, y) \leq f_i(x, y)$, and
(A4) for each $x \in C, y \longmapsto f_i(x, y)$ is convex and lower semi-continuous.

Let $S : C \longrightarrow C$ be a nonexpansive mapping with $\Omega = (\cap_{i=1}^{k} EP(f_i)) \cap F(S) \neq \emptyset$. Let $\lambda, \rho \in (0, 1)$ and let $g : C \longrightarrow C$ be a ρ-contraction. Define $T^i_{r_n} : H \to C$ by

$$T^i_{r_n} x = \{z \in C : f_i(z, y) + \frac{1}{r_n}\langle y - z, z - x\rangle, \ \forall y \in C\},$$

and let $\{x_n\}$ be a sequence generated as follows:

$$\begin{cases} x_1 \in C \\ u_n^i = T^i_{r_n} x_n, \ \forall i \in I \\ x_{n+1} = \alpha_n g(x_n) + (1-\alpha_n) y_n, \\ y_n = (1-\lambda)x_n + \lambda S z_n, \\ z_n = \frac{u_n^1 + \ldots + u_n^k}{k}, \ \forall n \in \mathbb{N}. \end{cases} \tag{1.7}$$

If the control coefficient sequence $\{\alpha_n\} \subset (0, 1)$, and $\{r_n\} \subset (0, \infty)$ satisfy the following conditions:

(D1) $\lim_{n\to\infty} \alpha_n = 0$, $\Sigma_{n=1}^{\infty} \alpha_n = \infty$ and $\lim_{n\to\infty} |\alpha_{n+1} - \alpha_n| = 0$,
(D2) $\liminf_{n\to\infty} r_n > 0$ and $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$,

then the sequence $\{x_n\}$ and $\{u_n^i\}$, for all $i \in I$, converge strongly to an element $c = P_\Omega g(c) \in \Omega$.

Theorem 1.1 contains many important results in the literature and has many important applications as demonstrated in [17].

It is our purpose in this paper to extend Theorem 1.1 to the case where S belongs to a certain class of asymptotically nonexpansive maps.

2. PRELIMINARY

We shall need the following results:

Lemma 2.1 [18] (Demiclosednes Principle). Let E be a uniformly convex Banach space, C a nonempty closed convex subset of E and $T : C \to C$ an asymptotically nonexpansive mapping with a sequence $\{k_n\} \subseteq [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$. Then $(I - T)$ is demiclosed at zero.

Lemma 2.2 ([19]) Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space E and let $\{\beta_n\}$ be a sequence in [0,1] with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = \beta_n y_n + (1 - \beta_n)x_n$ for all integers $n \geq 0$ and $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$, then $\lim_{n\to\infty} \|y_n - x_n\| = 0$

Lemma 2.3 ([6]). Let C be a nonempty closed convex subset of H and let f be a bi-function of $C \times C$ into $\Re$ satisfying the following conditions:

(A1) $f(x, x) = 0$ for all $x \in C$,
(A2) f is monotone,
(A3) for each $x, y, z \in C$, $\lim_{t\downarrow 0} f(tz + (1 - t)x, y) \leq f(x, y)$,
(A4) for each $x \in C, y \longmapsto f(x, y)$ is convex and lower semi-continuous.

Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that $f(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0$, for all $y \in C$.

Lemma 2.4 ([8]). Let C be a nonempty closed convex subset of H and let f be a bi-function of $C \times C$ into $\Re$ satisfying (A1) - (A4). For $r > 0$ and each $x \in H$, define a mapping $T_r : H \to C$ as follows: $T_r(x) = \{z \in C : f(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0,\ \forall y \in C\}$. Then the following hold:

(i) T_r is single valued;
(ii) T_r is firmly nonexpansive, that is, for any $x, y \in H$,

$$\|T_r(x) - T_r(y)\|^2 \leq \langle T_r(x) - T_r(y), x - y\rangle$$

(iii) $F(T_r) = EP(f)$
(iv) $EP(f)$ is closed and convex.

If $I = \{1, 2, \dots k,\ k \in \mathbb{N}\}$ is a finite index set, $f_i : C \times C \to \Re$ bi-functions satisfying conditions (A1)-(A4) and for each $i \in I$, $T^i_{r_n} : H \to C$ is given by

$$T^i_{r_n} x = \{z \in C : f_i(z, y) + \frac{1}{r_n}\langle y - z, z - x\rangle,\ \forall y \in C\},$$

then for each $i \in I,\ n \in \mathbb{N}$ it follows from Lemmas 2.3 and 2.4 that $T^i_{r_n}$ is a firmly nonexpansive single-valued mapping such that $F(T^i_{r_n}) = EP(f_i)$ is closed and convex.

