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WEAK AND STRONG CONVERGENCE THEOREMS FOR APPROXIMATING FIXED POINTS OF NONEXPANSIVE MAPPINGS USING A COMPOSITE HYBRID ITERATION METHOD

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ABSTRACT. We prove that the recent results of Miao and Li [Applicable Analysis and Discrete Mathematics 2(2008), 197-204, http://pefmath.etf.bg.ac.yu] concerning the iterative approximation of fixed points of nonexpansive mappings in Hilbert spaces using a composite hybrid iteration method can be extended to arbitrary Banach spaces without the strong monotonicity assumption imposed on the hybrid operators.

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1. INTRODUCTION

Let E be an arbitrary Banach space. A mapping $T : E \longrightarrow E$ is said to be $L-$*Lipschitzian* if there exists $L > 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\|, \forall x, y \in E. \tag{1}$$

T is said to be *nonexpansive* if $L = 1$ in (1). Let H be a Hilbert space, A mapping $T : H \longrightarrow H$ is said to be $\eta-$*strongly monotone* if there exists $\eta > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq \eta\|x - y\|^2, \forall x, y \in H. \tag{2}$$

Nonexpansive mappings are intimately connected with several nonlinear mappings that are of interest in ordinary and partial differential equations (see for example [2, 4, 5, 6, 12]). Bruck [7] remarked that the intimate connection of nonexpansive mapping with the important class of accretive operators (i.e. operator $T : D(T) \subseteq E \longrightarrow E$ such that $\|x-y\| \leq \|x-y+\lambda(Tx-Ty)\|, \forall x, y \in D(T)$ for all $\lambda \geq 0$, $E-$Banach space) account for in part the importance of nonexpansive mappings which is one of the first classes of mappings

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for which fixed point results were obtained by using the geometric structure of the underlying Banach space instead of compactness property.

The interest and importance of construction of fixed points of nonexpansive mappings stem mainly from the fact that it may be applied in many areas such as image recovering, signal processing (see for example [3, 8, 23]), solving convex minimization problems (see for example [10, 27, 28, 30, 31]). Hence iterative techniques for approximating fixed points of nonexpansive mappings have been studied by several authors (see for example [1, 13, 16, 17, 18]) using the famous Mann [14] iteration scheme. As a generalization of the Mann process, Xu and Kim [29], Yamada [31] and Wang [26] introduced the so called hybrid iteration method which has been used in solving certain variational inqualities:

The Hybrid Iteration Method (Wang [26]): Let H be a Hilbert space, $T : H \longrightarrow H$ a nonexpansive mapping with $F(T) = \{x \in H : Tx = x\} \neq \emptyset$ and $F : H \longrightarrow H$ an $\eta-$*strongly monotone* and *Lipschitzian* mapping. Let $\{a_n\}_{n=1}^{\infty} \subset (0, 1)$ and $\{\lambda_n\}_{n=1}^{\infty}$ be real sequences in $[0, 1)$, and $\mu > 0$, then the sequence $\{x_n\}_{n=1}^{\infty}$ is generated from an arbitrary $x_1 \in H$ by

$$x_{n+1} = a_n x_n + (1 - a_n)T^{\lambda_{n+1}} x_n, n \geq 1 \tag{3}$$

where $T^{\lambda_{n+1}}x = Tx - \lambda_{n+1}\mu F(Tx), \mu > 0$. Using this result, Wang [26] obtained weak and strong convergence of (3) to the fixed point of T. Observe that if either $\lambda_n = 0, \forall n \geq 1$ or $F \equiv 0$, then (3) reduces to the well known Mann iteration method

$$x_{n+1} = a_n x_n + (1 - a_n)Tx_n, n \geq 1 \tag{4}$$

which has been used by several authors for the approximation of fixed points of operators or operator equations.

Motivated by the work of Wang [26] and earlier results of Xu and Kim [29] and Yamada [31], Miao and Li generalized (3) by developing the following Composite Hybrid Iteration Process:

Let H be a Hilbert space, $T : H \longrightarrow H$ a nonexpansive mapping with $F(T) \neq \emptyset$ and $f(resp.g) : H \longrightarrow H$ an $\eta_f(resp.\eta_g)-$*strongly monotone* and $k_f(resp.k_g)-$*Lipschitzian* mapping. For any $x_1 \in H$. $\{x_n\}$ is defined by

$$\left.\begin{array}{lcl} x_{n+1} & = & a_n x_n + (1 - a_n)T_f^{\lambda_{n+1}} y_n \\ y_n & = & b_n x_n + (1 - b_n)T_g^{\beta_n} x_n \end{array}\right\} \quad , \quad n \geq 1 \tag{5}$$

$$\text{where} \quad T_f^{\lambda_{n+1}}x = Tx - \lambda_{n+1}\mu_f f(Tx), \quad \mu_f > 0, \quad \forall x \in H,$$
$$T_g^{\beta_n}x = Tx - \beta_n\mu_g g(Tx), \quad \mu_g > 0, \quad \forall x \in H,$$

and $\{a_n\} \subset (0,1), \{b_n\} \subset (0,1)$ and $\lambda_n \subset [0,1), \beta_n \subset [0,1)$ satisfy the following conditions:

(i) $\alpha \leq a_n \leq 1-\alpha, \beta \leq b_n \leq 1-\beta$, for some $\alpha, \beta \in (0, \frac{1}{2})$;
(ii) $\Sigma_{n=1}^{+\infty}\lambda_n < +\infty, \quad \Sigma_{n=1}^{+\infty}\beta_n < +\infty$.
(iii) $0 < \mu_f < \frac{2\eta_f}{k_f^2}, 0 < \mu_g < \frac{2\eta_g}{k_g^2}$.

