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# ITERATIVE PROCEDURES FOR FINITE FAMILY OF TOTAL ASYMPTOTICALLY NONEXPANSIVE MAPPINGS

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ABSTRACT. It is our aim in this paper to introduce an explicit iterative scheme for a finite family of total asymptotically nonexpansive mappings and prove its strong convergence to a common fixed point of these mappings in smooth reflexive real Banach spaces which admits weakly sequentially continuous duality mappings. In addition, we proved path existence theorem for finite family of asymptotically nonexpansive mappings; and further showed that the convergence of the path guarantees that the set of common fixed points of finite family of asymptotically nonexpansive mappings is a sunny nonexpansive retract. Our theorems improve, generalize and unify several recently announced results in the literature.

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## 1. Introduction

Approximation of solutions of equations involving nonexpansive mappings and their generalizations by iteration process has been of increasing research interest to numereous mathematicians in recent years. One of the first results of this nature was obtained by Browder [6] for nonexpansive self-mappings in Hilbert spaces. If $K$ is a nonempty closed and convex subset of a Hilbert space $H$. Browder [6] studied the path

$$u \in K,\ x_t = tu + (1-t)Tz_t,\ t \in (0,1), \tag{1}$$

where $T : K \to K$ is a nonexpansive mapping (that is, $\|Tx-Ty\| \leq \|x-y\|\ \forall\ x, y \in K$), with nonempty fixed points set $F(T) = \{x \in K : Tx = x\}$. In [6], Browder proved that $\lim\limits_{t\to 0} x_t$ exists and that $\lim\limits_{t\to 0} x_t \in F(T)$. The result was extended by Reich [21] to uniformly

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smooth real Banach spaces. In fact, it was shown in [21] that $\lim_{t\to 0} x_t$ is a sunny nonexpansive retraction of $K$ onto $F(T)$.

In [17], Halpern discussed the convergence of the explicit iteration process

$$x_1 \in K,\ x_{n+1} = \alpha_n u + (1-\alpha_n)Tx_n,\ n \geq 1, \tag{2}$$

in the frame work of real Hilbert spaces. Under appropriate conditions on the iterative parameter $\{\alpha_n\}_{n\geq 1}$, it has been shown by Halpern [17], Lions [19], Wittmann [23] and Bauschke [4] that $\{x_n\}_{n\geq 1}$ converges strongly to $P_{F(T)}u$, the projection of $u$ to the fixed point set $F(T)$ of $T$.
Browder and Halpern iterative algorithms had motivated different iteration processes for approximation of fixed points of asymptotically nonexpansive mappings (where a mapping $T : K \to K$ is called *asymptotically nonexpansive* (see Goebel and Kirk [16]) if and only if there exists a sequence $\{\mu_n\}_{n\geq 1} \subset [0, +\infty)$ with $\lim_{n\to\infty} \mu_n = 0$ such that for all $x, y \in K$, $\|T^n x - T^n y\| \leq (1+\mu_n)\|x-y\|$ for all $n \in \mathbb{N}$). In this regard, Lim and Xu [18] introduced and studied the following implicit iteration scheme for asymptotically nonexpansive mapping $T$

$$z_n = \alpha_n u + (1-\alpha_n)T^n z_n,\ n \geq 1. \tag{3}$$

They showed that the sequence $\{z_n\}_{n\geq 1}$ generated by (3) converges strongly to a fixed point of $T$ in the frame work of uniformly smooth Banach spaces, under suitable conditions on the iterative parameters.

In [15], Chidume *et al.* proved the strong convergence of the explicit iteration scheme generated by

$$x_1, u \in K, x_{n+1} = \alpha_n u + (1-\alpha_n)T^n x_n,\ n \geq 1, \tag{4}$$

where $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = +\infty$ and $T$ is asymptotically nonexpansive.

Recently, Alber *et al.* [2] obtained strong convergence of (4) for a total asymptotically nonexpansive mapping $T$ in the setting of smooth reflexive Banach space with weakly sequentially continuous duality mapping.

Motivated by the results of these authors, it is our aim in this paper to introduce an explicit iterative scheme for finite family of total asymptotically nonexpansive mappings and prove its strong convergence to a common fixed point of these mappings in smooth reflexive real Banach spaces $E$ which admits weakly sequentially continuous duality mappings. We also proved path existence theorem for finite family of asymptotically nonexpansive mappings; and further showed that the convergence of the path guarantees the existence of sunny nonexpansive retraction of a nonempty closed and convex subset $K$ of $E$ onto the set of common fixed points finite family of asymptotically nonexpansive mappings.

## 2. Preliminaries

A real normed space $(E, \ \|.\|)$ (with dual space $E^*$) is said to be *smooth* if and only if for all $x \in E$ with $\|x\| = 1$, there exists a unique $f^* \in E^*$ such that $\|f^*\| = 1$ and $\langle x, f^* \rangle = \|x\|$, where $\langle \ , \ \rangle$ denotes the duality pairing between members of $E$ and $E^*$. The space $E$ is said to be *uniformly smooth* if and only if for all $\epsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in E$ with $\|x\| = 1$ and $\|y\| \leq \delta$, the inequality

$$\frac{\|x + y\| + \|x - y\|}{2} - 1 < \epsilon \|y\|$$

holds. It is well known that every uniformly smooth real Banach space is *smooth and reflexive real Banach space* (see e.g., [12]).

We denote by $J_q$ *the generalized duality mapping* from $E$ to $2^{E^*}$ defined by

$$J_q x := \{f^* \in E^* : \langle x, f^* \rangle = \|x\|^q, \|f^*\| = \|x\|^{q-1}\} \ \forall \ x \in E.$$

For $q = 2$, the mapping $J = J_2$ from $E$ to $2^{E^*}$ is called *the normalized duality mapping*. It is well known that if $E$ is uniformly smooth or $E^*$ is strictly convex (that is, for all $f, g \in E^*$ such that $\|f\| = 1 = \|g\|$, we have that $\frac{\|f+g\|}{2} < 1$), then the duality mapping is single-valued. If $E = H$ is a Hilbert space then the duality mapping becomes the identity map of $H$ (see e.g., [12]). In the sequel, we shall denote the single-valued normalized duality mapping by $j$.

A real normed space $E$ with strictly convex dual is said to have a *weakly sequentially continuous normalized duality mapping* $j$ if and only if for each sequence $\{x_n\}_{n \geq 1}$ in $X$ such that $\{x_n\}_{n \geq 1}$ converges weakly to $x^*$ in $X$, we have that $\{j(x_n)\}_{n \geq 1}$ converges in the weak$^*$

topology to $j(x^*)$. Apart from Hilbert spaces, it was noted in [6] that the most significant class of Banach spaces having a weakly sequentially continuous generalized duality mapping are the sequence spaces $\ell_p$ for $1 < p < +\infty$ (see [7] and [8], where it is also made known that $L_p(\mathbb{R})$ has no weakly sequentially continuous duality mapping for $p \neq 2$).

