Journal of the Nigerian Mathematical Society
Vol. 33, pp. 113-127, 2014

MODIFIED ITERATIVE ALGORITHM FOR FAMILY OF ASYMPTOTICALLY NONEXPANSIVE MAPPINGS IN BANACH SPACES

BASHIR ALI[1] AND G.C. UGWUNNADI

ABSTRACT. In this paper we introduce a new modified iterative scheme for approximation of common fixed points of countably infinite family of asymptotically nonexpansive mappings and solutions of some variational inequality problems. We prove strong convergence theorem that extend and generalize some recent results. Our Theorem particularly, is applicable in l_p spaces $(1 < p < \infty)$.

Keywords and phrases: Fixed point, strongly accretive mapping, Asymptotically nonexpansive mapping, strictly pseudocontractive map
2010 Mathematical Subject Classification: 47H09, 47J25

1. INTRODUCTION

Let E be a real Banach space and E^* be the dual space of E. A mapping $\varphi : [0,\infty) \to [0,\infty)$ is called a guage function if it is strictly increasing, continuous and $\varphi(0) = 0$. Let φ be a gauge function, a generalized duality mapping with respect to φ, $J_\varphi : E \to 2^{E^*}$ is defined by, $x \in E$,

$$J_\varphi x = \{x^* \in E^* : \langle x, x^*\rangle = ||x||\varphi(||x||), ||x^*|| = \varphi(||x||)\},$$

where $\langle .,. \rangle$ denotes the duality pairing between element of E and that of E^*. If $\varphi(t) = t$, then J_φ is simply called the normalized duality mapping and is denoted by J. For any $x \in E$, an element of $J_\varphi x$ is denoted by $j_\varphi x$.

The space E is said to have weakly (sequentially) continuous duality map if there exists a gauge function φ such that J_φ is single valued and (sequentially) continuous from E with weak topology to E^* with weak* topology.

A Banach space E is said to satisfy *Opial's condition* [11] if for any

Received by the editors November 16 2013; Revised: January 17, 2014; Accepted: January 20, 2014

[1]Corresponding author

sequence $\{x_n\} \subset E$, $x_n \rightharpoonup x$ as $n \to \infty$ implies that

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|, \ \forall y \in E, y \neq x.$$

All Hilbert spaces and l_p spaces, $1 < p < \infty$ satisfy Opial's condition.

The space E is said to have *uniform Opial's condition* [13], if for each $c > 0$, there exists an $r > 0$ such that

$$1 + r \leq \underset{n\to\infty}{liminf} \|x + x_n\|$$

for each $x \in E$ with $\|x\| \geq c$ and each sequence $\{x_n\} \subset E$ satisfying $x_n \rightharpoonup 0$ as $n \to \infty$, and $\underset{n\to\infty}{liminf} \|x_n\| \geq 1$.

E is said to satisfy local uniform Opial's condition [6] if for any weak null sequence $\{x_n\}$ in E with $\underset{n\to\infty}{liminf} \|x_n\| \geq 1$ and any $c > 0$, there exists $r > 0$ such that

$$1 + r \leq \underset{n\to\infty}{liminf} \|x + x_n\|$$

for all $x \in E$ with $\|x\| \geq c$.

Remark 1.1 Observe that uniform Opial's condition implies local uniform Opial's condition which in turn implies Opial's condition. It is also known that every Banach space with weakly sequentially continuous duality mapping satisfies uniform Opial's condition (see [6]). Every l_p space, $(1 < p < \infty)$ has a weakly sequentially continuous duality map.

A mapping $T : E \to E$ is $L-$*Lipschitzian* if for some $L > 0, ||Tx - Ty|| \leq L||x - y|| \ \forall \ x, y \in E$. If $L \in [0, 1)$, then T is called contraction, and if $L = 1$, then T is called nonexpansive. A mapping $T : E \to E$ is called *asymptotically nonexpansive* if there exists a sequence $\{v_n\} \subset [0, \infty)$, $\lim\limits_{n\to\infty} v_n = 0$ such that for all $x, y \in E$

$$||T^n x - T^n y|| \leq (1 + v_n)||x - y|| \quad \text{for all} \quad n \in \mathbb{N}. \tag{1}$$

A point $x \in E$ is called a *fixed point of* T provided $Tx = x$. We denote by $F(T)$ the set of all fixed point of T (i.e., $F(T) = \{x \in E : Tx = x\}$).

The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [4] as an important generalization of the class of nonexpansive mappings. They (Goebel and Kirk [4]) proved that if K is a nonempty, bounded, closed and convex subset of a real uniformly convex Banach space and T is a self asymptotically nonexpansive mapping of K, then T has a fixed point in K.

The mapping T is said to be asymptotically regular if

$$\lim_{n\to\infty}||T^{n+1}x - T^n x|| = 0$$

for all $x \in K$. It is said to be uniformly asymptotically regular if for any bounded subset C of K,

$$\lim_{n\to\infty}\sup_{x\in C}||T^{n+1}x - T^n x|| = 0.$$

A mapping T is said to be *demiclosed* at a point p if whenever $\{x_n\}$ is a sequence in E such that $x_n \rightharpoonup x^* \in E$ and $Tx_n \to p$ then $Tx^* = p$.