Lemma 2.5 ([17]). Let H be a real Hilbert space. Then for any $x_1, x_2, \dots, x_k \in H$ and $a_1, a_2, \dots, a_k \in [0, 1]$ with $\sum_{i=1}^k a_i = 1, k \in \mathbb{N}$, we have

$$\|\sum_{i=1}^{k} a_i x_i\|^2 = \sum_{i=1}^{k} a_i\|x_i\|^2 - \sum_{i=1}^{k-1}\sum_{j=i+1}^{k} a_i a_j\|x_i - x_j\|^2. \tag{2.1}$$

Lemma 2.6 ([20]). Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of non negative real numbers such that

$$a_{n+1} \leq (1 - \lambda_n)a_n + \lambda_n\beta_n + \sigma_n,\ n \geq 1,$$

where $\{\lambda_n\} \subseteq (0, 1), \{\beta_n\} \subseteq \Re$, $\{\sigma_n\}$ is a sequence of nonnegative real numbers and

(i) $\Sigma_{n=0}^{\infty}\lambda_n = \infty,$ or equivalently $\Pi_{n=0}^{\infty}(1 - \lambda_n) = 0,$
(ii) $\limsup_{n\to\infty} \beta_n \leq 0,$ and
(iii) $\Sigma_{n=0}^{\infty}\sigma_n < \infty.$

Then $\lim_{n\to\infty} a_n = 0$.

It is also well known that in real Hilbert spaces H, we have,

$$||x + y||^2 \leq ||y||^2 + 2\langle x, x + y\rangle,\ \forall x, y \in H \tag{2.2}$$

3. MAIN RESULTS

We now prove the following:

Theorem 3.1 Let C be a nonempty closed convex subset of a real Hilbert space H and $I = \{1, 2, 3, \dots, k\}$ be a finite index set. For each $i \in I$, let f_i be a bi-function from $C \times C$ into $\Re$ satisfying $(A_1)-(A_4)$. Let $S : C \longrightarrow C$ be an asymptotically nonexpansive with sequence $\{k_n\}_{n=1}^{\infty} \subseteq [1, \infty)$, let S be also uniformly asymptotically regular. Let $\Omega = (\cap_{i=1}^{k} EP(f_i)) \cap F(S) \neq \emptyset$, $\lambda, \rho \in (0, 1)$, and let $g : C \longrightarrow C$ be a ρ-contraction. Let $\{x_n\}$ be the sequence generated as follows:

$$\begin{cases} x_1 \in C \\ u_n^i = T_{r_n}^i x_n, \ \forall i \in I \\ x_{n+1} = \alpha_n g(x_n) + (1 - \alpha_n) y_n, \\ y_n = (1 - \lambda) x_n + \lambda S^n z_n, \\ z_n = \frac{u_n^1 + \ldots + u_n^k}{k}, \ \forall n \in \mathbb{N}. \end{cases} \tag{3.1}$$

If the control coefficient sequence $\{\alpha_n\} \subset (0, 1)$, and the sequence $\{r_n\} \subset (0, +\infty)$ satisfy the following conditions:

(D1) $\lim_{n \longrightarrow \infty} \alpha_n = 0$, $\Sigma_{n=1}^{\infty} \alpha_n = \infty$ and $\lim_{n \longrightarrow \infty} |\alpha_{n+1} - \alpha_n| = 0$,
(D2) $\liminf_{n \longrightarrow \infty} r_n > 0$ and $\lim_{n \longrightarrow \infty} |r_{n+1} - r_n| = 0$,
(D3) $\Sigma_{n=1}^{\infty} (k_n - 1) < \infty$.

Then the sequence $\{x_n\}$ and $\{u_n^i\}$, for all $i \in I$, converge strongly to an element $c = P_\Omega g(c) \in \Omega$.

Proof. Since $g : C \to C$ is a ρ-contraction, and $P_\Omega : H \to \Omega \subseteq H$ is nonexpansive, then $P_\Omega g : H \to \Omega \subseteq H$ is a ρ-contraction and hence has a unique fixed point. Thus, there exists unique $c \in C \subseteq H$ such that $P_\Omega g(c) = c \in \Omega$.

