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A GENERALIZATION OF AFUWAPE-BARBASHIN-EZEILO PROBLEM FOR CERTAIN THIRD ORDER NONLINEAR VIBRATIONS

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ABSTRACT. Using frequency-domain results of Afuwape [2], a theorem of Corduneanu on abstract Volterra operators [4] we have given conditions under which the third order system

$$\begin{aligned} \mu'(x)\dot{x} &= y - x \\ \alpha\dot{y} &= g(z-y) \\ \dot{z} &= f(y-x) - gy \end{aligned}$$

has a periodic solution - where the nonlinearity f satisfies a sector condition, α and g are constants. Our result constitutes an application of [4] (pp. 124) and a generalization of [2].

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1. INTRODUCTION

The system

$$\begin{aligned} \mu'(x)\dot{x} &= y - x \\ \alpha\dot{y} &= g(z-y) \\ \dot{z} &= (1-g)y - x \end{aligned}$$

which was studied by K. O. Friedrichs [7] in the investigation of nonlinear vibrations was recently investigated by Afuwape [2] using frequency domain method to prove the existence of a non-trivial periodic solution under the following restrictions: α, g are constants and μ is differentiable function with $\mu(0) = 0, 0 < g < 1$ and $g\eta[(1-g)\eta - 1] + \alpha(\eta - 1) > 0$ for $\eta = \mu'(0)$. The Afuwape's [2] frequency domain considerations for the K. O. Friedrichs' studies constitute further generalization and extension of Afuwape's [1] earlier generalization on the Barbashin-Ezeilo problem [6]. It is

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worth mentioning that in a very recent development, Afuwape et al [3] considered and introduce the iterative problem of approximating cylcles of the second kind for the Barbashin-Ezeilo problem in Hilbert spaces. In this work we modify the right hand side of the state system by replacing the term $y - x$ with the nonlinearity $f = f(y-x)$. We note that if f is the identity function, we shall obtain the earlier result of Afuwape [2]. The main aim of this research is to use the frequency-domain method, a result of Corduneanu ([4] pp. 119 - 125) and the following transformation in Afuwape [2]:

$$\sigma \;=\; \mu(x),\;\; z_1 = y,\;\; z_2 = z \tag{1}$$

to prove the existence of non-trivial periodic solution for the system below:

$$\begin{array}{ll} \mu'(x)\dot{x} & = y - x \\ \alpha\dot{y} & = g(z-y) \\ \dot{z} & = f(y-x) - gy \end{array} \tag{2}$$

where the nonlinearity f satisfies the sector condition

$$0 < \frac{f(\xi_1) - f(\xi_2)}{\xi_1 - \xi_2} \;\leq\; k,\;\; \xi_1 \neq \xi_2,\;\; f(0) \;=\; 0 \tag{3}$$

for all $\xi_1, \xi_2 \in \mathbb{R}$.

Observe that on application of (1) to (2) we obtain the system

$$\begin{array}{ll} \dot{z}_1 & = -\frac{g}{\alpha}z_1 + \frac{g}{\alpha}z_2 \\ \dot{z}_2 & = -gz_1 + f(\dot{\sigma}) \\ \dot{\sigma} & = z_2 - \varphi(\sigma),\;\; \varphi(\sigma) \;=\; \mu^{-1}(\sigma). \end{array} \tag{4}$$

Next, let $L_{loc}(\mathbb{R}_+, \mathbb{R})$ denote the space of locally integrable functions defined on $\mathbb{R}_+$ and $L_{loc}([0,\tau), \mathbb{R})$ denote the space of locally integrable functions defined on $[o, \tau), \tau \leq \infty$.

Definition 1: An operator $V : C([0,\tau),\mathbb{R}) \longrightarrow C_{loc}([0,\tau),\mathbb{R})$ is called an abstract Volterra operator if for any $x, y \in C([0,\tau),\mathbb{R})$ such that $x(s) = y(s)$ for $s \in [t_0, t], t_0 < t \leq \tau$, then (Vx)(t) = (Vy)(t).

Apart from the classical Volterra integral operators a good and more general example of abstract Volterra operators are the Niemytzki operators defined below:

Definition 2: Let $F = F(t, x(t))$ satisfy the Caratheodory conditions:

$(C1)$ For a fixed x the function g is continuous in t.

$(C2)$ For a fixed t the function g is measurable in x.