A mapping $G : D(G) \subset E \to E$ is said to be *accretive* if for all $x, y \in D(G)$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle Gx - Gy, j(x-y)\rangle \geq 0, \tag{2}$$

where $D(G)$ denote the domain of G. For some $\eta, \mu \in (0,1)$, G is called $\eta-$ *strongly accretive* if for all $x, y \in D(G)$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle Gx - Gy, j(x-y)\rangle \geq \eta||x-y||^2, \tag{3}$$

and μ-*strictly pseudocontractive* if

$$\langle Gx - Gy, j(x-y)\rangle \leq \|x-y\|^2 - \mu\|(I-G)x - (I-G)y\|^2$$

holds $\forall x, y \in E$. It is known that if G is μ-strictly pseudocontractive then it is $(1+\frac{1}{\mu})-$Lipschitzian.
Let K be a nonempty, closed and convex subset of E and $G : K \to E$ be a nonlinear mapping. The variational inequality problem is to:

$$\text{find } u \in K \text{ such that } \langle Gu, j(v-u)\rangle \geq 0,\ \forall v \in K,$$

for some $j(v-u) \in J(v-u)$. The set of solution of variational inequality problem is denoted by $VI(K,G)$. In Hilbert spaces H, accretive operators are called monotone where inequality (2) and (3) hold with j replaced by the identity map on H, in that case the variational inequality problem reduces to:

$$\text{find } u \in K \text{ such that } \langle Gu, v-u\rangle \geq 0,\ \forall v \in K,$$

which was first studied by Stampacchia [14].

Variational inequality theory has emerged as an important tool in studying a wide class of related problems in Mathematical, Physical, Engineering and Nonlinear Optimization sciences (see, for example, [5, 7, 10],[19]-[21]).

In 2000, Moudafi [9] introduced the viscosity approximation method for nonexpansive mappings. Let f be a contraction on H, starting with an arbitrary $x_0 \in H$, define a sequence $\{x_n\}$ recursively by

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Tx_n, \ n \geq 0, \tag{4}$$

where $\{\alpha_n\}$ is a sequence in (0,1). He proved that under certain appropriate conditions on $\{\alpha_n\}$, the sequence $\{x_n\}$ generated by (4) strongly converges to the unique solution x^* in F of the variational inequality

$$\langle (I-f)x^*, x-x^*\rangle \geq 0, \text{ for all } x \in F.$$

Xu [17] in 2003, proved, under some condition on the real sequence $\{\alpha_n\}$, that the sequence $\{x_n\}$ defined by $x_0 \in H$ chosen arbitrarily,

$$x_{n+1} = \alpha_n b + (I-\alpha_n A)Tx_n, \ n \geq 0, \tag{5}$$

converges strongly to $x^* \in F$ which is the unique solution of the minimization problem

$$\min_{x\in F} \frac{1}{2}\langle Ax, x\rangle - \langle x, b\rangle,$$

where A is a strongly positive bounded linear operator (i.e. $\exists \, \bar{\gamma} > 0$ such that $\langle Ax, x\rangle \geq \bar{\gamma}||x||^2, \ \forall x \in H$).

Combining the iterative method (4) and (5), Marino and Xu [8] studied the following general iterative method:

$$x_{n+1} = \alpha_n f(x_n) + (I-\alpha_n A)Tx_n, \ n \geq 0, \tag{6}$$

they proved that if the sequence $\{\alpha_n\}$ of parameters satisfies appropriate conditions, then the sequence $\{x_n\}$ generated by (6) converges strongly to $x^* \in F$ which solves the variational inequality problem

$$\langle (\gamma f - A)x^*, x-x^*\rangle \leq 0 \ \forall x \in F,$$

which is the optimality condition for the minimization problem

$$\min_{x\in F} \frac{1}{2}\langle Ax, x\rangle - h(x),$$

where h is a potential function for γf (i.e. $h'(x) = \gamma f(x)$ for $x \in H$).

On the other hand, Yamada [19] in 2001 introduced the following hybrid iterative method:

$$x_{n+1} = Tx_n - \lambda_n \mu G T x_n, \ n \geq 0, \tag{7}$$

where G is a κ-Lipschitzian and η-strongly monotone operator with $\kappa > 0, \eta > 0$ and $0 < \mu < 2\eta/\kappa^2$. Under some appropriate conditions, he proved that the sequence $\{x_n\}$ generated by (7) converges strongly to the unique solution of the variational inequality problem

$$\langle Gx^*, x - x^* \rangle \geq 0, \ \forall x \in F.$$

Recently, combining (6) and (7), Tian [16] considered the following general iterative method:

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n \mu G)T(x_n), \tag{8}$$

and proved that the sequence $\{x_n\}$ generated by (8) converges strongly to the unique solution $x^* \in F$ of the variational inequality problem

$$\langle (\gamma f - \mu G)x^*, x - x^* \rangle \leq 0, \ \forall x \in F.$$

Ali [1] studied a modified scheme for approximation of a common fixed point of family of nonexpansive mappings in a real q-uniformly smooth Banach space which is also uniformly convex. He proved the following theorem.