We proceed to prove that the sequences $\{x_n\}, \{y_n\}, \{z_n\}$ and $\{u_n^i\}$ are all bounded. Observe that (3.1) is equivalent to:

$$\begin{cases} x_1 \in C \\ f_i(u_n^i, y) + \frac{1}{r_n} \langle y - u_n^i, u_n^i - x_n \rangle \geq 0, \ \forall y \in C, \text{ and } \forall i \in I \\ x_{n+1} = \alpha_n g(x_n) + (1 - \alpha_n) y_n, \\ y_n = (1 - \lambda) x_n + \lambda S^n z_n, \\ z_n = \frac{u_n^1 + \ldots + u_n^k}{k}, \ \forall n \in \mathbb{N}. \end{cases} \tag{3.2}$$

For each $i \in I$ we have

$$||u_n^i - c|| = ||T_{r_n}^i x_n - T_{r_n}^i c|| \leq ||x_n - c||, \ \forall n \in \mathbb{N}. \tag{3.3}$$

From (3.2) we have

$$||z_n - c|| \leq \frac{1}{k} \sum_{i=1}^{k} ||u_n^i - c|| \leq ||x_n - c||. \text{ (using (3.3))} \tag{3.4}$$

Using (3.4) we obtain

$$\begin{aligned}
\|y_n - c\| &= \|(1-\lambda)(x_n - c) + \lambda(S^n z_n - c)\| \\
&\leq (1-\lambda)\|x_n - c\| + \lambda\|S^n z_n - c\| \\
&\leq (1-\lambda)\|x_n - c\| + \lambda k_n\|z_n - c\| \\
&\leq (1-\lambda)\|x_n - c\| + \lambda k_n\|x_n - c\| \\
&= (1+\lambda(k_n - 1))\|x_n - c\|. \qquad (3.5)
\end{aligned}$$

Observe that application of (3.5) now yields

$$\begin{aligned}
\|x_{n+1} - c\| &\leq \alpha_n\|g(x_n) - c\| + (1-\alpha_n)\|y_n - c\| \\
&\leq \alpha_n\|g(x_n) - g(c)\| + \alpha_n\|g(c) - c\| \\
&\quad +(1-\alpha_n)\|y_n - c\| \\
&\leq \alpha_n\rho\|x_n - c\| + \alpha_n\|g(c) - c\| \\
&\quad +(1-\alpha_n)(1+\lambda(k_n - 1))\|x_n - c\| \\
&= (1-\alpha_n(1-\rho) + \lambda(k_n - 1)(1-\alpha_n))\|x_n - c\| \\
&\quad +\alpha_n\|g(c) - c\| \\
&\leq (1+\frac{\lambda}{\rho}(k_n - 1))[1-\alpha_n(1-\rho)]\|x_n - c\| \\
&\quad +\alpha_n(1-\rho)\frac{\|g(c) - c\|}{(1-\rho)} \\
&\leq (1+\frac{\lambda}{\rho}(k_n - 1))\max\{\|x_n - c\|, \frac{\|g(c) - c\|}{(1-\rho)}\} \\
&\vdots \\
&\leq \Pi_{j=1}^{n}(1+\frac{\lambda}{\rho}(k_j - 1))\max\{\|x_1 - c\|, \frac{\|g(c) - c\|}{(1-\rho)}\}.
\end{aligned}$$

Since $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, it follows that $\{x_n\}$ is bounded. Thus $\{y_n\}, \{z_n\}$, and $\{u_n^i\},\ i \in I$ are all bounded.

Next we verify that $\lim ||x_{n+1} - x_n|| = 0$.

Since $u_{n-1}^i, u_n^i \epsilon C$ for each $i \epsilon I$, we obtain from (3.2) that

$$f_i(u_n^i, u_{n-1}^i) + \frac{1}{r_n}\langle u_{n-1}^i - u_n^i, u_n^i - x_n\rangle \geq 0, \qquad (3.6)$$

$$f_i(u_{n-1}^i, u_n^i) + \frac{1}{r_{n-1}}\langle u_n^i - u_{n-1}^i, u_{n-1}^i - x_{n-1}\rangle \geq 0. \qquad (3.7)$$

It follows from (3.6), (3.7) and condition (A2) that

$$\begin{aligned} 0 &\leq r_n[f_i(u_n^i, u_{n-1}^i) + f_i(u_{n-1}^i, u_n^i] + \langle u_{n-1}^i - u_n^i, u_n^i - x_n \\ &\quad -\frac{r_n}{r_{n-1}}\langle u_{n-1}^i - x_{n-1}\rangle \\ &\leq \langle u_{n-1}^i - u_n^i, u_n^i - x_n - \frac{r_n}{r_{n-1}}(u_{n-1}^i - x_{n-1})\rangle. \end{aligned} \tag{3.8}$$

From (3.8) we obtain

$$\begin{aligned} &\langle u_{n-1}^i - u_n^i, u_{n-1}^i - u_n^i + x_n - x_{n-1} + x_{n-1} - u_{n-1}^i \\ &+\frac{r_n}{r_{n-1}}(u_{n-1}^i - x_{n-1})\rangle \leq 0. \end{aligned} \tag{3.9}$$