A mapping $T : K \to K$ is said to be *total asymptotically nonexpansive* (see e.g., [1]) if there exist nonnegative real sequences $\{\mu_n\}_{n\geq1}$ and $\{l_n\}_{n\geq1}$ with $\mu_n \to 0,\ l_n \to 0$ as $n \to \infty$ and nondecreasing continuous function $\phi : [0, +\infty) \to [0, +\infty)$ with $\phi(0) = 0$ such that for all $x, y \in K$,

$$\|T^n x - T^n y\| \leq \|x - y\| + \mu_n \phi(\|x - y\|) + l_n,\ n \geq 1. \tag{5}$$

**Remark 1.** If $\phi \equiv 0$, the zero operator, then (5) reduces to

$$\|T^n x - T^n y\| \leq \|x - y\| + l_n,\ n \geq 1,$$

so that if $K$ is bounded and $T^N$ is continuous for some integer $N \geq 1$, then the mapping $T$ is of asymptotically nonexpansive type (the class of mappings which are of asymptotically nonexpansive type includes the class of mappings which are asymptotically nonexpansive in the intermediate sense and the class of nearly asymptotically nonexpansive mappings which had been studied by several authors, see e.g. [13, 14, 22]). If $\phi(t) = t$, then (5) becomes

$$\|T^n x - T^n y\| \leq (1 + \mu_n)\|x - y\| + l_n,\ n \geq 1.$$

In addition, if $l_n = 0$ for all $n \geq 1$, then total asymptotically nonexpansive mappings coincide with asymptotically nonexpansive mappings. If $\mu_n = 0$ and $l_n = 0$ for all $n \geq 1$, we obtain from (5) the class of mappings that contains the class of nonexpansive mappings.

Alber *et al.* [1] introduced the class of total asymptotically nonexpansive mappings as a more general class of asymptotically nonexpansive mappings. The idea behind the introduction of the class of total asymptotically nonexpansive mappings is to unify various definitions of classes of mappings associated with the class of asymptotically nonexpansive mappings and to prove general convergence theorems applicable to all these classes of mappings.

It is worthy to mention that (to the best of our knowledge) in the earlier work done on total asymptotically nonexpansive mappings, no example of such mapping which is more general than asymptotically nonexpansive mappings has been provided. As part of our contributions in this paper, we at this juncture give the following example which seems new to show that the class of total asymptotically nonexpansive mappings properly contains that of asymptotically nonexpanssive mappings:

**Example 2.** Let $E := \mathbb{R} \times \ell_1$ be endowed with the norm $\|.\|_E = |.| + \|.\|_{\ell_1}$. Let $K$ be a subset of $E$ defined by $K := [0,1] \times B$, where $B$ is the closed unit ball of $\ell_1$. For all $u \in [0,1]$ and $\overline{x} = (x_1, x_2, x_3, ...) \in B$ define $T : K \to K$ by

$$T(u,\overline{x}) = \begin{cases} \left(\frac{1}{3}, \left(0, \frac{|x_1|^2}{3}, \frac{x_2}{3}, \frac{x_3}{3}, \frac{x_4}{3}, \dots\right)\right), \text{ if } u \in \left[0, \frac{1}{3}\right] \\ \\ \left(0, \left(0, \frac{|x_1|^2}{3}, \frac{x_2}{3}, \frac{x_3}{3}, \frac{x_4}{3}, \dots\right)\right), \text{ if } u \in \left(\frac{1}{3}, 1\right]. \end{cases} \tag{6}$$

We can easily check that $T$ given by (6) is well defined; not continuous and thus cannot be asymptotically nonexpansive (since every asymptotically nonexpansive mapping is uniformly $L$-Lipschitzian, so Lipschitz and every Lipschitz mapping is continuous). Next, let $\{l_n\}_{n\geq 1}$ be a sequence of real numbers such that $l_1 = \frac{1}{3}$ and $\lim\limits_{n\to\infty} l_n = 0$. Observe that for all $(u,\overline{x}), (v,\overline{y}) \in K$ (i.e., $u, v \in [0,1]$, $\overline{x} = (x_1, x_2, x_3, ...), \overline{y} = (y_1, y_2, y_3, ...) \in B$),

$$\left\|T(u,\overline{x}) - T(v,\overline{y})\right\|_E \leq |u-v| + l_1 + \frac{1}{3}\max\left\{|x_1| + |y_1|, 1\right\}\|\overline{x} - \overline{y}\|_{\ell_1}.$$

Moreover, we can equally check easily that for all $n \geq 2$ and for all $(u,\overline{x}), (u,\overline{y}) \in K$,

$$T^n(u,\overline{x}) = \Big(\frac{1}{3}, \Big(\underbrace{0, 0, \dots, 0, 0}_{n-\text{times}}, \frac{|x_1|^2}{3^n}, \frac{x_2}{3^n}, \frac{x_3}{3^n}, \frac{x_4}{3^n}, \dots\Big)\Big)$$

and

$$\left\|T^n(u,\overline{x}) - T^n(v,\overline{y})\right\|_E \leq \frac{1}{3^n}\max\left\{|x_1| + |y_1|, 1\right\}\|\overline{x} - \overline{y}\|_{\ell_1}.$$

So, for all $n \geq 1$,

$$\begin{aligned} \left\|T^n(u,\overline{x}) - T^n(v,\overline{y})\right\|_E \;\leq\;& |u-v| + \|\overline{x} - \overline{y}\|_{\ell_1} \\ & + \frac{2}{3^n}\left[|u-v| + \|\overline{x} - \overline{y}\|_{\ell_1}\right] + l_n \end{aligned} \tag{7}$$

Thus, with $\phi : [0,+\infty) \to [0,+\infty)$ defined by $\phi(t) = 2t$, $\mu_n = \frac{1}{3^n}$ for all $n \geq 1$ and $\{l_n\}_{n\geq 1}$ any null sequence with $l_1 = \frac{1}{3}$, we obtain from (7) that

$$\Big\|T^n(u,\overline{x})-T^n(v,\overline{y})\Big\|_E \leq \Big\|(u,\overline{x})-(v,\overline{y})\Big\|_E+\mu_n\phi\Big(\Big\|(u,\overline{x})-(v,\overline{y})\Big\|_E\Big)+l_n.$$

Thus, the operator $T$ given by (6) is total asymptotically nonexpansive but not asymptotically nonexpansive.