Observe that if $b_n = 1, \forall n \geq 1, (5)$ reduces to (3) and if $b_n = 1, \forall n \geq 1, \lambda_n = 0$ or $f \equiv 0, (5)$ reduces to the well known Mann iteration Method. Using (5), Miao and Li [15] proved the following:

Lemma 1 ([15], Page 200): The iterative process $\{x_n\}$ as in (5) satisfies

(1) $\lim_{n\to+\infty} \|x_n - p\|$ exists for each $p \in F(T)$,
(2) $\lim_{n\to+\infty} \|x_n - Tx_n\| = 0$.

Theorem 1 ([15], Page 202): The iterative process $\{x_n\}$ as in (5) converges weakly to a fixed point of T.

Theorem 2 ([15], Page 203): Let T be completely continuous or demicompact. The iterative process $\{x_n\}$ as in (5) converges strongly to a fixed point of T.

It is our purpose in this paper to extend Lemma 1, Theorem 1 and Theorem 2 from Hilbert spaces to arbitrary Banach spaces. Our results are much more general and also more applicable than the results of Miao and Li [15] because the strong monotonicity condition imposed on f and g is not required in our results.

2. PRELIMINARY

In this section we will state the following Lammas. A Banach space E is said to satisfy *Opial's condition* (see for example [19]) if for each sequence $\{x_n\}_{n=1}^{\infty} \in E$ which converges weakly to a point $x \in E$ we have

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|, \forall y \in E.$$

Let E be a Banach space. A mapping T with domain $D(T)$ and range $R(T)$ in E is said to be *demiclosed* at a point $p \in D(T)$ if whenever $\{x_n\}_{n=1}^{\infty}$ is a sequence in E which converges weakly to a point $x \in E$ and $\{Tx_n\}_{n=1}^{\infty}$ converges strongly to p, then $Tx = p$.

Furthermore, T is said to be *demicompact* if whenever $\{x_n\}_{n=1}^{\infty}$ is a bounded sequence in $D(T)$ such that $\{x_n - Tx_n\}_{n=1}^{\infty}$ converges strongly, then $\{x_n\}_{n=1}^{\infty}$ has a subsequence which converges strongly. T is said to satisfy *condition* (A) if $F(T) \neq \emptyset$ and there exist non-decreasing functions $f : [0,\infty) \to [0,\infty)$ and $g : [0,\infty) \to [0,\infty)$ with $f(0) = 0, g(0) = 0, f(t) > 0$ and $g(t) > 0$ $\forall t \in (0,\infty)$ such that $\|x - Tx\| \geq f(d(x, F(T)))$ and $\|x - Tx\| \geq g(d(x, F(T)))$ for all $x \in D(T)$ where $d(x, F(T)) := inf\{\|x - p\| : p \in F(T)\}$.

Lemma 2 ([11]): Let E be a reflexive Banach space satisfying Opial's condition and let K be a nonempty closed convex subset of E. Let $T : K \to E$ be a nonexpansive mapping. Then $(I - T)$ is demiclosed on K, where I is the identity mapping.

Lemma 3 ([20], page 1184, see also [21]): Let $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ and $\{\delta_n\}_{n=1}^{\infty}$ be sequences of nonnegative real numbers satisfying the inequality $a_{n+1} \leq (1 + \delta_n)a_n + b_n, n \geq 1$. If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim\limits_{n\to\infty} a_n$ exists. If in addition $\{a_n\}_{n=1}^{\infty}$ has a subsequence which converges strongly to zero, then $\lim\limits_{n\to\infty} a_n = 0$.

Lemma 4 ([1], page 223, see also [9], page 770): Let E be an arbitrary normed space and let $\{t_n\}_{n=1}^{\infty}$ be a real sequence satisfying the conditions:

(i) $0 \leq t_n \leq t \leq 1$, and for some $t \in (0, 1)$,
(ii) $\Sigma_{n=1}^{+\infty} t_n = +\infty$,
Let $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ be two sequences in E such that
(iii) $u_{n+1} = (1 - t_n)u_n + t_n v_n, \quad \forall n \geq 1$
(iv) $\lim\limits_{n\to\infty} \|u_n\| = d$ for some $d \in [0,\infty)$,
(v) $\limsup\limits_{n\to\infty} \|v_n\| \leq d$,
(vi) $\{\Sigma_{j=1}^{n} t_j v_j\}_{n=1}^{+\infty}$ is bounded. Then $d = 0$.

3. MAIN RESULTS

Theorem 3 : Let E be an arbitrary real Banach space, $T : E \longrightarrow E$ a nonexpansive mapping with $F(T) \neq \emptyset$ and $f(resp.g) : E \longrightarrow E$ are $L_f(resp.L_g)-$Lipschitzian mappings. For any $x_1 \in E, \{x_n\}$ is generate by

$$\left.\begin{array}{lcl} x_{n+1} & = & a_n x_n + (1 - a_n)T_f^{\lambda_{n+1}} y_n \\ y_n & = & b_n x_n + (1 - b_n)T_g^{\beta_n} x_n \end{array}\right\} \quad , \quad n \geq 1 \tag{6}$$

$$\text{where}\quad \begin{aligned} T_f^{\lambda_{n+1}}x &= Tx - \lambda_{n+1}\mu_f f(Tx), \quad \mu_f > 0, \quad \forall x \in E, \\ T_g^{\beta_n}x &= Tx - \beta_n\mu_g g(Tx), \quad \mu_g > 0, \quad \forall x \in E, \end{aligned}$$

and $\{a_n\} \subset (0,1), \{b_n\} \subset (0,1)$ and $\lambda_n \subset [0,1), \beta_n \subset [0,1)$ satisfying the following conditions:

(i) $0 < \alpha < a_n < 1$, for some $\alpha \in (0,1)$;
(ii) $\Sigma_{n=1}^{+\infty}(1-a_n) = +\infty$
(iii) $\Sigma_{n=1}^{+\infty}(1-b_n) < +\infty$
(iv) $\Sigma_{n=1}^{+\infty}\lambda_n < +\infty$
(v) $\Sigma_{n=1}^{+\infty}\beta_n < +\infty$.