Hence

$$\begin{aligned} \|u_n^i - u_{n-1}^i\| &\leq \|x_n - x_{n-1}\| \\ &\quad +|\frac{r_n - r_{n-1}}{r_{n-1}}|\|x_{n-1} - u_{n-1}^i\| \ \forall n \in \mathbb{N}. \end{aligned} \tag{3.10}$$

Let $M := \frac{1}{k}\Sigma_{i=1}^k\|x_{n-1} - u_{n-1}^i\|$. Then using (3.10) we obtain

$$\begin{aligned} \|z_n - z_{n-1}\| &= ||\frac{1}{k}(u_n^1 + u_n^2 + \ldots u_n^k) \\ &- \frac{1}{k}(u_{n-1}^1 + u_{n-1}^2 + \ldots u_{n-1}^k)|| \\ &\leq \frac{1}{k}\Sigma_{i=1}^k\|u_n^i - u_{n-1}^i\| \\ &\leq \|x_n - x_{n-1}\| + M|\frac{r_n - r_{n-1}}{r_{n-1}}|. \end{aligned} \tag{3.11}$$

Set $v_n = \frac{x_{n+1}-(1-\beta_n)x_n}{\beta_n}$, where $\beta_n = 1-(1-\lambda)(1-\alpha_n), n \in \mathbb{N}$. Then for each $n \in \mathbb{N}$

$$x_{n+1} - x_n = \beta_n(v_n - x_n), \text{ and}$$

$$v_n = \frac{\alpha_n g(x_n) + \lambda(1-\alpha_n)S^n z_n}{\beta_n}.$$

Thus for any $n \in \mathbb{N}$ we have

$$\begin{aligned} v_{n+1} - v_n &= \frac{\alpha_{n+1} g(x_{n+1})}{\beta_{n+1}} - \frac{\alpha_n g(x_n)}{\beta_n} - \frac{\lambda(1-\alpha_n)S^n z_n}{\beta_n} \\ &\quad + \frac{\lambda(1-\alpha_{n+1})S^{n+1} z_{n+1}}{\beta_{n+1}} \\ &= \frac{\alpha_{n+1} g(x_{n+1})}{\beta_{n+1}} - \frac{\alpha_n g(x_n)}{\beta_n} \\ &\quad - \frac{\lambda(1-\alpha_n)}{\beta_n}[S^n z_n - S^{n+1} z_n] \\ &\quad - \lambda[\frac{1-\alpha_n}{\beta_n} - \frac{1-\alpha_{n+1}}{\beta_{n+1}}]S^{n+1} z_{n+1} \\ &\quad - \frac{\lambda(1-\alpha_n)}{\beta_n}[s^{n+1} z_n - s^{n+1} z_{n+1}]. \end{aligned}$$

Hence

$$\begin{aligned} \|v_{n+1} - v_n\| - \|x_{n+1} - x_n\| &\leq \frac{\alpha_{n+1}}{\beta_{n+1}} \|g(x_{n+1})\| + \frac{\alpha_n}{\beta_n} ||g(x_n)|| \\ &\quad + \frac{\lambda(1-\alpha_n)}{\beta_n} ||S^n z_n - S^{n+1} z_n|| \\ &\quad + \lambda\Big|\frac{1-\alpha_n}{\beta_n} - \frac{1-\alpha_{n+1}}{\beta_{n+1}}\Big| ||S^{n+1} z_{n+1}|| \\ &\quad + \frac{\lambda(1-\alpha_n)}{\beta_n} ||S^{n+1} z_n - S^{n+1} z_{n+1}|| \\ &\quad - \|x_{n+1} - x_n\| \\ &\leq \frac{\alpha_{n+1}}{\beta_{n+1}} \|g(x_{n+1})\| + \frac{\alpha_n}{\beta_n} ||g(x_n)|| \\ &\quad + \frac{\lambda(1-\alpha_n)}{\beta_n} ||S^n z_n - S^{n+1} z_n|| \\ &\quad + \lambda\Big|\frac{1-\alpha_n}{\beta_n} - \frac{1-\alpha_{n+1}}{\beta_{n+1}}\Big| ||S^{n+1} z_{n+1}|| \\ &\quad + \frac{\lambda(1-\alpha_n)}{\beta_n} (k_{n+1} - 1) ||z_n - z_{n+1}|| \\ &\quad + \frac{\lambda(1-\alpha_n)}{\beta_n} ||z_n - z_{n+1}|| - ||x_{n+1} - x_n|| \qquad (3.12) \\ &\leq \frac{\alpha_{n+1}}{\beta_{n+1}} \|g(x_{n+1})\| + \frac{\alpha_n}{\beta_n} ||g(x_n)|| \\ &\quad + \frac{\lambda(1-\alpha_n)}{\beta_n} ||S^n z_n - S^{n+1} z_n|| \\ &\quad + \lambda\Big|\frac{1-\alpha_n}{\beta_n} - \frac{1-\alpha_{n+1}}{\beta_{n+1}}\Big| ||S^{n+1} z_{n+1}|| \\ &\quad + \frac{\lambda(1-\alpha_n)}{\beta_n} (k_{n+1} - 1) ||z_n - z_{n+1}|| \\ &\quad + \Big[\frac{\lambda(1-\alpha_n)}{\beta_n} - 1\Big] ||x_{n+1} - x_n|| \end{aligned}$$