Then the operator T given by $(Tx)(t) = F(t, x(t))$ is called Niemytzki operator.

Abstract Volterra operators have been studied by various authors some of whom are Tychonoff [11], Tonelli [10] and Corduneanu [4] in the investigations of certain evolutionary processes dependent on heredity like feedback control equations in automatic control theory. In [4, 5] Corduneanu have treated many properties of abstract Volterra equations including properties of Niemytzki operators.

2. PRELIMINARY

In the sequel we shall make use of the following assumptions (see, for example, [4]):
$(A1)$ T is a continuous Volterra operator from the space $C([0,\tau),\mathbb{R})$ into the Banach space $C_{loc}([0,\tau),\mathbb{R})$
$(A2)$ There exists two functions $B_1, B_2 : [0,\tau) \longrightarrow \mathbb{R}$ such that B_1 is continuous, positive and nondecreasing while B_2 is a locally integrable and non-negative function such that

$$x \in C_{loc}([0,\tau),\mathbb{R}) \quad \text{and} \quad |x(t)| \leq B_1(t) \; t \in [0,\tau) \tag{5}$$

implies

$$|(Tx)(t)| \;\; \leq \;\; B_2(t) \; a.e \text{ on } [0,\tau) \tag{6}$$

where

$$B_1(t) - B_1(0) \;\; \geq \;\; \int_0^t B_2(s)ds. \tag{7}$$

Observe that the system (4) is in the form

$$\begin{aligned} \dot{z} &= Az + bf(\dot{\sigma}) \\ \dot{\sigma} &= c^* z + \rho\varphi(\sigma), \end{aligned} \tag{8}$$

with

$$A = \begin{pmatrix} \frac{-g}{\alpha} & \frac{g}{\alpha} \\ -g & 0 \end{pmatrix}, \; b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \; c = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \; \rho = -1.$$

The system (8) yields the integro-differential equation

$$\dot{\sigma}(t) \;\; = \;\; c^*\Phi(t)z(0) + \rho\varphi(\sigma(t)) + c^* \int_0^t \Phi(t-s)bf(\dot{\sigma}(s))ds \tag{9}$$

where Φ is some relevant fundamental matrix of the corresponding homogeneous problem with transfer function $W(p)$. Since the transfer function $W(p) = c^*(A - pI)^{-1}b = \frac{-g}{\alpha p^2 + gp + g^2} = \frac{m(p)}{n(p)}$ is nondegenerate the form (8) or (9) allows us to apply the following theorems of Corduneanu and Afuwape:

Theorem 1. *(Corduneanu* [4]*) Let* $x, F \in \mathbb{R}, t \in [0, \tau)$, *and* $\Phi(t-s)$ *is a matrix-valued kernel defined on* $D = \{(t,s) \mid 0 \leq s \leq t < \tau\}$ *and moreover* F *satisfies*

$$\|F(t,x,y)\| \;\; \leq \;\; \alpha(t)|x| + \beta|y| + \gamma(t) \tag{10}$$

whrer α, β, γ *are nonnegative locally integrable functions on* $[0, \tau)$. *Suppose that the operator* $T : C([0,\tau), \mathbb{R}) \longrightarrow C_{loc}([0,\tau), \mathbb{R})$ *satisfies conditions* $(A1)$ *and* $(A2)$ *and is given by*

$$(Tx)(t) \;\; = \;\; F\left(t, x(t), \int_0^t \Phi(t-s)x(s)ds\right). \tag{11}$$

Then equation (9) i.e (4) has a solution $x(t) \in C_{loc}([0,\tau), \mathbb{R})$ *such that (5) holds provided* $|x^0| < A(0)$.

Theorem 2. *(Afuwape* [2] *pp. 7)*
Suppose in the system

$$\begin{aligned} \dot{z}_1 &= -\tfrac{g}{\alpha} z_1 + \tfrac{g}{\alpha} z_2 \\ \dot{z}_2 &= (1-g)z_1 + \varphi(\sigma) \\ \dot{\sigma} &= z_1 - \varphi(\sigma), \;\; \varphi(\sigma) \;=\; \mu^{-1}(\sigma) \end{aligned} \tag{12}$$

the folowing hold

(1) $0 < \alpha < (2-g)g$ *and* $0 < g < 1$;

(2) there exists a number λ *such that* $\varphi'(0) < \lambda < \frac{g + \sqrt{g^2 - 4\alpha(g^2 - g)}}{2\alpha}$ *and* $\alpha\lambda^2 - g\lambda + (g^2 - g) < 0$;

(3) if $\eta = \varphi'(0) > 0$ *with* $\varphi(0) = 0$ *and* $g\eta[-g\eta 1] + \alpha(\eta - 1) > 0$.