Then,

(a) $\lim\limits_{n\to+\infty} \|x_n - p\|$ exists for each $p \in F(T)$,
(b) $\lim\limits_{n\to+\infty} \|x_n - Tx_n\| = 0$
(c) $\{x_n\}$ converges strongly to a fixed point of T if and only if $\liminf\limits_{n\to+\infty} d(x_n, F(T)) = 0$.

Proof: Let $p \in F(T)$ be arbitrary. Set $L := \max\{L_f, L_g\}$. Then

$$\begin{aligned} \|x_{n+1} &- p\| = \|a_n(x_n - p) + (1-a_n)(T_f^{\lambda_{n+1}}y_n - p)\| \\ &= \|a_n(x_n-p) + (1-a_n)(Ty_n - p) \\ &\quad -(1-a_n)\lambda_{n+1}\mu_f f(Ty_n)\| \\ &\le a_n\|x_n - p\| + (1-a_n)\|Ty_n - p\| \\ &\quad +(1-a_n)\lambda_{n+1}\mu_f\|f(Ty_n)\| \\ &\le a_n\|x_n - p\| + (1-a_n)\|y_n - p\| \\ &\quad +(1-a_n)\lambda_{n+1}\mu_f\{\|f(Ty_n) - f(p)\| + \|f(p)\|\} \\ &= a_n\|x_n - p\| + (1-a_n)\|y_n - p\| \\ &\quad +(1-a_n)\lambda_{n+1}\mu_f\|f(Ty_n) - f(p)\| \\ &\quad +(1-a_n)\lambda_{n+1}\mu_f\|f(p)\| \\ &\le a_n\|x_n - p\| + (1-a_n)\|y_n - p\| \\ &\quad +(1-a_n)\lambda_{n+1}\mu_f L\|y_n - p\| \\ &\quad +(1-a_n)\lambda_{n+1}\mu_f\|f(p)\| \\ &= a_n\|x_n - p\| + \{(1-a_n) \\ &\quad +(1-a_n)\lambda_{n+1}\mu_f L\}\|y_n - p\| \\ &\quad +(1-a_n)\lambda_{n+1}\mu_f\|f(p)\| \end{aligned} \tag{7}$$

$$\begin{aligned}
|y_n \quad - \quad & p\| = \|b_n(x_n-p)+(1-b_n)(T_g^{\beta_n}x_n-p)\| \\
= \quad & \|b_n(x_n-p)+(1-b_n)(Tx_n-p) \\
& -(1-b_n)\beta_n\mu_g g(Tx_n)\| \\
\leq \quad & b_n\|x_n-p\|+(1-b_n)\|Tx_n-p\| \\
& +(1-b_n)\beta_n\mu_g\|g(Tx_n)\| \\
\leq \quad & b_n\|x_n-p\|+(1-b_n)\|x_n-p\| \\
& +(1-b_n)\beta_n\mu_g\{\|g(Tx_n)-g(p)\|+\|g(p)\|\} \\
= \quad & b_n\|x_n-p\|+(1-b_n)\|x_n-p\| \\
& +(1-b_n)\beta_n\mu_g\|g(Tx_n)-g(p)\| \\
& +(1-b_n)\beta_n\mu_g\|g(p)\| \\
\leq \quad & b_n\|x_n-p\|+(1-b_n)\|x_n-p\| \\
& +(1-b_n)\beta_n\mu_g L\|x_n-p\| \\
& +(1-b_n)\beta_n\mu_g\|g(p)\| \\
= \quad & \{1+(1-b_n)\beta_n\mu_g L\}\|x_n-p\| \\
& +(1-b_n)\beta_n\mu_g\|g(p)\|. \qquad (8)
\end{aligned}$$

Substitute (8) into (7) to obtain

$$\begin{aligned}
\|x_{n+1}-p\| = {} & a_n\|x_n-p\|+\{(1-a_n)+(1-a_n)\lambda_{n+1}\mu_f L\} \\
& \times[\{1+(1-b_n)\beta_n\mu_g L\}\|x_n-p\| \\
& +(1-b_n)\beta_n\mu_g\|g(p)\|]+(1-a_n)\lambda_{n+1}\mu_f\|f(p)\| \\
= {} & a_n\|x_n-p\|+\{(1-a_n)+(1-a_n)\lambda_{n+1}\mu_f L\} \\
& \times\{1+(1-b_n)\beta_n\mu_g L\}\|x_n-p\| \\
& +\{(1-a_n)+(1-a_n)\lambda_{n+1}\mu_f L\} \\
& \times(1-b_n)\beta_n\mu_g\|g(p)\|+(1-a_n)\lambda_{n+1}\mu_f\|f(p)\| \\
= {} & [a_n+(1-a_n)+(1-a_n)\lambda_{n+1}\mu_f L \\
& +(1-a_n)(1-b_n)\beta_n\mu_g L \\
& +(1-a_n)(1-b_n)\lambda_{n+1}\beta_n\mu_f\mu_g L^2]\|x_n-p\| \\
& +\{(1-a_n)+(1-a_n)\lambda_{n+1}\mu_f L\}(1-b_n)\beta_n\mu_g\|g(p)\|] \\
& +(1-a_n)\lambda_{n+1}\mu_f\|f(p)\|
\end{aligned}$$