$$+\frac{\lambda(1-\alpha_n)}{\beta_n}M\Big|\frac{r_{n+1}-r_n}{r_n}\Big|.$$

Since S is uniformly asymptotically regular, it follows from (3.12) and conditions (D1) and (D2) that

$$\limsup_{n\to\infty}\{||v_{n+1}-v_n||-||x_{n+1}-x_n||\}=0. \tag{3.13}$$

It now follows from (3.13) and Lemma 2.2 that $\lim\limits_{n\to\infty}||v_n-x_n||=0$. This with the relation $x_{n+1}-x_n=\beta_n(v_n-x_n)$ yields

$$\lim_{n\to\infty}||x_{n+1}-x_n||=0.$$

Next we show that $\lim\limits_{n\to\infty}||Su_n^i-u_n^i||=0$. We first prove that $||S^nu_n^i-u_n^i||=0$. From (3.10) we obtain

$$\lim_{n\to\infty}||u_{n+1}^i-u_n^i||=0,\ \forall i\in I \tag{3.14}$$

Also

$$\lim_{n\to\infty}||x_{n+1}-y_n||=\lim_{n\to\infty}\alpha_n||g(x_n)-y_n||=0. \tag{3.15}$$

Hence

$$||x_n-y_n||\le||x_n-x_{n+1}||+||x_{n+1}-y_n||\to 0 \text{ as } n\to\infty. \tag{3.16}$$

It follows that

$$\lim_{n\to\infty}||S^nz_n-x_n||=\lim_{n\to\infty}\frac{1}{\lambda}||y_n-x_n||=0. \tag{3.17}$$

From Lemma 2.4 we obtain

$$\begin{aligned}
||u_n^i-c||^2 &= ||T_{r_n}^ix_n-T_{r_n}^ic||^2\\
&\le \langle T_{r_n}^ix_n-T_{r_n}^ic, x_n-c\rangle\\
&= \langle u_n^i-c, x_n-c\rangle\\
&= \frac{1}{2}\big[||u_n^i-c||^2+||x_n-c||^2-||u_n^i-x_n||^2\big].
\end{aligned}$$

Hence

$$||u_n^i-c||^2\le||x_n-c||^2-||u_n^i-x_n||^2. \tag{3.18}$$

It follows from (3.18) and Lemma 2.5 that

$$\begin{aligned}
||z_n - c||^2 &= \left\|\sum_{i=1}^{k} \frac{1}{k}(u_n^i - c)\right\|^2 \\
&\le \frac{1}{k}\sum_{i=1}^{k} ||u_n^i - c||^2 \\
&\le \frac{1}{k}\sum_{i=1}^{k} \Big[||x_n - c||^2 - ||u_n^i - x_n||^2\Big] \\
&= ||x_n - c|| - \frac{1}{k}\sum_{i=1}^{k} ||u_n^i - x_n||^2. \qquad (3.19)
\end{aligned}$$