Let $K$ be a nonempty closed convex subset of $E$ and let $P$ be a mapping of $E$ onto $K$. Then $P$ is said to be sunny if $P\Big(Px+t(x-Px)\Big) = Px \;\forall\; x \in E$ and $t > 0$. A mapping $P$ of $E$ into $E$ is said to be a retraction if $P^2 = P$. If a mapping $P$ is a retraction, then $Px = x \;\forall\; x \in R(P)$, where $R(P)$ denotes the range of $P$. A subset $K$ of $E$ is said to be sunny nonexpansive retract of $E$ if there exists a sunny nonexpansive retraction of $E$ onto $K$ and it is said to be a nonexpansive retract of $E$ if there exists a nonexpansive retraction of $E$ onto $K$. If $E = H$, where $H$ denotes a Hilbert space, the metric projection $P_K$ is a sunny nonexpansive retraction from $H$ to any nonempty closed and convex subset of $H$. This, however, is not true for larger Banach spaces since nonexpansivity of projections $P_K$ characterizes Hilbert spaces. On the other hand, a sunny nonexpansive retraction can play a similar role in a Banach space as a projection does in Hilbert spaces. For more details on nonexpansive retractions, see, for example, [3, 5, 9, 10, 11, 20, 21].

**Proposition 3.** *(see e.g. [9, 10]) Suppose that $K$ is a nonempty closed and convex subset of a smooth Banach space $E$ and $\Omega$ is a subset of $K$, then a nonexpansive retraction $P : K \to \Omega$ is at most one and it is sunny retraction if and only if for $x \in K$ and for all $\tilde{x} \in \Omega$,*

$$\Big\langle x - Px, j(\tilde{x} - Px)\Big\rangle \leq 0. \tag{8}$$

In what follows, we shall make use of the following lemmas.

**Lemma 4.** *Let $E$ be a real normed space, then*

$$\|x + y\|^2 \leq \|x\|^2 + 2\Big\langle y, j(x + y)\Big\rangle \;\forall\; x, y \in E,\; j(x + y) \in J(x + y).$$

**Lemma 5.** *(see [2]) Let $\{\lambda_n\}_{n\geq1}$ and $\{\gamma_n\}_{n\geq1}$ be nonnegative real sequences. Let $\{\alpha_n\}_{n\geq1}$ be a sequence of positive real numbers such that*

$$\lambda_{n+1} \leq \lambda_n - \alpha_n\lambda_n + \gamma_n,\; \forall\; n \geq 1.$$

*Suppose that for* $n \geq 1$,

$$\frac{\gamma_n}{\alpha_n} \leq c_1 \text{ and } \alpha_n \leq \alpha,$$

*for some constants* $c_1, \alpha > 0$, *then* $\lambda_n \leq \max\{\lambda_1, (1+\alpha)c_1\}$. *Moreover, if* $\sum_{n=1}^{\infty} \alpha_n = +\infty$ and $\gamma_n = o(\alpha_n)$, *then* $\lim_{n\to\infty} \lambda_n = 0$.

**Lemma 6.** *(see e.g., [2]) Let* $E$ *be a reflexive Banach space with weakly continuous normalized duality mapping. Let* $K$ *be a closed convex subset of* $E$ *and* $T : K \to K$ *a uniformly continuous total asymptotically nonexpansive mapping with bounded orbits. Then* $I - T$ *is demiclosed at zero.*

## 3. Main results

Let $K$ be a nonempty closed and convex subset of a real normed space $E$. Let $T_1, T_2, ..., T_m : K \to K$ be $m$ total asymptotically nonexpansive mappings and $\{\alpha_n\}_{n\geq 1} \subset (0,1)$. We define the explicit iteration process $\{x_n\}$ by

$$\begin{array}{l}
x_1, u \in K,\ x_2 = \alpha_1 u + (1-\alpha_1)T_1 x_1 \\
x_3 = \alpha_2 u + (1-\alpha_2)T_2 x_2 \\
\vdots \\
x_m = \alpha_{m-1} u + (1-\alpha_{m-1})T_{m-1} x_{m-1} \\
x_{m+1} = \alpha_m u + (1-\alpha_m)T_m x_m \\
x_{m+2} = \alpha_{m+1} u + (1-\alpha_{m+1})T_1^2 x_{m+1} \\
\vdots \\
x_{2m} = \alpha_{2m-1} u + (1-\alpha_{2m-1})T_{m-1}^2 x_{2m-1} \\
x_{2m+1} = \alpha_{2m} u + (1-\alpha_{2m})T_m^2 x_{2m} \\
x_{2m+2} = \alpha_{2m+1} u + (1-\alpha_{2m+1})T_1^3 x_{2m+1} \\
\vdots
\end{array} \tag{9}$$

Since for all $z \in \mathbb{Z}$ (where $\mathbb{Z}$ is the set of integers), there exists $j(z) \in \{1, 2, ..., m\}$ such that $z - j(z)$ is divisible by $m$ (that is $j(z) = z \bmod m$), then there exists $q(z) \in \mathbb{Z}$ with $\lim_{z\to+\infty} q(z) = +\infty$ such that

$$z = \big(q(z) - 1\big)m + j(z). \tag{10}$$

We may write (9) in a more compact form as

$$x_1,\ u \in K,\ x_{n+1} = \alpha_n u + (1-\alpha_n)T_{j(n)}^{q(n)} x_n,\ \forall\, n \geq 1. \tag{11}$$

By similar procedure as in (9), the following implicit iteration process is generated:

$$z_1,\ u \in K,\ z_n = \alpha_n u + (1-\alpha_n)T_{j(n)}^{q(n)} z_n,\ \forall\, n \geq 1. \tag{12}$$

**Remark 7.** Since $n-m \in \mathbb{Z}$ for all $n \in \mathbb{N}$ (where $\mathbb{N}$ denotes the set of positive integers), we obtain from (10) (for the particular case $n-m \in \mathbb{Z}$) that

$$n - m = \big(q(n-m)-1\big)m + j(n-m). \tag{13}$$

Also substituting $n \in \mathbb{N}$ for $z$ in (10) and subtracting $m$ from both sides of the resulting equation gives

$$n - m = \Big(\big(q(n)-1\big)-1\Big)m + j(n). \tag{14}$$

Comparing (13) and (14) we obtain (by unique representation theorem) that

$$q(n-m) = q(n) - 1 \quad \text{and } j(n-m) = j(n)\ \forall\, n \in \mathbb{N}. \tag{15}$$

**Proposition 8.** *Let $K$ be a nonempty subset of a real normed space $E$ and $T_1, T_2, ..., T_m : K \to K$ be $m$ total asymptotically nonexpansive mappings, then there exist sequences $\{\mu_n\}$, $\{\ell_n\} \subset [0,+\infty)$ with $\lim\limits_{n\to\infty} \mu_n = 0 = \lim\limits_{n\to\infty} \ell_n$ and a nondecreasing continuous function $\phi : [0,+\infty) \to [0,+\infty)$, with $\phi(0) = 0$ such that for all $x, y \in K$,*

$$\|T_i^n x - T_i^n y\| \leq \|x-y\| + \mu_n \phi(\|x-y\|) + \ell_n\ \forall\, n \geq 1,\ i = 1,2,...,m.$$