$$
\begin{aligned}
&= [1 + (1 - a_n)\lambda_{n+1}\mu_f L + (1 - a_n)(1 - b_n)\beta_n\mu_g L \\
&\quad +(1 - a_n)(1 - b_n)\lambda_{n+1}\beta_n\mu_f\mu_g L^2\}\|x_n - p\| \\
&\quad +\{(1 - a_n) + (1 - a_n)\lambda_{n+1}\mu_f L\}(1 - b_n)\beta_n\mu_g\|g(p)\|] \\
&\quad +(1 - a_n)\lambda_{n+1}\mu_f\|f(p)\| \\
&= [1 + \delta_n]\|x_n - p\| + \sigma_n
\end{aligned} \tag{9}
$$

where

$$
\begin{aligned}
\delta_n &= (1 - a_n)\lambda_{n+1}\mu_f L + (1 - a_n)(1 - b_n)\beta_n\mu_g L \\
&\quad +(1 - a_n)(1 - b_n)\lambda_{n+1}\beta_n\mu_f\mu_g L^2 \qquad \text{and} \\
\sigma_n &= \{(1 - a_n) + (1 - a_n)\lambda_{n+1}\mu_f L\}(1 - b_n)\beta_n\mu_g\|g(p)\| \\
&\quad +(1 - a_n)\lambda_{n+1}\mu_f\|f(p)\| \\
&= (1 - a_n)(1 - b_n)\beta_n\mu_g\|g(p)\| \\
&\quad +(1 - a_n)(1 - b_n)\lambda_{n+1}\mu_f\beta_n\mu_g L\|g(p)\| \\
&\quad +(1 - a_n)\lambda_{n+1}\mu_f\|f(p)\|.
\end{aligned}
$$

Since $\sum_{n=1}^{\infty}\delta_n < \infty$ and $\sum_{n=1}^{\infty}\sigma_n < \infty$, it follows from Lemma 3 that $\lim_{n\to\infty}\|x_n - p\|$ exists. It follows that $\{x_n\}$ is bounded, completing the proof of (a). Since $\{\|x_n - p\|\}_{n=1}^{\infty}$ is bounded, then there exists $M > 0$ such that

$$
\|x_n - p\| \le M \ \forall n \ge 1. \tag{10}
$$

Observe that

$$
\begin{aligned}
\|x_{n+1} - Tx_{n+1}\| &= \|a_n(x_n - Tx_{n+1}) + (1 - a_n)(Ty_n - Tx_{n+1}) \\
&\quad -(1 - a_n)\lambda_{n+1}\mu_f f(Ty_n)\| \\
&\le a_n\|x_n - Tx_{n+1}\| + (1 - a_n)\|Ty_n - Tx_{n+1}\| \\
&\quad +(1 - a_n)\lambda_{n+1}\mu_f\|f(Ty_n)\| \\
&\le a_n\|x_n - Tx_{n+1}\| + (1 - a_n)\|y_n - x_{n+1}\| \\
&\quad +(1 - a_n)\lambda_{n+1}\mu_f L\|y_n - p\| \\
&\quad +(1 - a_n)\lambda_{n+1}\mu_f\|f(p)\| \\
&\le a_n\|x_n - x_{n+1}\| + a_n\|x_{n+1} - Tx_{n+1}\| \\
&\quad +(1 - a_n)\|y_n - x_n\| + (1 - a_n)\|x_n - x_{n+1}\| \\
&\quad +(1 - a_n)\lambda_{n+1}\mu_f L\|y_n - p\| + (1 - a_n)\lambda_{n+1}\mu_f\|f(p)\| \\
&= \|x_n - x_{n+1}\| + a_n\|x_{n+1} - Tx_{n+1}\| \\
&\quad +(1 - a_n)\|(1 - b_n)(Tx_n - x_n) - (1 - b_n)\beta_n\mu_g g(Tx_n)\| \\
&\quad +(1 - a_n)\lambda_{n+1}\mu_f L\|y_n - p\| + (1 - a_n)\lambda_{n+1}\mu_f\|f(p)\|
\end{aligned}
$$

$$
\begin{aligned}
&\leq \|x_n - x_{n+1}\| + a_n\|x_{n+1} - Tx_{n+1}\| \\
&\quad +(1-a_n)(1-b_n)\|Tx_n - x_n\| \\
&\quad +(1-a_n)(1-b_n)\beta_n\mu_g\|g(Tx_n)\| \\
&\quad +(1-a_n)\lambda_{n+1}\mu_f L\|y_n - p\| + (1-a_n)\lambda_{n+1}\mu_f\|f(p)\| \\
&\leq \|x_n - x_{n+1}\| + a_n\|x_{n+1} - Tx_{n+1}\| \\
&\quad +(1-a_n)(1-b_n)\|Tx_n - x_n\| \\
&\quad +(1-a_n)(1-b_n)\beta_n\mu_g L\|x_n - p\| \\
&\quad +(1-a_n)(1-b_n)\beta_n\mu_g\|g(p)\| + (1-a_n)\lambda_{n+1}\mu_f L\|y_n - p\| \\
&\quad +(1-a_n)\lambda_{n+1}\mu_f\|f(p)\|. \qquad (11)
\end{aligned}
$$