Furthermore,

$$\begin{aligned}
||x_{n+1} - c||^2 &= ||\alpha_n(g(x_n) - c) + (1-\alpha_n)(y_n - c)||^2 \\
&\le \alpha_n||g(x_n) - c||^2 + (1-\alpha_n)||y_n - c||^2 \\
&\le 2\alpha_n||g(x_n) - g(c)||^2 + 2\alpha_n||g(c) - c||^2 \\
&\quad +(1-\alpha_n)\Big[(1-\lambda)||x_n - c||^2 + \lambda||S^n z_n - c||^2\Big] \\
&\le 2\rho^2\alpha_n||x_n - c||^2 + 2\alpha_n||g(c) - c||^2 \\
&\quad +(1-\alpha_n)\Big[(1-\lambda)||x_n - c||^2 + \lambda k_n^2||z_n - c||^2\Big] \\
&\le (1-\alpha_n)(1-\lambda)||x_n - c||^2 + \alpha_n[2\rho^2||x_n - c||^2 \\
&\quad +2||g(c) - c||^2] + \lambda(k_n^2 - 1)||z_n - c||^2 \\
&\quad +\lambda(1-\alpha_n)||z_n - c||^2 \\
&\le (1-\alpha_n)(1-\lambda)||x_n - c||^2 + \alpha_n[2\rho^2||x_n - c||^2 \\
&\quad +2||g(c) - c||^2] + \lambda(k_n^2 - 1)||z_n - c||^2 \\
&\quad +\lambda(1-\alpha_n)\Big[||x_n - c||^2 - \frac{1}{k}\sum_{i=1}^{k}||u_n^i - x_n||^2\Big] \quad (3.20)
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{\lambda(1-\alpha_n)}{k}\sum_{i=1}^{k}||u_n^i - x_n||^2 &\le ||x_n - c||^2 - ||x_{n+1} - c||^2 \\
&\quad +\alpha_n\Big[2\rho^2||x_n - c||^2 + 2||g(c) - c||^2\Big] \\
&\quad +\lambda(k_n^2 - 1)||z_n - c||^2 \\
&\le (||x_n - c|| + ||x_{n+1} - c||)||x_{n+1} - x_n|| \\
&\quad +\alpha_n\Big[2\rho^2||x_n - c||^2 + 2||g(c) - c||^2\Big] \\
&\quad +\lambda(k_n^2 - 1)||z_n - c||^2 \to 0 \text{ as } n \to \infty.
\end{aligned}$$

Hence

$$\lim_{n\to\infty} ||u_n^i - x_n|| = 0. \tag{3.21}$$

Furthermore,

$$||z_n - x_n|| \leq \frac{1}{k}\sum_{i=1}^{k} ||u_n^i - x_n|| \to 0 \text{ as } n \to \infty, \tag{3.22}$$

and hence

$$||u_n^i - z_n|| \leq ||u_n^i - x_n|| + ||x_n - z_n|| \to 0 \text{ as } n \to \infty. \tag{3.23}$$

For all $i \in I$, we obtain from (3.17), (3.2) and (3.23) that

$$\begin{aligned}
||S^n u_n^i - u_n^i|| &\leq ||S^n u_n^i - S^n z_n|| + ||S^n z_n - x_n|| + ||x_n - u_n^i|| \\
&\leq k_n ||u_n^i - z_n|| + ||S^n z_n - x_n|| \\
&\quad + ||x_n - u_n^i|| \to 0 \text{ as } n \to \infty. \qquad (3.24)
\end{aligned}$$

Observe that

$$\begin{aligned}
||x_n - Sz_n|| &\leq ||x_n - S^n z_n|| + ||S^n z_n - Sz_n|| \\
&\leq ||x_n - S^n z_n|| + L||S^{n-1} z_n - z_n|| \\
&\leq ||x_n - S^n z_n|| + L||S^{n-1} z_n - S^{n-1} z_{n-1}|| \\
&\quad + L||S^{n-1} z_{n-1} - z_n|| \\
&\leq ||x_n - S^n z_n|| + L^2||z_n - z_{n-1}|| \\
&\quad + L||S^{n-1} z_{n-1} - x_{n-1}|| + L||x_{n-1} - z_n|| \\
&\leq ||x_n - S^n z_n|| + L^2||z_n - x_n|| + L^2||x_n - x_{n-1}|| \\
&\quad + L^2||x_{n-1} - z_{n-1}|| + L||S^{n-1} z_{n-1} - x_{n-1}|| \\
&\quad + L||x_{n-1} - x_n|| \\
&\quad + L||x_n - z_n|| \to 0 \text{ as } n \to \infty. \qquad (3.25)
\end{aligned}$$

Hence

$$\begin{aligned}
||Su_n^i - u_n^i|| &\leq ||Su_n^i - Sz_n|| + ||Sz_n - x_n|| + ||x_n - u_n^i|| \\
&\leq L||u_n^i - z_n|| + ||Sz_n - x_n|| \\
&\quad + ||x_n - u_n^i|| \to 0 \text{ as } n \to \infty. \qquad (3.26)
\end{aligned}$$

We proceed to prove that $\limsup_{n\to\infty} \langle g(c) - c, x_n - c\rangle \leq 0$. Since $\{x_n\}$ is bounded, let $\{x_{n_j}\}_{j=1}^{\infty}$ be a subsequence of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \langle g(c) - c, x_n - c\rangle = \lim_{j\to\infty} \langle g(c) - c, x_{n_j} - c\rangle. \tag{3.27}$$