**Proof.** Let $I := \{1,2,...,m\}$. Since $T_1, T_2, ..., T_m : K \to K$ are $m$ total asymptotically nonexpansive mappings, then there exist sequences $\{\mu_{in}\}$, $\{\ell_{in}\} \subset [0,+\infty)$ with $\lim\limits_{n\to\infty} \mu_{in} = 0 = \lim\limits_{n\to\infty} \ell_{in}$ and nondecreasing continuous function $\phi_i : [0,+\infty) \to [0,+\infty)$, with $\phi_i(0) = 0$ such that for all $x, y \in K$,

$$\|T_i^n x - T_i^n y\| \leq \|x-y\| + \mu_{in}\phi_i(\|x-y\|) + \ell_{in}\ \forall\, n \geq 1,\ \forall\, i \in I.$$

Setting $\mu_n := \max\limits_{i\in I}\{\mu_{in}\}$, $\ell_n := \max\limits_{i\in I}\{\ell_{in}\}$ and defining $\phi : [0,+\infty) \to [0,+\infty)$ by $\phi(t) = \max\limits_{i\in I}\{\phi_i(t)\},\ \forall\ t \in [0,+\infty)$, then $\phi$ is nondecreasing continuous with $\phi(0) = 0$; the sequences $\{\mu_n\}$, $\{\ell_n\}$ belong to $[0,+\infty)$ and are such that $\lim\limits_{n\to\infty} \mu_n = 0 = \lim\limits_{n\to\infty} \ell_n$ and for all $x, y \in K$,

$$\|T_i^n x - T_i^n y\| \leq \|x-y\| + \mu_n\phi(\|x-y\|) + \ell_n\ \forall\, n \geq 1,\ i \in I.$$

This completes the proof. $\square$

**Remark 9.** In what follows, $\mu_n := \max_{i\in I}\{\mu_{in}\}$, $\ell_n := \max_{i\in I}\{\ell_{in}\}$ and $\phi(t) = \max_{i\in I}\{\phi_i(t)\}$, $\forall\ t \in [0,+\infty)$. We shall assume that $\{\alpha_n\}_{n\geq 1}$ is a sequence in $(0,1)$ such that $\mu_{q(n)} = o(\alpha_n)$, $\ell_{q(n)} = o(\alpha_n)$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = +\infty$. We shall furher assume that there exists constants $M_0 > 0, M_1 > 0$ such that $\phi(t) \leq M_0 t$ for all $t > M_1$.

### 3.1. CONVERGENCE OF EXPLICIT ITERATION SCHEME

We now state and prove the following theorem.

**Theorem 10.** *Let $K$ be a nonempty closed and convex subset of a smooth reflexive real Banach space $E$ which admits weakly sequentially continous normalized duality mapping and $T_i : K \to K,\ i = 1,2,...,m$ be $m$ uniformly continuous total asymptotically nonexpansive mappings such that $F := \bigcap_{i=1}^{m} F(T_i) \neq \emptyset$. Let $\{x_n\}_{n\geq 1}$ be given by (11), then $\{x_n\}_{n\geq 1}$ is bounded. Moreover, if $P : K \to F$ is a sunny nonexpansive retraction of $K$ onto $F$, then the following are equivalent:*

*(i)* $\{x_n\}_{n\geq 1}$ *converges to the common fixed point $Pu$ of $\{T_i\}_{i=1}^m$.*

*(ii)* $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0.$

**Proof.** It is easy to see from (11) that $\{x_n\}_{n\geq 1}$ is well defined since $K$ is convex and nonempty. We first show that $\{x_n\}_{n\geq 1}$ is bounded. Let $p \in F$, then we obtain using (11) that

$$\begin{aligned} \|x_{n+1} - p\| &= \Big\|\alpha_n(u-p) + (1-\alpha_n)\Big(T_{j(n)}^{q(n)}x_n - p\Big)\Big\| \\ &\leq \alpha_n\|u-p\| + (1-\alpha_n)\Big\|T_{j(n)}^{q(n)}x_n - p\Big\| \\ &\leq (1-\alpha_n)\|x_n - p\| + \alpha_n\|u-p\| \\ &\quad +(1-\alpha_n)\Big[\mu_{q(n)}\phi\Big(\|x_n-p\|\Big) + \ell_{q(n)}\Big]. \end{aligned} \tag{16}$$

Since $\phi$ is a continuous function, it follows that $\phi$ attains a maximum (say $M$) in the interval $[0, M_1]$ and (by our assumption, see Remark 9) $\phi(t) \leq M_0 t$ whenever $t > M_1$. In either case, we have that

$$\phi(t) \leq M + M_0 t \; \forall \; t \in [0, +\infty). \tag{17}$$

Thus, using (16) and (17), we get

$$\|x_{n+1} - p\| \leq \|x_n - p\| - \Big(\alpha_n - (1-\alpha_n)\mu_{q(n)} M_0\Big)\|x_n - p\| + \sigma_n, \tag{18}$$

where $\sigma_n = (1-\alpha_n)\big(\mu_{q(n)}M + \ell_{q(n)}\big) + \alpha_n\|u-p\|$. Since $\mu_{q(n)} = o(\alpha_n)$ and $\ell_{q(n)} = o(\alpha_n)$, we may assume without loss of generality that there exist $k_0 \in (0,1)$ and $M_2 > 0$ such that for all $n \geq 1$,

$$\frac{\mu_{q(n)}}{\alpha_n} \leq \frac{(1-k_0)}{M_0(1-\alpha_n)} \text{ and } \frac{\sigma_n}{\alpha_n} \leq M_2.$$

Thus, we obtain from (18) that

$$\|x_{n+1} - p\| \leq \|x_n - p\| - k_0\alpha_n\|x_n - p\| + \sigma_n;$$

and by Lemma 5, we have that

$$\|x_n - p\| \leq \max\{\|x_0 - p\|, \; (1+k_0)M_2\}.$$

Thus, the sequence $\{x_n\}_{n\geq 0}$ is bounded.