Substitute (8) into (11) to obtain

$$
\begin{aligned}
\|x_{n+1} - Tx_{n+1}\| &\leq \|x_n - x_{n+1}\| + a_n\|x_{n+1} - Tx_{n+1}\| \\
&\quad +(1-a_n)(1-b_n)\|Tx_n - x_n\| \\
&\quad +(1-a_n)(1-b_n)\beta_n\mu_g L\|x_n - p\| \\
&\quad +(1-a_n)(1-b_n)\beta_n\mu_g\|g(p)\| + (1-a_n)\lambda_{n+1}\mu_f L \\
&\quad \times[\{1 + (1-b_n)\beta_n\mu_g L\}\|x_n - p\| \\
&\quad +(1-b_n)\beta_n\mu_g\|g(p)\|] + (1-a_n)\lambda_{n+1}\mu_f\|f(p)\| \\
&= \|x_n - x_{n+1}\| + a_n\|x_{n+1} - Tx_{n+1}\| \\
&\quad +(1-a_n)(1-b_n)\|Tx_n - x_n\| \\
&\quad +(1-a_n)(1-b_n)\beta_n\mu_g L\|x_n - p\| \\
&\quad +(1-a_n)(1-b_n)\beta_n\mu_g\|g(p)\| \\
&\quad +(1-a_n)\lambda_{n+1}\mu_f L\{1 + (1-b_n)\beta_n\mu_g L\}\|x_n - p\| \\
&\quad +(1-a_n)(1-b_n)\lambda_{n+1}\mu_f\beta_n\mu_g L\|g(p)\| \\
&\quad +(1-a_n)\lambda_{n+1}\mu_f\|f(p)\| \qquad (12)
\end{aligned}
$$

$$
\begin{aligned}
\|x_{n+1} - Tx_{n+1}\| &\leq \frac{1}{(1-a_n)}\|x_n - x_{n+1}\| + (1-b_n)\|Tx_n - x_n\| \\
&\quad +(1-b_n)\beta_n\mu_g L\|x_n - p\| + (1-b_n)\beta_n\mu_g\|g(p)\| \\
&\quad +\lambda_{n+1}\mu_f L\{1 + (1-b_n)\beta_n\mu_g L\}\|x_n - p\| \\
&\quad +(1-b_n)\lambda_{n+1}\mu_f\beta_n\mu_g L\|g(p)\| + \lambda_{n+1}\mu_f\|f(p)\| \qquad (13)
\end{aligned}
$$

$$
\begin{aligned}
\|x_{n+1} - x_n\| &= (1-a_n)\|T_f^{\lambda_{n+1}} y_n - x_n\| \\
&= (1-a_n)\|Ty_n - x_n - \lambda_{n+1}\mu_f f(Ty_n)\|
\end{aligned}
$$

$$
\begin{aligned}
&\leq (1-a_n)\|Ty_n - x_n\| + (1-a_n)\lambda_{n+1}\mu_f\|f(Ty_n)\| \\
&\leq (1-a_n)\|y_n - x_n\| + (1-a_n)\|x_n - Tx_n\| \\
&\quad +(1-a_n)\lambda_{n+1}\mu_f L\|y_n - p\| \\
&\quad +(1-a_n)\lambda_{n+1}\mu_f\|f(p)\| \\
&\leq (1-a_n)(1-b_n)\|x_n - Tx_n\| + (1-a_n)\|x_n - Tx_n\| \\
&\quad +(1-a_n)(1-b_n)\beta_n\mu_g L\|x_n - p\| \\
&\quad +(1-a_n)(1-b_n)\beta_n\mu_g\|g(p)\| \\
&\quad +(1-a_n)\lambda_{n+1}\mu_f L\|y_n - p\| \\
&\quad +(1-a_n)\lambda_{n+1}\mu_f\|f(p)\|. \qquad (14)
\end{aligned}
$$

Substitute (8) into (14) to obtain

$$
\begin{aligned}
\|x_{n+1} - x_n\| &\leq (1-a_n)(1-b_n)\|x_n - Tx_n\| \\
&\quad +(1-a_n)\|x_n - Tx_n\| \\
&\quad +(1-a_n)(1-b_n)\beta_n\mu_g L\|x_n - p\| \\
&\quad +(1-a_n)(1-b_n)\beta_n\mu_g\|g(p)\| \\
&\quad +(1-a_n)\lambda_{n+1}\mu_f L[\{1+(1-b_n)\beta_n\mu_g L\}\|x_n - p\| \\
&\quad +(1-b_n)\beta_n\mu_g\|g(p)\|] + (1-a_n)\lambda_{n+1}\mu_f\|f(p)\| \\
&= (1-a_n)(2-b_n)\|x_n - Tx_n\| \\
&\quad +(1-a_n)(1-b_n)\beta_n\mu_g L\|x_n - p\| \\
&\quad +(1-a_n)(1-b_n)\beta_n\mu_g\|g(p)\| \\
&\quad +(1-a_n)\lambda_{n+1}\mu_f L\{1+(1-b_n)\beta_n\mu_g L\}\|x_n - p\| \\
&\quad +(1-a_n)(1-b_n)\lambda_{n+1}\mu_f\beta_n\mu_g L\|g(p)\| \\
&\quad +(1-a_n)\lambda_{n+1}\mu_f\|f(p)\| \qquad (15)
\end{aligned}
$$