Again since $\{x_{n_j}\}_{j=1}^{\infty}$ is bounded, it has a subsequence $\{x_{n_{j_m}}\}_{m=1}^{\infty}$ which converges weakly to a point $x^* \in C$. It follows from (3.21) that $\lim_{m\to\infty} ||u_{n_{j_m}}^i - x_{n_{j_m}}|| = 0$, and hence $\{u_{n_{j_m}}^i\}_{m=1}^{\infty}$ converges weakly to x^* for all $i \in I$. Since $(I - S)$ is demiclosed at 0 (Lemma 2.1), it follows from (3.26) that $x^* \in F(S) := \{x \in C : Sx = x\}$. Furthermore, since

$$f_i(u^i_{n_{jm}}, y) + \frac{1}{r_{n_{jm}}}\langle y - u^i_{n_{jm}}, u^i_{n_{jm}} - x_{n_{jm}}\rangle \geq 0,\ \forall y \in C,\ \text{and } \forall i \in I,$$

it follows from (A2) that

$$\begin{aligned} \frac{1}{r_{n_{jm}}}\langle y - u^i_{n_{jm}}, u^i_{n_{jm}} - x_{n_{jm}}\rangle &\geq f_i(y, u^i_{n_{jm}}) + f_i(u^i_{n_{jm}}, y) \\ &\quad + \frac{1}{r_{n_{jm}}}\langle y - u^i_{n_{jm}}, u^i_{n_{jm}} - x_{n_{jm}}\rangle \\ &\geq f_i(y, u^i_{n_{jm}}). \end{aligned}$$

Thus

$$\langle y - u^i_{n_{jm}}, \frac{u^i_{n_{jm}} - x_{n_{jm}}}{r_{n_{jm}}}\rangle \geq f_i(y, u^i_{n_{jm}}),\ \forall y \in C.$$

It follows from (3.21) and (D2) that

$$f_i(y, x^*) \leq 0,\ \forall y \in C. \tag{3.28}$$

For arbitrary but fixed $y \in C$, let $y_t = ty + (1-t)x^*,\ t \in [0,1]$. Then $y_t \in C$ and hence $f_i(y_t, x^*) \leq 0,\ \forall i \in I$. From (A1)-(A4) we obtain for all $i \in I$

$$0 = f_i(y_t, y_t) \leq t f_i(y_t, y) + (1-t) f_i(y_t, x^*) \leq t f_i(y_t, y),\ \forall i \in I;$$

and hence,

$$f_i(x^*, y) \geq \lim_{t \downarrow 0} f_i(ty + (1-t)x^*, y) = \lim_{t \downarrow 0} f_i(y_t, y) \geq 0. \tag{3.29}$$

It follows from (3.29) that $x^* \in \cap_{i=1}^k EP(f_i)$, and hence $x^* \in \Omega = (\cap_{i=1}^k EP(f_i)) \cap F(S)$. From (3.27) we obtain

$$\begin{aligned} \limsup_{n\to\infty}\langle g(c) - c, x_n - c\rangle &= \lim_{j\to\infty}\langle g(c) - c, x_{n_j} - c\rangle \\ &= \lim_{m\to\infty}\langle g(c) - c, x_{n_{jm}} - c\rangle \\ &= \langle g(c) - c, x^* - c\rangle \leq 0. \end{aligned} \tag{3.30}$$

Finally we prove that $\{x_n\}$, $\{u^i_n\}$, $\forall i \in I$ converge strongly to $c = P_\Omega g(c) \in \Omega$.

Using (2.2) we obtain

$$\begin{aligned}
||x_{n+1}-c||^2 &= ||(1-\alpha_n)(y_n-c)+\alpha_n(g(x_n)-c)||^2\\
&\leq (1-\alpha_n)^2||y_n-c||^2+2\alpha_n\langle g(x_n)-c, x_{n+1}-c\rangle\\
&\leq (1-\alpha_n)^2\Big[||(1-\lambda)(x_n-c)+\lambda(S^nz_n-c)||^2\Big]\\
&\quad +2\alpha_n\langle g(c)-c, x_{n+1}-c\rangle\\
&\quad +2\alpha_n\rho||x_n-c||||x_{n+1}-c||\\
&\leq (1-\alpha_n)^2\Big[1+\lambda(k_n-1)\Big]||x_n-c||^2\\
&\quad +2\alpha_n\langle g(c)-c, x_{n+1}-c\rangle\\
&\quad +2\alpha_n\rho||x_n-c||\Big[||x_{n+1}-x_n||+||x_n-c||\Big]\\
&= \Big[1-2\alpha_n(1-\rho)\Big]||x_n-c||^2\\
&\quad +\alpha_n\Big[\alpha_n||x_n-c||+2\langle g(c)-c, x_{n+1}-c\rangle\\
&\quad +2\rho||x_n-c||||x_{n+1}-x_n||\Big]\\
&\quad +(1-\alpha_n)^2\lambda(k_n-1)||x_n-c||^2. \qquad (3.31)
\end{aligned}$$