Next, it is easy to see that if $x_n \to Pu$ as $n \to \infty$, then $\lim\limits_{n\to\infty}\|x_{n+1} - x_n\| = 0$. It therefore suffices to prove the converse. Suppose that $\lim\limits_{n\to\infty}\|x_{n+1} - x_n\| = 0$, we show that $\{x_n\}_{n\geq 1}$ congerges strongly to $Pu \in F$. Now, since $\{x_n\}_{n\geq 1}$ is bounded, there exists a constant $R_1 > 0$ such that for fixed $p \in F$, $x_n \in \overline{B_{R_1}(p)} = \{x \in E : \|x-p\| \leq R_1\}$ for all $n \geq 1$, Thus, $\|x_n - p\| \leq R_1 \; \forall \; n \geq 1$. This implies that

$$\|x_n\| \leq \|x_n - p\| + \|p\| \leq R_1 + \|p\| \; \forall \; n \geq 1.$$

Observe that if $R_1 \leq M_1$, then since $\phi$ is a continuous nondecreasing function, we have that $\phi(\|x_n - p\|) \leq \phi(R_1) \leq M$, where $M = \max\limits_{t\in[0,M_1]}\phi(t)$. On the other hand, if $R_1 > M_1$, then $\phi(R_1) \leq M_0R_1$ so that $\phi(\|x_n - p\|) \leq \phi(R_1) \leq M_0R_1$. In either case, we have that

$$\phi(\|x_n - p\|) \leq M + M_0R_1 := M_3 \; \forall \; n \geq 1.$$

So,

$$\begin{aligned} \left\|T^{q(n)}_{j(n)}x_n\right\| &\leq \left\|T^{q(n)}_{j(n)}x_n - p\right\| + \|p\| \\ &\leq \|x_n - p\| + \mu_{q(n)}\phi(\|x_n - p\|) + \ell_{q(n)} + \|p\| \\ &\leq R_1 + \mu_{q(n)}M_3 + \ell_{q(n)} + \|p\|; \end{aligned} \tag{19}$$

and since $\{\mu_n\}_{n\geq 0}$ and $\{\ell_n\}_{n\geq 0}$ are null sequences, we obtain from (19) that $\left\{T_{j(n)}^{q(n)}x_n\right\}_{n\geq 1}$ is bounded. From the recursion formula (11), using the boundedness of $\{x_n\}_{n\geq 1}$, $\left\{T_{j(n)}^{q(n)}x_n\right\}_{n\geq 1}$ and the fact that $\alpha_n \to 0$ as $n \to \infty$, we obtain from (11) that

$$\lim_{n\to\infty}\left\|x_{n+1} - T_{j(n)}^{q(n)}x_n\right\| = 0. \tag{20}$$

But,

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{21}$$

Thus, using (20) and (21) we get

$$\left\|x_n - T_{j(n)}^{q(n)}x_n\right\| \leq \|x_n - x_{n+1}\| + \left\|x_{n+1} - T_{j(n)}^{q(n)}x_n\right\| \to 0 \quad \text{as } n\to\infty. \tag{22}$$

Furthermore, we get using (21) that

$$\lim_{n\to\infty}\|x_n - x_{n-i}\| = 0 = \lim_{n\to\infty}\|x_n - x_{n+i}\|, i = 1,2,...,m. \tag{23}$$

By uniform continuity of $T_i,\ i = 1,2,...,m$, there exists a continuous increasing function $\pi_i : \mathbb{R} \to \mathbb{R}$ with $\pi_i(0) = 0$ such that $\|T_ix - T_iy\| \leq \pi_i(\|x-y\|)\ \forall\, x,y \in K,\ i = 1,2,...,m$. Thus, defining $\pi_0 : \mathbb{R} \to \mathbb{R}$ by $\pi_0(t) = \max_{1\leq i\leq m}\pi_i(t)\ \forall\ t \in \mathbb{R}$, we have that $\pi_0$ is a continuous increasing function with $\pi_0(0) = 0$ and

$$\begin{aligned}\left\|x_n - T_{j(n)}x_n\right\| &\leq \left\|x_n - T_{j(n)}^{q(n)}x_n\right\| + \left\|T_{j(n)}^{q(n)}x_n - T_{j(n)}x_n\right\| \\ &\leq \left\|x_n - T_{j(n)}^{q(n)}x_n\right\| + \pi_0\left\|T_{j(n)}^{q(n)-1}x_n - x_n\right\|. \end{aligned}\tag{24}$$

Observe that from the second summand on the right hand side (second line) of (24), we get

$$\begin{aligned}\left\|T_{j(n)}^{q(n)-1}x_n - x_n\right\| \leq\ & \left\|T_{j(n)}^{q(n)-1}x_n - T_{j(n-m)}^{q(n)-1}x_{n-m}\right\| \\ &+\left\|T_{j(n-m)}^{q(n)-1}x_{n-m} - x_{n-m}\right\| \\ &+\|x_{n-m} - x_n\|.\end{aligned}\tag{25}$$

But by (15),

$$q(n-m) = q(n) - 1 \ \text{ and } \ j(n-m) = j(n).$$

Considering the first two summands on the right hand side of (25), it then follows that the first summand,

$$\begin{aligned}\left\|T_{j(n)}^{q(n)-1}x_n - T_{j(n-m)}^{q(n)-1}x_{n-m}\right\| &= \left\|T_{j(n)}^{q(n)-1}x_n - T_{j(n)}^{q(n)-1}x_{n-m}\right\| \\ &\leq \|x_n - x_{n-m}\| \\ &\quad +\mu_{q(n)-1}\phi(\|x_n - x_{n-m}\|) + \\ &\quad \ell_{q(n)-1}. \end{aligned} \tag{26}$$

Thus, (26) implies that

$$\lim_{n\to\infty}\left\|T_{j(n)}^{q(n)-1}x_n - T_{j(n-m)}^{q(n)-1}x_{n-m}\right\| = 0. \tag{27}$$

Moreover, the second summand,

$$\left\|T_{j(n-m)}^{q(n)-1}x_{n-m} - x_{n-m}\right\| = \left\|T_{j(n-m)}^{q(n-m)}x_{n-m} - x_{n-m}\right\| \to 0 \tag{28}$$

as $n \to +\infty$

So, using (27) and (28) in (25), we obtain that

$$\lim_{n\to\infty}\|T_{j(n)}^{q(n)-1}x_n - x_n\| = 0.$$

As a result, we obtain from (22) and (24) (using the property of $\pi_0$) that

$$\lim_{n\to\infty}\|x_n - T_{j(n)}x_n\| = 0. \tag{29}$$

Furthermore, we obtain for $i = 1, 2, ..., m$ that

$$\begin{aligned}\|x_n - T_{j(n)+i}x_n\| &\leq \|x_n - x_{n+i}\| + \|x_{n+i} - T_{j(n)+i}x_{n+i}\| \\ &\quad +\|T_{j(n)+i}x_{n+i} - T_{j(n)+i}x_n\| \\ &\leq \|x_n - x_{n+i}\| + \|x_{n+i} - T_{j(n)+i}x_{n+i}\| \\ &\quad +\pi_0(\|x_{n+i} - x_n\|). \end{aligned} \tag{30}$$

So, using (23), (29) and (30), we have that

$$\lim_{n\to\infty}\|x_n - T_{j(n)+i}x_n\| = 0, \ i = 1, 2, ..., m. \tag{31}$$

But for all $i \in \{1, 2, ..., m\}$, there exists $\theta_i \in \{1, 2, ..., m\}$ such that

$$j(n) + \theta_i = i \ (mod\, m).$$

It therefore follows from (31) that

$$\lim_{n\to\infty}\|x_n - T_i x_n\| = \lim_{n\to\infty}\|x_n - T_{j(n)+\theta_i}x_n\| = 0.$$

Hence, $\lim_{n\to\infty}\|x_n - T_i x_n\| = 0$ for all $i \in \{1, 2, ..., m\}$.