Substitute (15) into (13) to obtain

$$
\begin{aligned}
\|x_{n+1} - Tx_{n+1}\| &\leq \frac{1}{(1-a_n)}\{(1-a_n)(2-b_n)\|x_n - Tx_n\| \\
&\quad +(1-a_n)(1-b_n)\beta_n\mu_g L\|x_n - p\| \\
&\quad +(1-a_n)(1-b_n)\beta_n\mu_g\|g(p)\| \\
&\quad +(1-a_n)\lambda_{n+1}\mu_f L\{1+(1-b_n)\beta_n\mu_g L\}\|x_n - p\| \\
&\quad +(1-a_n)(1-b_n)\lambda_{n+1}\mu_f\beta_n\mu_g L\|g(p)\| \\
&\quad +(1-a_n)\lambda_{n+1}\mu_f\|f(p)\|\} \\
&\quad +(1-b_n)\|Tx_n - x_n\| + (1-b_n)\beta_n\mu_g L\|x_n - p\|
\end{aligned}
$$

$$
\begin{aligned}
& +(1-b_n)\beta_n\mu_g\|g(p)\| \\
& +\lambda_{n+1}\mu_f L\{1+(1-b_n)\beta_n\mu_g L\}\|x_n-p\| \\
& +(1-b_n)\lambda_{n+1}\mu_f\beta_n\mu_g L\|g(p)\|+\lambda_{n+1}\mu_f\|f(p)\| \\
&= (2-b_n)\|x_n-Tx_n\|+(1-b_n)\beta_n\mu_g L\|x_n-p\| \\
& +(1-b_n)\beta_n\mu_g\|g(p)\| \\
& +\lambda_{n+1}\mu_f L\{1+(1-b_n)\beta_n\mu_g L\}\|x_n-p\| \\
& +(1-b_n)\lambda_{n+1}\mu_f\beta_n\mu_g L\|g(p)\| \\
& +\lambda_{n+1}\mu_f\|f(p)\|+(1-b_n)\|Tx_n-x_n\| \\
& +(1-b_n)\beta_n\mu_g L\|x_n-p\|+(1-b_n)\beta_n\mu_g\|g(p)\| \\
& +\lambda_{n+1}\mu_f L\{1+(1-b_n)\beta_n\mu_g L\}\|x_n-p\| \\
& +(1-b_n)\lambda_{n+1}\mu_f\beta_n\mu_g L\|g(p)\|+\lambda_{n+1}\mu_f\|f(p)\| \\
&= (3-2b_n)\|x_n-Tx_n\|+2[(1-b_n)\beta_n\mu_g L+\lambda_{n+1}\mu_f L\{1 \\
& +(1-b_n)\beta_n\mu_g L\}]\|x_n-p\|+2\lambda_{n+1}\mu_f\|f(p)\| \\
& +2(1+\lambda_{n+1}\mu_f L)(1-b_n)\beta_n\mu_g\|g(p)\| \\
\|x_{n+1} & - Tx_{n+1}\| \leq (3-2b_n)\|x_n-Tx_n\|+\varphi_n \\
&= [1+2(1-b_n)]\|x_n-Tx_n\|+\varphi_n \\
&= [1+\omega_n]\|x_n-Tx_n\|+\varphi_n \qquad (16)
\end{aligned}
$$

where $\omega_n = 2(1-b_n)$ and

$$
\begin{aligned}
\varphi_n &= 2[(1-b_n)\beta_n\mu_g L+\lambda_{n+1}\mu_f L\{1 \\
& \quad +(1-b_n)\beta_n\mu_g L\}]\|x_n-p\|+2\lambda_{n+1}\mu_f\|f(p)\| \\
& \quad +2(1+\lambda_{n+1}\mu_f L)(1-b_n)\beta_n\mu_g\|g(p)\| \\
&\leq 2[(1-b_n)\beta_n\mu_g L+\lambda_{n+1}\mu_f L\{1 \\
& \quad +(1-b_n)\beta_n\mu_g L\}]M+2\lambda_{n+1}\mu_f\|f(p)\| \\
& \quad +2(1+\lambda_{n+1}\mu_f L)(1-b_n)\beta_n\mu_g\|g(p)\|
\end{aligned}
$$

From conditions (iii)- (v), it follows that $\sum_{n=1}^{\infty}\omega_n < \infty$ and $\sum_{n=1}^{\infty}\varphi_n < \infty$. Also it follows from Lemma 3 that $\lim_{n\to\infty}\|x_n - Tx_n\|$ exists. Let $\lim_{n\to\infty}\|x_n - Tx_n\| = d$ and set $u_n = x_n - Tx_n$, so that

$$u_{n+1} = (1-t_n)u_n + t_n v_n \qquad (17)$$

where $t_n = 1 - a_n$, $\quad v_n = \frac{1}{(1-a_n)}(Tx_n - Tx_{n+1}) + (Ty_n - Tx_n) - \lambda_{n+1}\mu_f f(Ty_n)$

$$\begin{aligned}
\|v_n\| &\leq \frac{1}{(1-a_n)}\|Tx_n - Tx_{n+1}\| + \|Ty_n - Tx_n\| \\
&\quad + \lambda_{n+1}\mu_f\|f(Ty_n)\| \\
&\leq \frac{1}{(1-a_n)}\|x_n - x_{n+1}\| + \|y_n - x_n\| \\
&\quad + \lambda_{n+1}\mu_f\|f(Ty_n) - f(p) + f(p)\| \\
&\leq \frac{1}{(1-a_n)}\|x_n - x_{n+1}\| + \|y_n - p\| + \|x_n - p\| \\
&\quad + \lambda_{n+1}\mu_f L\|y_n - p\| + \lambda_{n+1}\mu_f\|f(p)\| \\
&\leq \frac{1}{(1-a_n)}\|x_n - x_{n+1}\| + \|x_n - p\| \\
&\quad + (1 + \lambda_{n+1}\mu_f L)\|y_n - p\| + \lambda_{n+1}\mu_f\|f(p)\| \qquad (18)
\end{aligned}$$