Then the system (12) i.e (4) has a non-trivial periodic solution.

Remark 1. *Observe that the system* (12) *corresponds to the integro-differential equation equation*

$$\dot{\sigma}(t) \;\; = \;\; \Phi(t) + \rho\varphi(\sigma(t)) + c^* \int_0^t \Phi(t-s) b\varphi(\sigma(s))ds \tag{13}$$

which is, obviously, a generalization of the integro-differential equation corresponding to the system considered in [1]. *Therefore, our result for (9) modifies, extends* (13) *and so modifies and generalizes the results in* [1] *and* [2].

3. MAIN RESULTS

Our main result is an application of Theorem 1 and Theorem 2 to the system (4) and its proof is a consequence of the Lemma below:

Lemma 1. *Let f satisfy the sector condition (3) and a Lipschitz condition with Lipschitz constant $\lambda > 1$. Suppose $\sigma(t)$ is defined by system (4), then there exist a function $u(t)$ and a real numbers k_1 such that $\sigma(t) < u(t) \leq e^{k_1 t}$.*

Proof: The system (4) is equivalent to the integro-differential equation (9). So, clearly,

$$\begin{aligned} |\dot{\sigma}(t)| &\leq \lambda|\sigma(t)| + k\left|c^* \int_0^t \Phi(t-s)b\dot{\sigma}(s)ds\right| + |c^*\Phi(t)z(0)| \\ &\leq \int_0^t \lambda|\dot{\sigma}(t)| + k\,|c^*\Phi(t-s)b|\,|\dot{\sigma}(s)|\,ds + |c^*\Phi(t)z(0)| \\ &= \int_0^t [\lambda + k\,|c^*\Phi(t-s)b|]\,|\dot{\sigma}(s)|\,ds + \gamma(t) \end{aligned}$$

where $\gamma(t) = |c^*\Phi(t)z(0)|$. On application of Gronwall inequality to

$$|\dot{\sigma}(t)| \leq \int_0^t [\lambda + k\,|c^*\Phi(t-s)b|]\,|\dot{\sigma}(s)|\,ds + \gamma(t)$$

we obtain the inequality

$$|\dot{\sigma}(t)| \leq q e^{\int_0^t \lambda + k|c^*\Phi(t-s)b|ds} + \phi(\gamma(t))$$

for some function ϕ. So we can choose k_1 and $u(t)$ in such a way that

$$u(t) = q e^{\int_0^t \lambda + k|c^*\Phi(t-s)b|ds} + \phi(\gamma(t)) \leq e^{k_1 t} \tag{14}$$

which yields $|\dot{\sigma}(t)| \leq u(t) \leq e^{k_1 t}$.

Observing that since $\lambda > 1$ we must have $|\sigma(t)| < \lambda|\sigma|$ and so $|\sigma(t)| \leq |\dot{\sigma}(t)|$ which implies the desired result $|\sigma(t)| < u(t) \leq e^{k_1 t}$. End of proof.

Theorem 3. *Suppose in the system (4) the following hold*
(a) $0 < \alpha < (1-g)g, 0 < g < 1$
(b) There exists $\lambda > 1$ such that

$$\varphi'(\sigma) < \lambda < \frac{g + \sqrt{g^2 - 4\alpha g^2}}{2\alpha} \text{ and } \alpha \leq \frac{1}{4}$$

(c) If $\eta = \varphi'(0) > 0$ with $\varphi(0) = 0$ and $g\eta[\eta(1-g)-1] + \alpha(\eta-1) > 0$; then the system (4) has a periodic solution.