Next, let $\tilde{x} = Pu$, then we obtain from the recursion formula (11) using Lemma 4 that

$$\begin{aligned} \|x_{n+1} - \tilde{x}\|^2 &\leq (1-\alpha_n)^2 \Big\| T_{j(n)}^{q(n)} x_n - \tilde{x} \Big\|^2 \\ &+ 2\alpha_n \Big\langle u - \tilde{x}, j(x_{n+1} - \tilde{x}) \Big\rangle. \end{aligned} \tag{32}$$

Using the fact that $T_1, T_2, ..., T_m$ are total asymptotically nonexpansive and $\tilde{x} \in F$, we have that

$$\begin{aligned} \Big\| T_{j(n)}^{q(n)} x_n - \tilde{x} \Big\| &\leq \|x_n - \tilde{x}\| + \mu_{q(n)} \Big( M_0 \|x_n - \tilde{x}\| + M \Big) + \ell_{q(n)} \\ &\leq \|x_n - \tilde{x}\| + \delta_n, \end{aligned} \tag{33}$$

where $\delta_n = M_4 \Big( \mu_{q(n)} + \ell_{q(n)} \Big)$ for some $M_4 > 0$. Since $\{\delta_n\}_{n\geq 0}$ is a null sequence, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $\delta_n \in (0,1)$. So, for all $n \geq n_0$, we obtain from (33) that

$$\Big\| T_{j(n)}^{q(n)} x_n - \tilde{x} \Big\| \leq (1-\delta_n)\|x_n - \tilde{x}\| + \delta_n(\|x_n - \tilde{x}\| + 1). \tag{34}$$

Thus, since the function $f : [0, +\infty) \to [0, +\infty)$ defined by $f(t) = t^2$ is convex and nondecreasing function, we obtain from (34) that

$$\begin{aligned} \Big\| T_{j(n)}^{q(n)} x_n - \tilde{x} \Big\|^2 &\leq \Big( (1-\delta_n)\|x_n - \tilde{x}\| + \delta_n \big( \|x_n - \tilde{x}\| + 1 \big) \Big)^2 \\ &\leq (1-\delta_n)\|x_n - \tilde{x}\|^2 + \delta_n \big( \|x_n - \tilde{x}\| + 1 \big)^2 \\ &\leq \|x_n - \tilde{x}\|^2 + \delta_n M_5, \end{aligned} \tag{35}$$

for some $M_5 > 0$. Using (35) in (32), we get

$$\begin{aligned} \|x_{n+1} - \tilde{x}\|^2 &\leq (1-\alpha_n)\|x_n - \tilde{x}\|^2 + (1-\alpha_n)\delta_n M_5 \\ &+ 2\alpha_n \Big\langle u - \tilde{x}, j(x_n - \tilde{x}) \Big\rangle. \end{aligned} \tag{36}$$

Next, we show that

$$\limsup_{n\to\infty} \Big\langle u - \tilde{x}, j(x_n - \tilde{x}) \Big\rangle \leq 0.$$

Let $\{x_{n_r}\}_{r\geq 1}$ be a subsequence of $\{x_n\}_{n\geq 0}$ such that

$$\limsup_{n\to\infty} \Big\langle u - \tilde{x}, j(x_n - \tilde{x}) \Big\rangle = \lim_{r\to\infty} \Big\langle u - \tilde{x}, j(x_{n_r} - \tilde{x}) \Big\rangle.$$

Since $\{x_n\}_{n\geq 1}$ is bounded and the space $E$ is reflexive, there exists a subsequence $\{x_{n_{r_k}}\}_{k\geq 1}$ of $\{x_{n_r}\}_{r\geq 1}$ such that $\{x_{n_{r_k}}\}_{k\geq 1}$ converges weakly to some $x^* \in K$. Since $\lim_{k\to\infty} \|x_{n_{r_k}} - T_i x_{n_{r_k}}\| = 0,\ i =$

$1, 2, ..., m$, we obtain from Lemma 6 that $x^* \in F$. Since the normalized duality mapping $j$ is weakly sequentially continuous, we obtain from (8) that

$$\begin{aligned}\limsup_{n\to\infty}\Big\langle u-\tilde{x}, j(x_n-\tilde{x})\Big\rangle &= \lim_{r\to\infty}\Big\langle u-\tilde{x}, j(x_{n_r}-\tilde{x})\Big\rangle\\ &= \lim_{k\to\infty}\Big\langle u-\tilde{x}, j(x_{n_{r_k}}-\tilde{x})\Big\rangle\\ &= \Big\langle u-\tilde{x}, j(x^*-\tilde{x})\Big\rangle \le 0.\end{aligned}$$

Thus, defining $\xi_n := \max\Big\{0, \Big\langle u-\tilde{x}, j(x_n-\tilde{x})\Big\rangle\Big\}$, we easily see that $\lim\limits_{n\to\infty}\xi_n = 0$. Thus, we obtain from (36) that

$$\begin{aligned}\|x_{n+1}-\tilde{x}\|^2 &\le (1-\alpha_n)\|x_n-\tilde{x}\|^2 + (1-\alpha_n)\delta_n M_5 + 2\alpha_n\xi_{n+1}.\\ &\le \|x_n-\tilde{x}\|^2 - \alpha_n\|x_n-\tilde{x}\|^2 + \gamma_n, \qquad (37)\end{aligned}$$

where $\gamma_n = \delta_n M_5 + 2\alpha_n\xi_{n+1} = o(\alpha_n)$. Hence, by Lemma 5, we have that $\{x_n\}_{n\ge1}$ converges strongly to $\tilde{x} = Pu \in F$. This completes the proof. □

**Remark 11.** We note that if $T_1, T_2, ..., T_m$ in Theorem 10 were asymptotically nonexpansive mappings, the condition "there exist $M_0 > 0$ and $M_1 > 0$ such that $\phi(t) \le M_0 t$ for all $t > M_1$" is not needed. Besides, it is easy to see that every asymptitotically nonexpansive mapping $T : K \to K$ is uniformly $L$-Lipschitzian (that is, there exists a constant $L > 0$ such that $\|T^n x - T^n y\| \le L\|x-y\|$ for all $x, y \in K,\ n \ge 1$). So, every asymptotically nonexpasive mapping is uniformly continuous. Hence we have the following corollary.