Theorem 1.1 (Ali [1]) Let E be a real q-uniformly smooth Banach space which is also uniformly convex. Let K be a closed, convex and nonempty subset of E. For $\alpha > 0$, let $T_i : K \to K \ i \in \mathbb{N}$ and $A : K \to E$ be a family of nonexpansive maps and α-inverse strongly accretive map, respectively. Let P_K be a nonexpansive projection of E onto K. For some real numbers $\delta \in (0,1)$ and $\lambda \in (0, (\frac{q\alpha}{d_q})^{\frac{1}{q-1}})$ define a sequence $\{x_n\}$ iteratively by $x_1, u \in K$,

$$x_{n+1} = \alpha_n u + (1-\delta)(1-\alpha_n)x_n + \delta \sum_{i \geq 1} \sigma_{in} T_i P_K(x_n - \lambda A x_n), \quad n \geq 1 \tag{9}$$

where $\{\alpha_n\}$ and $\{\sigma_{in}\}$ are real sequences in $(0,1)$ satisfying the following conditions: (i) $\lim \alpha_n = 0$, (ii) $\sum \alpha_n = \infty$, (iii) $\sum_{i\geq 1} \sigma_{in} = 1 - \alpha_n$ and $\lim\limits_{n\to\infty} \sum_{i \geq 1} |\sigma_{i,n+1} - \sigma_{in}| = 0$. Let $F := [\cap_{i=1}^{\infty} F(T_i)] \cap VI(K,A) \neq \emptyset$. If either the duality map j of E admits weak sequentially continuity or for at least one $i \in \mathbb{N}$, $T_i P_K(I - \lambda A)$ is demicompact, then $\{x_n\}$ converges strongly to some element in F. Most recently, Ali et al. [3], extended the result of Tian [16] to q-uniformly smooth Banach space whose duality mapping is weakly sequentially continuous. Under some assumption on $\{\alpha_n\}, \gamma, \mu$ and G, they proved that the sequence $\{x_n\}$ generated by (8) converges strongly to the unique solution $x^* \in F$ of the variational inequality problem

$$\langle (\gamma f - \mu G)x^*, j(x - x^*)\rangle \leq 0, \ \forall x \in F.$$

Motivated by these result, it is our purpose in this paper to introduce a new modified iterative scheme for approximation of common fixed points of family of asymptotically nonexpansive mappings and solution of some variational inequality problem. We prove strong convergence theorem that extend and generalize some recent results. Our Theorem particularly, is applicable in l_p spaces $(1 < p < \infty)$.

2. PRELIMINARY

The following lemmas and Theorem are used for our main result.

Lemma 2.1 Let E be a real normed space. Then

$$||x + y||^2 \leq ||x||^2 + 2\langle y, j(x + y)\rangle,$$

for all $x, y \in E$ and for all $j(x + y) \in J(x + y)$.

Lemma 2.2 (Xu [18]) Let $\{a_n\}$ be a sequence of nonegative real numbers satisfying the following relation:

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n\sigma_n + \gamma_n, \ n \geq 0$$

where, (i) $\{\alpha_n\} \subset [0, 1]$, $\sum \alpha_n = \infty$; (ii) $\limsup \sigma_n \leq 0$;; (iii) $\gamma_n \geq 0$; $(n \geq 0)$, $\sum \gamma_n < \infty$. Then, $a_n \to 0$ as $n \to \infty$.

Lemma 2.3 (Suzuki [15]) Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space E and let $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf \beta_n \leq \limsup \beta_n < 1$. Suppose that $x_{n+1} = \beta_n y_n + (1 - \beta_n)x_n$ for all integer $n \geq 1$ and $\limsup_{n\to\infty}(||y_{n+1} - y_n|| - ||x_{n+1} - x_n||) \leq 0$. Then, $\lim_{n\to\infty} ||y_n - x_n|| = 0$.

Lemma 2.4 (Piri and Vaezi [12] see also [2]) Let E be a real Banach space and $G : E \to E$ be a mapping.
(i) If G is $\eta-$*strongly* accretive and $\mu-$*strictly* pseudo-contractive with $\eta + \mu > 1$, then $I - G$ is contractive with constant $\sqrt{\frac{1-\eta}{\mu}}$.
(ii) If G is $\eta-$*strongly* accretive and $\mu-$*strictly* pseudo-contractive with $\eta + \mu > 1$, then for any fixed number $\kappa \in (0, 1)$, $I - \kappa G$ is contractive with constant $1 - \kappa\Big(1 - \sqrt{\frac{1-\eta}{\mu}}\Big)$.

Theorem 2.5 (Lin et al. [6]) Suppose E is a Banach space satisfying the locally uniform Opial condition, C is a nonempty weakly compact convex subset of E, and $T : C \to C$ is an asymptotically nonexpansive mapping. Then (I-T) is demiclosed at zero.

3. THE MAIN RESULTS

In the sequel we assume for the sequences $\{\alpha_n\}, \{\sigma_{in}\} \subset (0,1)$, that $\sum\limits_{i\geq 1}\sigma_{in} := 1-\alpha_n$ for each $n \in \mathbb{N}$.

Theorem 3.1 Let E be a real Banach space whose duality map is weakly sequentially continuous. Let $G : E \to E$ be an η-strongly accretive and μ-strictly pseudocontractive with $\eta + \mu > 1$ and let $f : E \to E$ be a contraction with coefficient $\alpha \in (0,1)$. Let $\{T_i\}_{i=1}^{\infty}$ be a family of uniformly asymptotically regular asymptotically nonexpansive self mappings of E with sequences $\{v_{in}\}$ such that $v_{in} \to 0$ as $n \to \infty$ for each $i \geq 1$ and $F = \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Assume that $\gamma \in \Big(0, \min\{\frac{\tau}{2\alpha}, \eta\}\Big)$, where $\tau := (1-\sqrt{\frac{1-\eta}{\mu}})$. Let $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ be sequences in $(0,1)$, and suppose that the following conditions are satisfied:

(C1) $\lim\limits_{n\to\infty}\beta_n = 0$ and $\lim\limits_{n\to\infty}\frac{v_n}{\beta_n} = 0$, where $v_n := \sup\limits_{i\geq 1}\{v_{in}\}$
(C2) $\Sigma_{n=0}^{\infty}\beta_n = \infty$
(C3) $\lim\limits_{n\to\infty}\alpha_n = 0$.
(C4) $\mathop{lim}\limits_{n\to\infty} \sum_{i\geq 1} |\sigma_{in+1} - \sigma_{in}| = 0$

For some fixed $\delta \in (0,1)$, let $\{x_n\}_{n=1}^{\infty}$ be a sequence defined iteratively by $x_0 \in E$ chosen arbitrarily,

$$\begin{cases} x_{n+1} = [1-\delta(1-\alpha_n)]x_n + \delta\sum_{i\geq 1}\sigma_{in}T_i^n y_n, \\ y_n = \beta_n\gamma f(x_n) + (I-\beta_n G)x_n, \quad n \geq 0. \end{cases} \tag{10}$$

Then, $\{x_n\}_{n=1}^{\infty}$ converges strongly to $p \in F$, where p is the unique solution of the variational inequality problem

$$\langle \gamma f(p) - Gp, j(q-p)\rangle \leq 0, \ \forall q \in F. \tag{11}$$

Proof: From the choice of γ, $(G-\gamma f)$ is strongly accretive, then the variational inequality (11) has a unique solution in F. Now we show that $\{x_n\}$ is bounded. Let $p \in F$, since $\frac{v_n}{\beta_n} \to 0$ as $n \to \infty$, it implies that $\frac{v_n}{(1+v_n)\beta_n} \to 0$ as $n \to \infty$, then there exists $n_0 \in \mathbb{N}$ such that $\frac{v_n}{(1+v_n)\beta_n} < \frac{\tau-\gamma\alpha}{2}$, for all $n \geq n_0$.

$$\begin{aligned} ||y_n - p|| &= ||\beta_n(\gamma f(x_n) - Gp) + (I - \beta_n G)(x_n - p)|| \\ &\leq \beta_n||\gamma f(x_n) - Gp|| + (1 - \beta_n\tau)||x_n - p|| \\ &\leq \Big(1 - \beta_n(\tau - \gamma\alpha)\Big)||x_n - p|| + \beta_n||\gamma f(p) - Gp||. \quad (12) \end{aligned}$$

Using (12), we obtain

$$\begin{aligned} ||x_{n+1} - p|| &= ||[1 - \delta(1 - \alpha_n)](x_n - p) + \delta\sum_{i\geq 1}\sigma_{in}(T_i^n y_n - p)|| \\ &\leq [1 - \delta(1 - \alpha_n)]||x_n - p|| + \delta(1 - \alpha_n)(1 + v_n)||y_n - p|| \\ &\leq \Big[1 - \delta(1 - \alpha_n) + \delta(1 - \alpha_n)(1 + v_n)[1 - \beta_n(\tau - \gamma\alpha)]\Big]||x_n - p|| \\ &\quad +\delta(1 - \alpha_n)(1 + v_n)\beta_n||\gamma f(p) - Gp|| \\ &= \Big[1 + \delta(1 - \alpha_n)v_n - \beta_n\delta(1 - \alpha_n)(1 + v_n)(\tau - \gamma\alpha)\Big]||x_n - p|| \\ &\quad +\delta(1 - \alpha_n)(1 + v_n)\beta_n||\gamma f(p) - Gp|| \\ &\leq \Big[1 - \beta_n\delta(1 - \alpha_n)(1 + v_n)\Big((\tau - \gamma\alpha) - \frac{v_n}{(1 + v_n)\beta_n}\Big)\Big]||x_n - p|| \\ &\quad +\beta_n\delta(1 - \alpha_n)(1 + v_n)\Big((\tau - \gamma\alpha) - \frac{v_n}{(1 + v_n)\beta_n}\Big)\frac{2||\gamma f(p) - Gp||}{\tau - \gamma\alpha} \\ &\leq \max\Big\{||x_n - p||, \frac{2||\gamma f(p) - Gp||}{\tau - \gamma\alpha}\Big\} \end{aligned}$$

By induction, we obtain

$$||x_n - p|| \leq \max\Big\{||x_{n_0} - p||, \frac{2||\gamma f(p) - Gp||}{\tau - \gamma\alpha}\Big\} \ \forall n \geq n_0.$$

Hence $\{x_n\}$ is bounded. Also $\{f(x_n)\}, \{G(x_n)\}, \{y_n\}, \{T_i^n x_n\}$ and $\{T_i^n y_n\}$ are all bounded.

Define two sequences by $\gamma_n := (1-\delta)\alpha_n + \delta$ and $z_n := \frac{x_{n+1} - x_n + \gamma_n x_n}{\gamma_n}$. From the recursion formula (10), observe that

$$z_n = \frac{\delta\sum_{i\geq 1}\sigma_{in}T_i^n y_n + \alpha_n x_n}{\gamma_n}$$