Proof: As mentioned above the system (4) is equivalent to the integro-differential equation (9) and since Theorem 1 conforms with

(9) we shall use Theorem 1 to prove existence. Also, since Theorem 2 conforms with (4) we shall, finally, use Theorem 2 to prove periodicity. Observe That the integro-differential equation (9) i.e

$$\dot{\sigma}(t) \;=\; \varphi(\sigma(t)) + c^* \int_0^t \Phi(t-s) b f(\dot{\sigma}(s)) ds + c^* \Phi(t) z(0)$$

induces an operator $T : C([0,\tau),\mathbb{R}) \longrightarrow C_{loc}([0,\tau),\mathbb{R})$ for $\tau \leq \infty$ given by

$$(Tx)(t) \;=\; F\left(t,\ x(t),\ c^* \int_0^t \Phi(t-s) b f(\dot{x}(s)) ds\right) \tag{15}$$

where

$$F\left(t,\ x(t),\ c^* \int_0^t \Phi(t-s) b f(\dot{x}(s)) ds\right) = \varphi(x(t))$$

$$+c^* \int_0^t \Phi(t-s) b f(\dot{x}(s)) ds + c^* \Phi(t) z(0) \tag{16}$$

Clearly, F is continuous in t for fixed $\varphi(x(t)) + c^* \int_0^t \Phi(t-s) b f(\dot{x}(s)) ds$ and measurable in $\varphi(x(t)) \ + \ c^* \int_0^t \Phi(t-s) b f(\dot{x}(s)) ds$ for fixed t which implies that it satisfies the Caratheodory conditions $(C1)$ and $(C2)$ so that T is a Niemytzki operator. This means that the hypothesis $(A1)$ of Theorem 1 is satisfied since Nimytzki operators are abstract Volterra operators.

In order to construct B_1 and B_2 that verify (5), (6) and (7) in hypothesis $(A2)$, we put $B_1 = e^{k_1 t}$ for some k_1 spcified in the proof of Lemma 1. Observe that integrating both sides of (9) yields

$$\left.\int_0^t (Tx(s))(t) ds\right|_{t=0} \;=\; 0 \ \forall \ x \ \in \ C([0,\tau),\mathbb{R}). \tag{17}$$

Further, for each $x \in C([0,\tau)\mathbb{R})$ $(\tau \leq \infty)$ - making use of the sector condition (3) and taking into considerations the fact that the condition $\varphi'(x) < \lambda < \frac{g+\sqrt{g^2-4\alpha g^2}}{2\alpha}$ yields a Lipschitz condition

$|\varphi(x(t))| \leq \lambda|x(t)|$ - we must have:

$$\begin{aligned}
|(Tx)(t)| &\leq |\varphi(\sigma(t))| + \left|c^* \int_0^t \Phi(t-s)bf(\dot{x}(s))ds\right| + |c^*\Phi(t)z(0)| \\
&\leq \lambda|x(t)| + k\left|c^* \int_0^t \Phi(t-s)b\dot{x}(s)ds\right| + |c^*\Phi(t)z(0)| \\
&\leq \lambda|x(t)| + \left(k\int_0^t |c^*\Phi(t-s)b|\,ds\right)\left(\int_0^t |\dot{x}(s)|ds\right) \\
&+ |c^*\Phi(t)z(0)| \qquad (18) \\
&\leq \lambda\left(\int_0^t |\dot{x}(t)|ds\right) + \left(k\int_0^t |c^*\Phi(t-s)b|\,ds\right)\left(\int_0^t |\dot{x}(s)|ds\right) \\
&+ |c^*\Phi(t)z(0)| + \lambda|x(0)| \\
&\leq \lambda e^{k_1(t)} + \left(k\int_0^t |c^*\Phi(t-s)b|\,ds\right)e^{k_1 t} + \gamma(t). \\
&= \lambda B_1(t) + B_1(t)\left(k\int_0^t |c^*\Phi(t-s)b|\,ds\right) + \gamma(t). \qquad (19)
\end{aligned}$$

From (19), putting $B_2(t) = \lambda B_1(t) + \left(k\int_0^t |c^*\Phi(t-s)bB_1(s)|\,ds\right) + \|c^*\Phi(t)z(0)\|$ it is clear that if $|x(t)| \leq |B_1(t)|$ then $|(Tx)(t)| \leq |B_2(t)|$ verifying (5) and (6). Also, from (14) we obtain $B_1(t) - B_1(0) \geq \int_0^t B_2(s)ds$ which verifies (7). Since, by (17) $\sigma(0) = 0, \sigma(0) < B_1(0)$ and by (18) $|\sigma(t)| \leq |B_1(t)| \Longrightarrow |(T\sigma)(t)| = |\sigma(t)| \leq |B_2(t)|$ it is clear that $\sigma \in C_{loc}([0,\tau), \mathbb{R}), \tau \leq \infty$. This proves existence of a solution.