**Corollary 12.** *Let $K$ be a nonempty closed and convex subset of a smooth reflexive real Banach space $E$ which admits weakly sequentially continous normalized duality mapping and $T_i : K \to K,\ i = 1, 2, ..., m$ be $m$ asymptotically nonexpansive mappings such that $F := \bigcap\limits_{i=1}^{m} F(T_i) \neq \emptyset$. Let $\{x_n\}_{n\ge1}$ be given by (11) and let $P : K \to F$ be the sunny nonexpansive mapping obtained from Theorem 13, then $\{x_n\}_{n\ge1}$ converges strongly to the common fixed point $Pu$ of $\{T_i\}_{i=1}^m$ if and only if $\lim\limits_{n\to\infty}\|x_{n+1}-x_n\| = 0$.*

**3.2.** EXISTENCE OF IMPLICIT ITERATION PROCESS AND SUNNY NONEXPANSIVE RETRACTION

**Theorem 13.** *Let $K$ be a nonempty closed and convex subset of a smooth reflexive real Banach space $E$ which admits weakly sequentially continous normalized duality mapping and $T_i : K \to K,\ i = 1, 2, ..., m$ be $m$ asymptotically nonexpansive mappings such that $F := \bigcap_{i=1}^{m} F(T_i) \neq \emptyset$. Then,*

(1) *for $n \in \mathbb{N}$ sufficiently large, there exists unique $z_n \in K$ satifying* (12).
(2) *If, in addition we assume that $\lim_{n\to\infty} \|z_{n+1} - z_n\| = 0$, then $\{z_n\}$ converges strongly to a common fixed point of $T_1, T_2, ..., T_m$.*
(3) *Moreover, $F$ is a sunny nonexpansive retract of $K$.*

**Proof.** For each $n \in \mathbb{N}$, defining the mapping $T_n : K \to K$ by

$$T_n z = \alpha_n u + (1 - \alpha_n) T_{j(n)}^{q(n)} z \ \forall\ z \in K.$$

It is not difficult to see that for $n \in \mathbb{N}$ large enough, $T_n$ is a strict contraction. To be precise, let $x, y \in K$, then since $T_1, T_2, ..., T_m$ are asymptotically nonexpansive, we have that

$$\begin{aligned} \|T_n x - T_n y\| &= (1 - \alpha_n)\left\|T_{j(n)}^{q(n)} x - T_{j(n)}^{q(n)} y\right\| \\ &\leq (1 - \alpha_n)\big(1 + \mu_{q(n)}\big)\|x - y\|. \end{aligned} \tag{38}$$

But since $\mu_{q(n)} = o(\alpha_n)$, there exists $N_0 \in \mathbb{N}$ such that for all $n \geq N_0$, $\mu_{q(n)} < \alpha_n$. Thus,

$$\forall\ n \geq N_0,\ 0 < (1 - \alpha_n)(1 + \mu_{q(n)}) < (1 - \alpha_n)(1 + \alpha_n) = 1 - \alpha_n^2 < 1$$

So, $T_n$ is a strict contraction for $n \in \mathbb{N}$ sufficiently large. Thus, for $n \in \mathbb{N}$ large enough, we obtain (*Banach Contraction Mapping Principle*) that there exists unique $z_n \in K$ such that $T_n z_n = z_n$. This implies that

$$z_n = \alpha_n u + (1 - \alpha_n) T_{j(n)}^{q(n)} z_n.$$

Next, we show that $\{z_n\}$ is bounded. Fix $p \in F$, then

$$\begin{aligned} \|z_n - p\| &= \left\|\alpha_n u + (1 - \alpha_n) T_{j(n)}^{q(n)} z_n - p\right\| \\ &\leq \alpha_n \|u - p\| + (1 - \alpha_n)(1 + \mu_{q(n)})\|z_n - p\| \end{aligned}$$

This implies that

$$\begin{aligned}\|z_n - p\| &\leq \frac{\alpha_n}{1-(1-\alpha_n)(1+\mu_{q(n)})}\|u-p\| \\ &= \frac{\alpha_n}{\alpha_n-(1-\alpha_n)\mu_{q(n)}}\|u-p\| \\ &= \frac{1}{1-\frac{(1-\alpha_n)\mu_{q(n)}}{\alpha_n}}\|u-p\|. \end{aligned} \tag{39}$$

Since $\mu_{q(n)} = o(\alpha_n)$ and $\lim\limits_{n\to\infty}(1-\alpha_n) = 1$, we obtain from (39) that $\{z_n\}$ is bounded. Thus, we obtain from (12) that

$$\left\| z_n - T_{j(n)}^{q(n)} z_n \right\| \leq \alpha_n M_6$$

for some $M_6 > 0$; and this implies that $\lim\limits_{n\to\infty} \left\| z_n - T_{j(n)}^{q(n)} z_n \right\| = 0$. Moreover, by the hypothesis, $\lim\limits_{n\to\infty} \|z_{n+1} - z_n\| = 0$. Thus, following the method of proof of converse part of Theorem 10, we obtain that $\lim\limits_{n\to\infty} \|z_n - T_i z_n\| = 0,\ i = 1, 2, ..., m$. Since $\{z_n\}_{n\geq 1}$ is bounded, and $E$ is reflexive real Banach space, there exits a subsequence $\{z_{n_k}\}_{k\geq 1}$ of $\{z_n\}_{n\geq 0}$ such $\{z_{n_k}\}_{k\geq 1}$ converges weakly to some $\tilde{y} \in K$ and since $\lim\limits_{k\to\infty} \|z_{n_k} - T_i z_{n_k}\| = 0,\ i = 1, 2, ..., m$, we obtain from Lemma 6 that $\tilde{y} \in F$. We now show that $\{z_{n_k}\}_{k\geq 1}$ converges strongly to $\tilde{y}$. Observe that

$$z_{n_k} - \tilde{y} = \alpha_{n_k}(u - \tilde{y}) + (1-\alpha_{n_k})\Big(T_{j(n_k)}^{q(n_k)} z_{n_k} - \tilde{y}\Big).$$

So, by Lemma 4, we get

$$\begin{aligned}\|z_{n_k} - \tilde{y}\|^2 &\leq (1-\alpha_{n_k})^2 \Big\| T_{j(n_k)}^{q(n_k)} z_{n_k} - \tilde{y} \Big\|^2 \\ &+ 2\alpha_{n_k}\Big\langle u - \tilde{y}, j(z_{n_k} - \tilde{y})\Big\rangle \\ &\leq \Big((1-\alpha_{n_k})(1+\mu_{q(n_k)})\Big)^2 \|z_{n_k} - \tilde{y}\|^2 \\ &+ 2\alpha_{n_k}\Big\langle u - \tilde{y}, j(z_{n_k} - \tilde{y})\Big\rangle. \end{aligned} \tag{40}$$