which implies

$$
\begin{aligned}
z_{n+1}-z_n &= \frac{\delta\sum_{i\geq 1}\sigma_{in+1}T_i^{n+1}y_{n+1}+\alpha_{n+1}x_{n+1}}{\gamma_{n+1}} \\
&\quad -\frac{\delta\sum_{i\geq 1}\sigma_{in}T_i^{n}y_{n}+\alpha_{n}x_{n}}{\gamma_{n}} \\
&= \frac{\delta\sum_{i\geq 1}\sigma_{in+1}\Big(T_i^{n+1}y_{n+1}-T_i^{n+1}y_n\Big)}{\gamma_{n+1}} \\
&\quad +\frac{\delta\sum_{i\geq 1}\sigma_{in+1}\Big(T_i^{n+1}y_{n}-T_i^{n}y_n\Big)}{\gamma_{n+1}} \\
&\quad +\Big(\frac{\delta\sum_{i\geq 1}\sigma_{in+1}T_i^{n}y_{n}}{\gamma_{n+1}}-\frac{\delta\sum_{i\geq 1}\sigma_{in}T_i^{n}y_{n}}{\gamma_{n}}\Big) \\
&\quad +\frac{\alpha_{n+1}}{\gamma_{n+1}}x_{n+1}-\frac{\alpha_n}{\gamma_n}x_n
\end{aligned}
$$

therefore,

$$
\begin{aligned}
||z_{n+1}-z_n|| &\leq \frac{\delta\sum_{i\geq 1}\sigma_{in+1}\Big\|T_i^{n+1}y_{n+1}-T_i^{n+1}y_n\Big\|}{\gamma_{n+1}} \\
&\quad +\frac{\delta\sum_{i\geq 1}\sigma_{in+1}\Big\|T_i^{n+1}y_{n}-T_i^{n}y_n\Big\|}{\gamma_{n+1}} \\
&\quad +\Big\|\frac{\delta\sum_{i\geq 1}\sigma_{in+1}T_i^{n}y_{n}}{\gamma_{n+1}}-\frac{\delta\sum_{i\geq 1}\sigma_{in}T_i^{n}y_{n}}{\gamma_{n}}\Big\| \\
&\quad +\frac{\alpha_{n+1}}{\gamma_{n+1}}||x_{n+1}||+\frac{\alpha_n}{\gamma_n}||x_n|| \\
&\leq \frac{\delta(1-\alpha_{n+1})}{\gamma_{n+1}}(1+v_{n+1})\|y_{n+1}-y_n\| \\
&\quad +\frac{\delta\sum_{i\geq 1}\sigma_{in+1}\Big\|T_i^{n+1}y_{n}-T_i^{n}y_n\Big\|}{\gamma_{n+1}} \\
&\quad +\frac{\delta}{\gamma_{n+1}\gamma_n}\Big\|\sum_{i\geq 1}\Big(\gamma_n\sigma_{in+1}-\gamma_{n+1}\sigma_{in}\Big)T_i^n y_n\Big\| \\
&\quad +\frac{\alpha_{n+1}}{\gamma_{n+1}}||x_{n+1}||+\frac{\alpha_n}{\gamma_n}||x_n||. \qquad (13)
\end{aligned}
$$

But

$$\begin{aligned} y_{n+1} - y_n &= \beta_{n+1}\gamma\Big(f(x_{n+1}) - f(x_n)\Big) + \Big(\beta_{n+1} - \beta_n\Big)\gamma f(x_n) \\ &\quad +\Big((I - \beta_{n+1}G)x_{n+1} - (I - \beta_{n+1}G)x_n\Big) \\ &\quad +\Big((I - \beta_{n+1}G)x_n - (I - \beta_n G)x_n\Big), \end{aligned}$$

so that

$$\begin{aligned} ||y_{n+1} - y_n|| &\leq \beta_{n+1}\gamma\alpha||x_{n+1} - x_n|| + |\beta_{n+1} - \beta_n|||\gamma f(x_n)|| \\ &\quad +(1 - \beta_{n+1}\tau)||x_{n+1} - x_n|| + |\beta_{n+1} - \beta_n|||G(x_n)|| \\ &= [1 - \beta_{n+1}(\tau - \gamma\alpha)]||x_{n+1} - x_n|| \\ &\quad +|\beta_{n+1} - \beta_n|\big[||\gamma f(x_n)|| + ||G(x_n)||\big] \end{aligned} \tag{14}$$

Using (14) in (13), we obtain that

$$\begin{aligned} &||z_{n+1} - z_n|| - ||x_{n+1} - x_n|| \\ &\quad\leq \Big(\frac{\delta(1 - \alpha_{n+1})}{\gamma_{n+1}}(1 + v_{n+1})[1 - \beta_{n+1}(\tau - \gamma\alpha)] - 1\Big)||x_{n+1} - x_n|| \\ &\qquad+\frac{\delta(1 - \alpha_{n+1})}{\gamma_{n+1}}(1 + v_{n+1})|\beta_{n+1} - \beta_n|\big[||\gamma f(x_n)|| + ||G(x_n)||\big] \\ &\qquad+\frac{\delta\sum_{i\geq 1}\sigma_{in+1}\Big\|T_i^{n+1}y_n - T_i^n y_n\Big\|}{\gamma_{n+1}} \\ &\qquad+\frac{\delta}{\gamma_{n+1}\gamma_n}\Big\|\sum_{i\geq 1}\Big(\gamma_n\sigma_{in+1} - \gamma_{n+1}\sigma_{in}\Big)T_i^n y_n\Big\| \\ &\qquad+\frac{\alpha_{n+1}}{\gamma_{n+1}}||x_{n+1}|| + \frac{\alpha_n}{\gamma_n}||x_n|| \\ &\quad\leq \Big(\frac{\delta(1 - \alpha_{n+1})}{\gamma_{n+1}}(1 + v_{n+1})[1 - \beta_{n+1}(\tau - \gamma\alpha)] - 1\Big)||x_{n+1} - x_n|| \\ &\qquad+\frac{\delta(1 - \alpha_{n+1})}{\gamma_{n+1}}(1 + v_{n+1})|\beta_{n+1} - \beta_n|\big[||\gamma f(x_n)|| + ||G(x_n)||\big] \\ &\qquad+\frac{\delta\sum_{i\geq 1}\sigma_{in+1}\Big\|T_i^{n+1}y_n - T_i^n y_n\Big\|}{\gamma_{n+1}} \\ &\qquad+\frac{\delta M^*}{\gamma_{n+1}\gamma_n}\Big[\gamma_n\sum_{i\geq 1}|\sigma_{in+1} - \sigma_{in}| + |\gamma_n - \gamma_{n+1}|(1 - \alpha_n)\Big] \\ &\qquad+\frac{\alpha_{n+1}}{\gamma_{n+1}}||x_{n+1}|| + \frac{\alpha_n}{\gamma_n}||x_n||, \end{aligned}$$