Let $a_n := ||x_n-c||^2,\ \lambda_n := 2\alpha_n(1-\rho)$,
$\beta_n := \frac{1}{2(1-\rho)}\Big[\alpha_n||x_n-c||+2\langle g(c)-c, x_{n+1}-c\rangle$
$+2\rho||x_n-c||||x_{n+1}-x_n||\Big]$, $\sigma_n := \lambda(k_n-1)||x_n-c||^2$. Then it follows from (3.31) that

$$a_{n+1}\leq[1-\lambda_n]a_n+\lambda_n\beta_n+\sigma_n,$$

and it follows from Lemma 2.6 that $\lim\limits_{n\to\infty}a_n=\lim\limits_{n\to\infty}|||x_n-c||=0$, and it also follows from (3.21) that $\lim\limits_{n\to\infty}||u_n^i-c||=0,\ \forall i\in I$. $\square$

Corollary 3.1 Let C be a nonempty closed convex subset of a real Hilbert space H and Let f be a bi-function from $C\times C$ into $\Re$ satisfying $(A_1)-(A_4)$. Let $S: C\longrightarrow C$ be an asymptotically nonexpansive mapping with sequence $\{k_n\}_{n=1}^{\infty}\subseteq[1,\infty)$. Let S uniformly asymptotically regular with $\Omega = EP(f)\cap F(S)\neq\emptyset$. Let $\lambda,\rho\in(0,1)$ and $g: C\longrightarrow C$ is a ρ-contraction. Let $\{x_n\}$ be a sequence generated as follows:

$$\begin{cases} x_1\in C\\ u_n=T_{r_n}x_n,\\ x_{n+1}=\alpha_ng(x_n)+(1-\alpha_n)y_n,\\ y_n=(1-\lambda)x_n+\lambda S^nu_n,\end{cases}$$

If the above control coefficient sequence $\{\alpha_n\} \subset (0,1)$ and $\{r_n\} \subset (0,+\infty)$ satisfying the following restrictions:

(D1) $\lim_{n\longrightarrow\infty} \alpha_n = 0$, $\Sigma_{n=1}^{\infty}\alpha_n = +\infty$ and $\lim_{n\longrightarrow\infty} |\alpha_{n+1} - \alpha_n| = 0$,

(D2) $\liminf_{n\longrightarrow\infty} r_n > 0$ and $\lim_{n\longrightarrow\infty} |r_{n+1} - r_n| = 0$.

(D3) $\Sigma_{n=1}^{\infty}(k_n - 1) < \infty$,

Then the sequence $\{x_n\}$ and $\{u_n\}$ converge strongly to an element $c = P_\Omega g(c) \in \Omega$.

If $f_i \equiv 0$, $\forall i$ in Theorem 3.1, then from the algorithm (3.1) we obtain $u_n^i \equiv P_C(x_n)$, $\forall i$. Thus we have the following:

Corollary 3.2 Let C be a nonempty closed convex subset of a real Hilbert space H and let $S : C \longrightarrow C$ be an asymptotically nonexpansive mapping with sequence $\{k_n\}_{n=1}^{\infty} \subseteq [1,\infty)$, and let S be uniformly asymptotically regular with $F(S) \neq \emptyset$. Let $\lambda, \rho \in (0,1)$ and $g : C \longrightarrow C$ is a ρ-contraction. Let $\{x_n\}$ be a sequence generated as follows:

$$\begin{cases} x_1 \in C \\ x_{n+1} = \alpha_n g(x_n) + (1-\alpha_n) y_n, \\ y_n = (1-\lambda)x_n + \lambda S^n P_C(x_n), \end{cases}$$

If the above control coefficient sequence $\{\alpha_n\} \subset (0,1)$ satisfy the conditions:

(D1) $\lim_{n\longrightarrow\infty} \alpha_n = 0$, $\Sigma_{n=1}^{\infty}\alpha_n = +\infty$ and $\lim_{n\longrightarrow\infty} |\alpha_{n+1} - \alpha_n| = 0$, and

(D2) $\Sigma_{n=1}^{\infty}(k_n - 1) < \infty$.

Then the sequence $\{x_n\}$ converges strongly to an element $c = P_\Omega g(c) \in F(S)$.

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