We shall now apply Theorem 2 to prove periodicity of solution which invloves showing that conditions $(a), (b),$ and (c) of Theorem 2 are satisfied. Condition (a) is trivially satisfied since the hypothesis $0 < \alpha < (1-g)g$ implies $0 < \alpha < (2-g)g$. We shall accomplish the remainig part of the proof by comparing the solution of (9) to a majoring solution to the Afuwape-Friedrich-type problem with transfer function given by $-W(p) = c^*(A - pI)^{-1}b = \frac{g}{\alpha p^2 + gp + g^2}$.

The system (1) studied in [1] corresponds to the integro-differential equation

$$\dot{\sigma}(t) = \varphi(\sigma(t)) - c^* \int_0^t e^{m_2(t-s)} b\varphi(\sigma(s))ds + c^*\xi(t)z(0) \quad (20)$$

with $m_2 = \frac{-(g + \sqrt{g^2 + 4\alpha(g - g^2)})}{2\alpha}$, while the system (4) studied in this work corresponds to (9) with $c^*\Phi(t-s)b = e^{p_2(t-s)}$ where $p_2 = \frac{-g(1+\sqrt{1-4\alpha})}{2\alpha}$. Our method of proof consists in showing that (9) is dominated by a solution of the Afuwape-type problem (20) which is periodic by Theorem 2.

Observe that $c^*\Phi(t-s)b = e^{p_2(t-s)}$ is a bounded and decreasing function for all $0 \le s \le t \le \tau$. Next, from above and using the fact that by (17) $\sigma(0) = 0$ (9) yields

$$\begin{aligned}\dot{\sigma}(t) &= \varphi(\sigma(t)) + \int_0^t e^{p_2(t-s)} f(\dot{\sigma(s)})ds + c^* e^{p_2 t} z(0) \\ &\le \varphi(\sigma(t)) + k\int_0^t |e^{p_2(t-s)}||\dot{\sigma(s)}|ds + c^* e^{p_2 t} z(0) \\ &\le \varphi(\sigma(t)) + |\sigma(t)| - \frac{k}{\lambda}\int_0^t |e^{p_2(t-s)}||\varphi(\sigma(s))|ds + c^* e^{p_2 t} z(0).\end{aligned}$$

Now, by Lemma 1 $|\sigma(t)| < |v(t)|$ where $v(t)$ is the bounded solution of the Afuwape-Friedrichs-type problem below:

$$\dot{v}(t) = \varphi(v(t)) - \frac{k}{\lambda}\int_0^t |e^{p_2(t-s)}||\varphi(v(s))|ds + \xi(t)z(0). \quad (21)$$

where $\xi(t) = c^* e^t z(0) + sup|\sigma(t)|$.

We shall now show the problem (21) satisfies coditions (b) and (c) of Theorem 2 based on our hypothesis. To verify condition (b) in Theorem 2 we observe that our hypothesis $\alpha \le \frac{1}{4}$ ensures the solvability of the inequality $\alpha\lambda^2 - g\lambda + g^2$ which corresponds to the inequality $\alpha\lambda^2 - g\lambda + (g^2 - g) < 0$ for $\varphi'(\sigma) < \lambda < \frac{g+\sqrt{g^2-4\alpha(g^2-g)}}{2\alpha}$ in condition (b) of Theorem 2. Therefore condition(b) is verified. Condition (c) of Theorem 2 follows directly by replacing $(1-g)$ by g. Therefore the system (4) has a periodic solution. This completes the proof of Theorem 3.

4. CONCLUDING REMARKS

It is of significance to observe that the estimates $\alpha\lambda^2 - g\lambda + (g^2 - g) < 0$ and $\varphi'(\sigma) < \lambda < \frac{g+\sqrt{g^2-4\alpha(g^2-g)}}{2\alpha}$ play vital roles in sovability of the ABE problem using operator theoretic approach. Similar estimates were invaluable in iterative construction of approximate cycles of the second kind in [3] for the ABE problem. It is recommended that these estimates be used to decide when the problem is solvable in Hilbert or Banach spaces. This would help to extend the results of Afuwape et al [3] and the ABE problem using operator analysis of the integro-differential equation (9) as a functional equation on these spaces.

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