for some $M^* > 0$ and this implies

$$\limsup_{n\to\infty}(||z_{n+1} - z_n|| - ||x_{n+1} - x_n||) \leq 0.$$

Consequently, by Lemma 2.3, we have

$$\lim_{n\to\infty} ||z_n - x_n|| = 0.$$

Hence

$$||x_{n+1} - x_n|| = (1 - \gamma_n)||z_n - x_n|| \to 0 \text{ as } n \to \infty. \tag{15}$$

From the recursion formula (10), we obtain

$$\delta \sum_{i\geq 1} \sigma_{in}||T_i^n y_n - x_n|| = ||x_{n+1} - x_n|| \to 0 \quad \text{as} \quad n \to \infty.$$

This implies that for each $i \geq 1$,

$$\lim_{n\to\infty} ||T_i^n y_n - x_n|| = 0. \tag{16}$$

Also, from the recursion formula (10), we obtain

$$||y_n - x_n|| = \beta_n||\gamma f(x_n) - G(x_n)|| \to 0 \quad \text{as} \quad n \to \infty. \tag{17}$$

which implies that for each $i \geq 1$,

$$||T_i^n y_n - y_n|| \leq ||T_i^n y_n - x_n|| + ||x_n - y_n|| \to 0 \text{ as } n \to \infty. \tag{18}$$

From (16) and (17), for each $i \geq 1$, we obtain

$$\begin{aligned} ||T_i^n x_n - x_n|| &\leq ||T_i^n x_n - T_i^n y_n|| + ||T_i^n y_n - x_n|| \\ &\leq (1 + v_n)||x_n - y_n|| + ||T_i^n y_n - x_n|| \to 0 \text{ as } n \to \infty. \end{aligned} \tag{19}$$

Therefore

$$\begin{aligned} ||T_i x_n - x_n|| &\leq ||T_i x_n - T_i^{n+1} x_n|| + ||T_i^{n+1} x_n - T_i^{n+1} x_{n+1}|| \\ &\quad + ||T_i^{n+1} x_{n+1} - x_{n+1}|| + ||x_{n+1} - x_n|| \\ &\leq L_i||x_n - T_i^n x_n|| + (2 + v_{n+1})||x_{n+1} - x_n|| \\ &\quad + ||T_i^{n+1} x_{n+1} - x_{n+1}||, \end{aligned}$$

for each $i \geq 1$, also by using (15) and (19), we obtain

$$\lim_{n\to\infty} ||T_i x_n - x_n|| = 0 \text{ for each } i \geq 1. \tag{20}$$

we also have

$$\begin{aligned} ||T_i y_n - y_n|| &\leq ||T_i y_n - T_i x_n|| + ||T_i x_n - x_n|| + ||x_n - y_n|| \\ &\leq (1 + L_i)||y_n - x_n|| + ||T_i x_n - x_n||. \end{aligned}$$

This implies that,

$$\lim_{n\to\infty} ||T_i y_n - y_n|| = 0 \text{ for each } i \geq 1. \tag{21}$$

Let $\{y_{n_k}\}$ be a subsequence of $\{y_n\}$ such that

$$\limsup_{n\to\infty}\langle\gamma f(p)-G(p),j(y_n-p)\rangle=\lim_{k\to\infty}\langle\gamma f(p)-G(p),j(y_{n_k}-p)\rangle \quad (22)$$

and assume without loss of generality that $y_{n_k}\rightharpoonup z\in E$. By Remark 1.1 and Theorem 2.5, $(I-T_i)$ is demiclosed at zero for each $i\geq 1$, so $z\in F$. Since the duality map of E is weakly sequentially continuous, we obtain

$$\begin{aligned}\limsup_{n\to\infty}\langle\gamma f(p)-G(p),j(y_n-p)\rangle &= \\ \lim_{k\to\infty}\langle\gamma f(p)-G(p),j(y_{n_k}-p)\rangle & \\ \leq \ \langle\gamma f(p)-G(p),j(z-p)\rangle\leq 0. & \quad (23)\end{aligned}$$