Substitute (8) and (15) into (18) to obtain

$$\begin{aligned}
\|v_n\| \leq\ & \frac{1}{(1-a_n)}[(1-a_n)(2-b_n)\|x_n - Tx_n\| \\
& + (1-a_n)(1-b_n)\beta_n\mu_g L\|x_n - p\| \\
& + (1-a_n)(1-b_n)\beta_n\mu_g\|g(p)\| \\
& + (1-a_n)\lambda_{n+1}\mu_f L\{1 + (1-b_n)\beta_n\mu_g L\}\|x_n - p\| \\
& + (1-a_n)(1-b_n)\lambda_{n+1}\mu_f\beta_n\mu_g L\|g(p)\| \\
& + (1-a_n)\lambda_{n+1}\mu_f\|f(p)\|] \\
& + \|x_n - p\| + (1 + \lambda_{n+1}\mu_f L)[\{1 + (1-b_n)\beta_n\mu_g L\}\|x_n - p\| \\
& + (1-b_n)\beta_n\mu_g\|g(p)\|] + \lambda_{n+1}\mu_f\|f(p)\| \\
\leq\ & (2-b_n)\|x_n - Tx_n\| + (1-b_n)\beta_n\mu_g L\|x_n - p\| \\
& + (1-b_n)\beta_n\mu_g\|g(p)\| \\
& + \lambda_{n+1}\mu_f L\{1 + (1-b_n)\beta_n\mu_g L\}\|x_n - p\| \\
& + (1-b_n)\lambda_{n+1}\mu_f\beta_n\mu_g L\|g(p)\| + \lambda_{n+1}\mu_f\|f(p)\| \\
& + \|x_n - p\| + (1 + \lambda_{n+1}\mu_f L)\{1 + (1-b_n)\beta_n\mu_g L\}\|x_n - p\| \\
& + (1 + \lambda_{n+1}\mu_f L)(1-b_n)\beta_n\mu_g\|g(p)\| + \lambda_{n+1}\mu_f\|f(p)\|
\end{aligned}$$

$$
\begin{aligned}
&= (2-b_n)\|x_n - Tx_n\| + [1 + (1-b_n)\beta_n\mu_g L]\|x_n - p\| \\
&\quad +(1-b_n)\beta_n\mu_g\|g(p)\| + \lambda_{n+1}\mu_f L\{1 \\
&\quad +(1-b_n)\beta_n\mu_g L\}\|x_n - p\| \\
&\quad +(1-b_n)\lambda_{n+1}\mu_f\beta_n\mu_g L\|g(p)\| + 2\lambda_{n+1}\mu_f\|f(p)\| \\
&\quad +(1+\lambda_{n+1}\mu_f L)\{1 + (1-b_n)\beta_n\mu_g L\}\|x_n - p\| \\
&\quad +(1+\lambda_{n+1}\mu_f L)(1-b_n)\beta_n\mu_g\|g(p)\| \\
&= (2-b_n)\|x_n - Tx_n\| + 2[1 + \lambda_{n+1}\mu_f L] \\
&\quad \times\{1 + (1-b_n)\beta_n\mu_g L\}\|x_n - p\| \\
&\quad +2\lambda_{n+1}\mu_f\|f(p)\| + 2[1 + \lambda_{n+1}\mu_f L](1-b_n)\beta_n\mu_g\|g(p)\| \\
\|v_n\| &\leq [1 + (1-b_n)]\|x_n - Tx_n\| + \vartheta_n \\
&= [1 + \psi_n]\|x_n - Tx_n\| + \vartheta_n \qquad (19)
\end{aligned}
$$

where $\psi_n = (1-b_n)$ and

$$
\begin{aligned}
\vartheta_n &= 2[1 + \lambda_{n+1}\mu_f L]\{1 + (1-b_n)\beta_n\mu_g L\}\|x_n - p\| \\
&\quad +2\lambda_{n+1}\mu_f\|f(p)\| + 2[1 + \lambda_{n+1}\mu_f L](1-b_n)\beta_n\mu_g\|g(p)\|
\end{aligned}
$$

From conditions (iii)- (v), $\lim_{n\to\infty}\psi_n = \lim_{n\to\infty}\vartheta_n = 0$. Since $\lim_{n\to\infty}\|x_n - Tx_n\|$ exists,

$$\|v_n\| \leq \|x_n - Tx_n\| + \psi_n D + Q,\ D > 0,\ Q > 0.$$

Therefore, $\limsup_{n\to\infty}\|v_n\| \leq d$. Observe that using the method of Deng [9], we have:

$$
\begin{aligned}
\|\sum_{j=1}^{n} t_j v_j\| \leq \|\sum_{j=1}^{n}(1-a_j)[&\frac{1}{(1-a_j)}(Tx_j - Tx_{j+1}) \\
&+(Ty_j - Tx_j) + \lambda_{j+1}\mu_f f(Ty_j)]\|
\end{aligned}
$$

$$
\begin{aligned}
&\leq \|\sum_{j=1}^{n}(Tx_j - Tx_{j+1})\| + \sum_{j=1}^{n}\|Ty_j - Tx_j\| + \sum_{j=1}^{n}\lambda_{j+1}\mu_f\|f(Ty_j)\| \\
&\leq \|Tx_1 - Tx_{n+1}\| + \sum_{j=1}^{n}\|Ty_j - Tx_j\| + \sum_{j=1}^{n}\lambda_{j+1}\mu_f\|f(Ty_j)\| \\
&\leq \|Tx_1 - Tx_{n+1}\| + \sum_{j=1}^{n}\|y_j - x_j\| + \sum_{j=1}^{n}\lambda_{j+1}\mu_f\|f(Ty_j)\| \\
&\leq \|Tx_1 - Tx_{n+1}\| + \sum_{j=1}^{n}(1-b_j)[\|x_j - Tx_j\| + \beta_j\mu_g\|g(Tx_j)\|] \\
&\quad + \sum_{j=1}^{n}\lambda_{j+1}\mu_f\|f(Ty_j)\| \\
&\leq \|x_1 - x_{n+1}\| + \sum_{j=1}^{n}(1-b_j)\|x_j - Tx_j\| + \sum_{j=1}^{n}\beta_j\mu_g\|g(Tx_j)\| \\
&\quad + \sum_{j=1}^{n}\lambda_{j+1}\mu_f\|f(Ty_j)\| \\
&\leq \|x_1 - x_{n+1}\| + K_1\sum_{j=1}^{n}(1-b_j) + \mu_g K_2\sum_{j=1}^{n}\beta_j \\
&\quad + \mu_f K_3\sum_{j=1}^{n}\lambda_{j+1} = W
\end{aligned}
$$