Since $(1-\alpha_{n_k})(1+\mu_{(qn_k)})<1$ for $k$ sufficiently large, we obatin from (40) that

$$\begin{aligned}
\|z_{n_k}-\tilde{y}\|^2 &\leq \frac{2\alpha_{n_k}}{1-(1-\alpha_{n_k})(1+\mu_{q(n_k)})}\Big\langle u-\tilde{y}, j(z_{n_k}-\tilde{y})\Big\rangle \\
&= \frac{2\alpha_{n_k}}{\alpha_{n_k}-(1-\alpha_{n_k})\mu_{q(n_k)})}\Big\langle u-\tilde{y}, j(z_{n_k}-\tilde{y})\Big\rangle \\
&= \frac{2}{1-\frac{\mu_{q(n_k)}}{\alpha_{n_k}}+\mu_{q(n_k)}}\Big\langle u-\tilde{y}, j(z_{n_k}-\tilde{y})\Big\rangle. \qquad (41)
\end{aligned}$$

Now, since $\{x_{n_k}\}_{k\geq 1}$ converges weakly to $\tilde{y}$ and $j$ is weakly sequentially continuous, we have that $\lim\limits_{n\to\infty}\Big\langle u-\tilde{y}, j(z_{n_k}-\tilde{y})\Big\rangle=0$. Also, $\lim\limits_{k\to\infty}\Big(1-\frac{\mu_{q(n_k)}}{\alpha_{n_k}}+\mu_{n_k}\Big)=1$. Thus, from (41), we have that

$$\lim_{k\to\infty}\|z_{n_k}-\tilde{y}\|=0.$$

Next, we show that $\{z_n\}_{n\geq 0}$ converges to $\tilde{y}$. Suppose that there is another subsequence $\{z_{n_i}\}_{i\geq 1}$ of $\{z_n\}_{n\geq 0}$ such that $z_{n_i}\to q^*,\ q^*\neq\tilde{y}$ as $i\to\infty$, then for $p\in F$,

$$\begin{aligned}
\Big\langle z_n-T^{q(n)}_{j(n)}z_n, j(z_n-p)\Big\rangle &= \Big\langle z_n-p, j(z_n-p)\Big\rangle \\
&+ \Big\langle p-T^{q(n)}_{j(n)}z_n, j(z_n-p)\Big\rangle \\
&\geq \|z_n-p\|^2-\Big\|p-T^{q(n)}_{j(n)}z_n\Big\|.\|z_n-p\| \\
&\geq \|z-p\|^2-(1+\mu_{q(n)})\|z_n-p\|^2. \qquad (42)
\end{aligned}$$

Since $\{z_n\}_{n\geq 0}$ is bounded and $z_n-T^{q(n)}_{j(n)}z_n=\frac{\alpha_n}{1-\alpha_n}(u-z_n)$, we get from (42) that

$$\begin{aligned}
\Big\langle u-z_n, j(z_n-p)\Big\rangle &\geq -\Big(\frac{(1-\alpha_n)\mu_{q(n)}}{\alpha_n}\|z_n-p\|^2\Big) \\
&\geq -\Big(\frac{(1-\alpha_n)\mu_{q(n)}}{\alpha_n}M_7\Big) \qquad (43)
\end{aligned}$$

for some $M_7>0$. So, (43) implies that

$$\Big\langle z_n-u, j(z_n-p)\Big\rangle\leq\frac{(1-\alpha_n)\mu_{q(n)}}{\alpha_n}M_7. \qquad (44)$$

Thus, since $\tilde{y} \in F$ and $q^* \in F$, we get from (44) that

$$\limsup_{k\to\infty} \Big\langle z_{n_k} - u, j(z_{n_k} - q^*)\Big\rangle \le 0$$
$$\text{and } \limsup_{i\to\infty} \Big\langle z_{n_i} - u, j(z_{n_i} - \tilde{y})\Big\rangle \le 0. \tag{45}$$

Since $j$ is weakly sequntially continuous, then careful usage of (45) gives

$$\Big\langle \tilde{y} - u, j(\tilde{y} - q^*)\Big\rangle \le 0 \text{ and } \Big\langle q^* - u, j(q^* - \tilde{y})\Big\rangle \le 0.$$

This implies that

$$\Big\langle \tilde{y} - u, j(\tilde{y} - q^*)\Big\rangle \le 0 \tag{46}$$

and

$$\Big\langle u - q^*, j(\tilde{y} - q^*)\Big\rangle \le 0. \tag{47}$$

So, adding (46) and (47), we have that

$$\Big\langle \tilde{y} - q^*, j(\tilde{y} - q^*)\Big\rangle \le 0 \Leftrightarrow \|\tilde{y} - q^*\| \le 0,$$

a contradiction. Hence, $\{z_n\}_{n\ge0}$ converges strongly to $\tilde{y}$.

Finally, observe that from (43),

$$\Big\langle z_n - u, j(z_n - p)\Big\rangle \;\le\; \frac{(1-\alpha_n)\mu_{q(n)}}{\alpha_n}\|z_n - p\|^2, \tag{48}$$

so that taking limit as $n \to \infty$ on both sides of (48) using the facts that $\{z_n\}_{n\ge0}$ converges strongly to $\tilde{y}$ (thus, converges weakly to $\tilde{y}$) and $j$ is weakly sequentially continuous, we obtain from (48) that for $u \in K$,

$$\Big\langle \tilde{y} - u, j(\tilde{y} - p)\Big\rangle \le 0 \;\forall\, p \in F.$$

Thus, defining $P : K \to F$ by $Pu = \tilde{y}$, we have by Proposition 3 that $P$ is a sunny nonexpansive retraction of $K$ onto $F = \displaystyle\bigcap_{i=1}^{m} F(T_i)$.

This completes the proof. □

**Remark 14.** A prototype for $\phi : [0,\infty) \to [0,\infty)$ satisfying the conditions of our theorems is $\phi(\lambda) = \lambda^s,\ 0 < s \le 1$.

**Remark 15.** Addition of *bounded* (or the so called mean) error terms to the iteration process studied in this paper leads to no further generalization.

**Remark 16.** If $f : K \to K$ is a contraction map and we replace $u$ by $f(x_n)$ in the recursion formulas of our theorems, we obtain what some authors now call *viscosity* iteration process. We observe that all our theorems in this paper carry over trivially to the so-called viscosity process. One simply replaces $u$ by $f(x_n)$, repeats the argument of this paper, using the fact that $f$ is a contraction map.

**Remark 17.** In [2], Alber *et al.* studied iteration processes for approximation of fixed point of *single* total asymptotically nonexpansive mapping in the setting of the results obtained in this paper; and under similar assumptions. Our theorems which hold for any *finite* family of total asymptotically nonexpansive mappings satisfying the conditions of our theorems thus augument and improve the corresponding results of Alber *et al.* [2]. Furthermore, our Example 2 is of independent interest.

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