We now conclude by showing that $x_n\to p$ as $n\to\infty$. Since $\frac{v_n}{\beta_n}\to 0$ as $n\to\infty$, if we denote by w_n the value $2v_n+v_n^2$, it implies that $\frac{w_n}{(1+w_n)\beta_n}\to 0$ as $n\to\infty$, then there exists $n_0\in\mathbb{N}$ such that $\frac{w_n}{(1+w_n)\beta_n}\leq\frac{\tau-2\gamma\alpha}{2}$, for all $n\geq n_0$. From recursion formula (10), we obtain

$$\begin{aligned}||x_{n+1}-p||^2 &= ||[1-\delta(1-\alpha_n)](x_n-p)+\delta\sum_{i\geq 1}\sigma_{in}(T_i^ny_n-p)||^2\\
&\leq [1-\delta(1-\alpha_n)]||x_n-p||^2+\delta\sum_{i\geq 1}\sigma_{in}||T_i^ny_n-p||^2\\
&\leq [1-\delta(1-\alpha_n)]||x_n-p||^2+\delta(1-\alpha_n)(1+v_n)^2||y_n-p||^2\\
&= [1-\delta(1-\alpha_n)]||x_n-p||^2\\
&\quad +\delta(1-\alpha_n)(1+w_n)\Big[||\beta_n(\gamma f(x_n)-Gp)+(I-\beta_nG)(x_n-p)||^2\Big]\\
&\leq [1-\delta(1-\alpha_n)]||x_n-p||^2+\delta(1-\alpha_n)(1+w_n)\Big[(1-\beta_n\tau)||x_n-p||^2\\
&\quad +2\beta_n\langle\gamma f(x_n)-Gp,j(y_n-p)\rangle\Big]\\
&\leq \Big[1-\delta(1-\alpha_n)+\delta(1-\alpha_n)(1+w_n)(1-\beta_n\tau)\Big]||x_n-p||^2\\
&\quad +2\beta_n\delta(1-\alpha_n)(1+w_n)\gamma\alpha||x_n-p||||y_n-p||\\
&\quad +2\beta_n\delta(1-\alpha_n)(1+w_n)\langle\gamma f(p)-Gp,j(y_n-p)\rangle\\
&\leq \Big[1-\delta(1-\alpha_n)+\delta(1-\alpha_n)(1+w_n)(1-\beta_n\tau)\Big]||x_n-p||^2\\
&\quad +2\beta_n\delta(1-\alpha_n)(1+w_n)\gamma\alpha||x_n-p||^2\\
&\quad +2\beta_n\delta(1-\alpha_n)(1+w_n)\gamma\alpha||x_n-p||||y_n-x_n||\end{aligned}$$

$$
\begin{aligned}
& +2\beta_n\delta(1-\alpha_n)(1+w_n)\langle\gamma f(p)-Gp, j(y_n-p)\rangle \\
= & \Big[1-\delta(1-\alpha_n)+\delta(1-\alpha_n)(1+w_n)(1-\beta_n[\tau-2\gamma\alpha])\Big]||x_n-p||^2 \\
& +2\beta_n\delta(1-\alpha_n)(1+w_n)\gamma\alpha||x_n-p||||y_n-x_n|| \\
& +2\beta_n\delta(1-\alpha_n)(1+w_n)\langle\gamma f(p)-Gp, j(y_n-p)\rangle \\
= & \Big[1-\beta_n\delta(1-\alpha_n)(1+w_n)\Big((\tau-2\gamma\alpha)-\frac{w_n}{(1+w_n)\beta_n}\Big)\Big]||x_n-p||^2 \\
& +2\beta_n\delta(1-\alpha_n)(1+w_n)\langle\gamma f(p)-Gp, j(y_n-p)\rangle \\
& +2\beta_n\delta(1-\alpha_n)(1+w_n)\gamma\alpha||x_n-p||||y_n-x_n|| \\
= & \Big[1-\beta_n\delta(1-\alpha_n)(1+w_n)\Big((\tau-2\gamma\alpha)-\frac{w_n}{(1+w_n)\beta_n}\Big)\Big]||x_n-p||^2 \\
& +\beta_n\delta(1-\alpha_n)(1+w_n)\Big((\tau-2\gamma\alpha)-\frac{w_n}{(1+w_n)\beta_n}\Big)\times \\
& \frac{2[\langle\gamma f(p)-Gp, j(y_n-p)\rangle+\gamma\alpha||x_n-p||||y_n-x_n||]}{\Big((\tau-2\gamma\alpha)-(w_n/(1+w_n)\beta_n)\Big)}.
\end{aligned}
$$

Observe that $\sum\beta_n\delta(1-\alpha_n)(1+w_n)\Big((\tau-2\gamma\alpha)-\frac{w_n}{(1+w_n)\beta_n}\Big)=\infty$ and

$$
\limsup\Big(\frac{2[\langle\gamma f(p)-Gp, j(y_n-p)\rangle+\gamma\alpha||x_n-p||||y_n-x_n||]}{\Big((\tau-2\gamma\alpha)-(w_n/(1+w_n)\beta_n)\Big)}\Big)\le 0
$$

Applying Lemma 2.2, we obtain $||x_n-p||\to 0$ as $n\to\infty$. This complete the proof.

The following corollaries follow from Theorem 3.1 .

Corollary 3.1 Let $E, G, f, F, \{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ be as in Theorem 3.1. Let $\{T_i\}_{i=1}^{\infty}$ be a family of nonexpansive self mappings of E. Assume that $\gamma\in\Big(0,\min\{\frac{\tau}{2\alpha},\eta\}\Big)$, where $\tau:=(1-\sqrt{\frac{1-\eta}{\mu}})$. For some fixed $\delta\in(0,1)$, let $\{x_n\}_{n=1}^{\infty}$ be a sequence defined iteratively by $x_0\in E$ chosen arbitrarily,

$$
\left\{\begin{array}{l} x_{n+1}=[1-\delta(1-\alpha_n)]x_n+\delta\sum_{i\ge1}\sigma_{in}T_iy_n, \\ y_n=\beta_n\gamma f(x_n)+(I-\beta_nG)x_n, \quad n\ge 0. \end{array}\right. \tag{24}
$$

Then, $\{x_n\}_{n=1}^{\infty}$ converges strongly to $p\in F$, where p is a solution of the variational inequality problem (11).