$\forall n \geq 1$ and for some $W > 0$. Hence $\{\sum_{j=1}^{n} t_j v_j\}_{n=1}^{\infty}$ is bounded. It now follows from Lemma 4 that $\lim_{n\to\infty}\|v_n\| = \lim_{n\to\infty}\|x_n - Tx_n\| = 0$, completing the proof of (b). From (3.3), we obtain that $\|x_{n+1}-p\| \leq \|x_n - p\| + \xi_n$ where $\xi_n = \delta_n M + \sigma_n$. Hence $d(x_{n+1}, F(T)) \leq d(x_n, F(T)) + \xi_n$. Since $\sum_{n=1}^{\infty}\xi_n < \infty$, it follows from Lemma 3 that $\lim_{n\to\infty} d(x_n, F(T))$ exists. If x_n converges strongly to a fixed point p of T then $\lim_{n\to\infty}\|x_n - p\| = 0$. Since $0 \leq d(x_n, F(T)) \leq \|x_n - p\|$, we have $\liminf_{n\to\infty} d(x_n, F(T)) = 0$. Conversely, suppose $\liminf_{n\to\infty} d(x_n, F(T)) = 0$, then we have $\lim_{n\to\infty} d(x_n, F(T)) = 0$. Thus for arbitrary $\epsilon > 0$, there exists a positive integer N_1 such that $d(x_n, F(T)) < \frac{\epsilon}{4}, \forall n \geq N_1$. Furthermore, $\sum_{n=1}^{\infty}\xi_n < \infty$ implies that there exists a positive integer N_2 such that $\sum_{j=n}^{\infty}\xi_j <$

$\frac{\epsilon}{4}, \forall n \geq N_2$. Choose $N = max\{N_1, N_2\}$, then $d(x_N, F(T)) < \frac{\epsilon}{4}$ and $\sum_{j=N}^{\infty} \xi_j < \frac{\epsilon}{4}$. It follows from Lemma 3 that $\forall n, m \geq N$ and for all $p \in F(T)$, we have

$$\begin{aligned}\|x_n - x_m\| &\leq \|x_n - p\| + \|x_m - p\| \\ &\leq \|x_N - p\| + \sum_{j=N+1}^{n} \xi_j + \|x_N - p\| + \sum_{j=N+1}^{m} \xi_j \\ &\leq 2\|x_N - p\| + 2\sum_{j=N+1}^{\infty} \xi_j\end{aligned}$$

Taking infimum over all $p \in F(T)$, we obtain

$$\|x_n - x_m\| \leq 2d(x_N, F(T)) + 2\sum_{j=N+1}^{\infty} \xi_j < \epsilon, \quad \forall n, m \geq N.$$

Thus $\{x_n\}_{n=1}^{\infty}$ is Cauchy. Suppose $\lim_{n\to\infty} x_n = u$, then since $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, we have $u \in F(T)$, completing the proof of Theorem 3.

Theorem 4: Let E be a real reflexive Banach space satisfying Opial's condition. $T : E \to E$ a nonexpansive mapping with $F(T) \neq \emptyset$, under the hypothesis of Theorem 3, the iteration scheme (6) converges weakly to a fixed point of T.

Proof:
From Lemma 2, $(I - T)$ is demiclosed at zero, and since the $\lim_{n\to+\infty} \|x_n - p\| = 0$ and E satisfies Opial's condition, it follows from standard argument that $\{x_n\}_{n=1}^{\infty}$ converges weakly to a fixed point of T.

4. CONCLUDING REMARKS

Remark 1: It follows from Lemma 3 and Theorem 3 that under the hypothesis of Theorem 3, $\{x_n\}_{n=1}^{\infty}$ converges strongly to a fixed point p of T if and only if $\{x_n\}_{n=1}^{\infty}$ has a subsequence $\{x_{n_j}\}_{j=1}^{\infty}$ which converges strongly to p . Thus, under the hypothesis of Theorem 3, if T is in addition completely continuous or demicompact, then $\{x_n\}_{n=1}^{\infty}$ converges strongly to a fixed point of T. Furthermore, if T satisfies condition (A), then $\liminf_{n\to+\infty} d(x_n, F(T)) = 0$, so under the conditions of Theorem 3, if T satisfies condition (A), then $\{x_n\}_{n=1}^{\infty}$

converges strongly to a fixed point of T.

Remark 2: Theorems 3 and 4 and Remark 1 extend the results of [15] from Hilbert spaces to much more general Banach spaces as considered here. Furthermore, the strong monotonicity condition imposed on f and g in [15] is not required in our results.

Remark 3: If $b_n = 1$ in (6), the results of Osilike, Isiogugu, and Nwokoro [22] become special cases of our results.

Remark 4: Our results extend, generalize and complement the results of Miao and Li [15], Wang [26] and others in literature.

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