Corollary 3.2 Let $E=H$ a real Hilbert space and $G, f, F, \{\alpha_n\}_{n=1}^{\infty}$, $\{T_i\}_{i=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ be as in Theorem 3.1. Let $\{T_i\}_{i=1}^{\infty}$ be a family of nonexpansive self mappings of E. Assume that $\gamma\in\Big(0,\min\{\frac{\tau}{2\alpha},\eta\}\Big)$, where $\tau:=(1-\sqrt{\frac{1-\eta}{\mu}})$. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence defined as in (10). Then, $\{x_n\}_{n=1}^{\infty}$ converges strongly to $p \in F$, where p is a solution of the variational inequality problem (11).

References

[1] B. Ali, *Iterative approximation of common fixed points for families of non-expansive mappings and solutions of Variational inequalities*, Advances in Nonlinear Variational Inqualities, 12 (2)(2009), 73 -89.

[2] B. Ali, *Common fixed points approximation for asymptotically nonexpansive semi group in Banach spaces,* ISRN Mathematical Anal., 2011 ID 684158, 14 pages, doi:10.5402/2011/684158.

[3] B. Ali, G.C. Ugwunnadi, Y. Shehu, *A general iterative algorithm for non-expansive mappings in Banach spaces*, Ann. Funct. Anal. 2 (2)(2011), 11-22.

[4] K. Goebel and W.A. Kirk, *A fixed point theorem for asymptotically non-expansive mappings*, Proc. Amer. Math. Soc., 35 (1972), 171-174.

[5] G.M. Korpelevich; *An extragradient method for finding saddle points and for other problems*, Ekonomika i Matematicheskie Metody, 12(1976), 747-756.

[6] P.K. Lin, K.K. Tan and H.K. Xu, *Demiclosedness principle and asymptotic behaviour for asymptotically nonexpansive mappings,* Nonlinear. Anal. 24 (1995), 929-946.

[7] P.E. Maingé; *Strong convergence of projected subgradient methods for nonsmooth and non- strictly convex minimization*. Set-Valued Anal., 16 (2008), 899-912.

[8] G. Marino and H.K. Xu, *A general iterative method for nonexpansive mappings in Hilbert spaces*, J. Math. Anal. Appl. 318 (1) (2006), 43-52.

[9] A. Moudafi, *Viscosity approximation methods for fixed-point problems*, J. Math. Anal. Appl. 241 (1) (2000), 46-55.

[10] M.A. Noor; *A class of new iterative methods for solving mixed variational inequalities*, Math. Computer Model. 31(2000), 11-19.

[11] Z. Opial, *Weak convergence of the sequence of successive approximations for nonexpansive mappings,* Bull. Amer. Math. Soc. 73 (1967), 591-597.

[12] H. Piri and H. Vaezi, *Strong convergence of a generalized iterative method for semigroups of nonexpansive mappings in Hilbert spaces,* Fixed Poit Theory and Applications, vol. 2010, Article ID 907275, 16 pages, 2010.

[13] S. Prus, *Banach spaces with the uniform Opial property,* Nonlinear. Anal. 18 (1992), 697-704.

[14] G. Stampacchi; *Formes bilineaires coercivites sur les ensembles convexes*, C. R. Acad. Sci- ences, Paris, 258 (1964), 4413-4416.

[15] T. Suzuki, *Strong convergence of Krasnoselskii and Manns type sequences for one-parameter nonexpansive semigroups without Bochner integrals,* Journal of Mathematical Analysis and Applications, vol. 305 (1) (2005), 227-239.

[16] M. Tian, *A general iterative method for nonexpansive mappings in Hilbert space,* Nonlinear Anal. 73 (2010), 689-694.

[17] H.K. Xu, *An iterative approach to quadratic optimization*, J. Optim. Theory Appl. 116 (2003), 659-678.

[18] H.K. Xu, *Iterative algorithms for nonlinear operators*, J. London Math. Soc. (2) 66 (2002), no. 1, 240-256.

[19] I. Yamada; *The hybrid steepest-descent method for variational inequality problems over the intersection of the fixed point sets of nonexpansive mappings,Inherently Parallel Algorithms in Feasibility and Optimization and Their Applications,* Edited by D. Butnariu, Y. Censor, and S. Reich, North-Holland, Amsterdam, Holland, 2001, 473 - 504.

[20] Y. Yao, Y.C. Liou, C.L. Li and H.T. Lin; *Extended extragradient methods for generalized varitional inequalities, Fixed Point Theory and Applications*, Volume 2012, Article ID 237083, 14 pages.

[21] Y. Yao and H.K. Xu; *Iterative methods for finding minimum-norm fixed points of nonexpan- sive mappings with applications,* Optimization, 60 (6) (2011), 645-658.

DEPARTMENT OF MATHEMATICAL SCIENCES, BAYERO UNIVERSITY KANO, NIGERIA

E-mail address: bashiralik@yahoo.com

DEPARTMENT OF MATHEMATICS, MICHAEL OKPARA UNIVERSITY OF AGRICULTURE, UMUDIKE, ABIA STATE, NIGERIA

E-mail addresses: ugwunnadi4u@yahoo.com