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ON SPAN OF FLAG MANIFOLDS $\mathbb{R}F(1,1,1,n-3)$

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ABSTRACT. We obtain bounds for the span of the incomplete flag manifold of length 3, $\mathbb{R}F(1,1,1,n-3)$, $n \geq 4$ using suitable fiberings where $\mathbb{R}F(1,1,1,n-3)$ is either a total space or base space. We obtain exact values for $n = 5$ and 6 using non-vanishing Stiefel-Whitney classes.

Keywords and phrases: real flag manifold, span, vector field problem, Stiefel-Whitney classes, fiber bundles
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1. INTRODUCTION AND STATEMENT OF THE RESULT

For vector bundle η over a base space X, *span* of η, (span(η)) is the largest integer r such that η admits r everywhere linearly independent sections. When X is a smooth manifold *span*(X) is defined to be the span of its tangent bundle, $\tau(X)$. The span is an important geometric characteristic of manifolds whose study has attracted some attention. Much is not known about the span of flag manifold $\mathbb{R}F(\underbrace{1,1,...1}_{k},n-k)$ [9].

For $\mathbb{R}F(1,n-1)$ which is the real projective space $\mathbb{R}P^{n-1}$, the span is known to coincide with the span of the $n-1$-sphere, S^{n-1}, that is,

$$\text{span}\mathbb{R}P^{n-1} = \text{span}S^{n-1} = \rho(n)$$

(cf.[8]) where $\rho(n) = 2^c + 8d - 1$ if n is expressed as $(2a+1)2^{c+4d}$, $a, c, d \geq 0$, $c \leq 3$.

In the case of $\mathbb{R}F(1,1,n-2)$, there are estimates of its span but the value has been determined only in the following cases: all n even except $n \equiv 0 \mod 16$ and for n odd, only when $n = 5$ and $n = 2^t + 3, t \geq 3$ (cf. [5, 7]).

The manifold $\mathbb{R}F(1,1,1,n-3)$ is a smooth connected compact homogeneous space of dimension $3n-6$. This manifold is non-orientable when n is odd. From the classical theorem of Hopf ([12]),

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we have that $\mathbb{R}F(1,1,1,n-3)$ has positive span since its Euler characteristic $\chi(\mathbb{R}F(1,1,1,n-3)) = 0$ (cf [8]).

Let ν_1, ν_2, ν_3 be the canonical line bundles, ν_4 the $n-3$ bundle, over $\mathbb{R}F(1,1,1,n-3)$, and $x = w_1(\nu_1), y = w_1(\nu_2), z = w_1(\nu_3)$ be the Stiefel-Whitney classes of ν_1, ν_2, ν_3 respectively. Let $\sigma_1 = x+y+z$, $\sigma_2 = xy+yz+xz$, $\sigma_3 = xyz$.

The $\mathbb{Z}_2$-cohomology algebra, $H^*(\mathbb{R}F(1,1,1,n-3), \mathbb{Z}_2)$ can be identified with $\mathbb{Z}_2[x,y,z]$ subject to the relations $\bar{\sigma}_{n-2} = \bar{\sigma}_{n-1} = \bar{\sigma}_n = 0$ where $\bar{\sigma}_i = \bar{\sigma}_i(x,y,z)$ is the i-th complete symmetric function in x, y and z so that $x^n = 0 = y^n = z^n$ [4].

The tangent bundle of $\mathbb{R}F(1,1,1,n-3)$, $\tau(\mathbb{R}F(1,1,1,n-3))$ can be expressed as

$$\tau(\mathbb{R}F(1,1,1,n-3)) \cong \oplus_{1\leq i,j\leq 4} \nu_i \otimes \nu_j. \tag{1}$$

A manifold closely related to $\mathbb{R}F(1,1,1,n-3)$ is the Grassmann manifold of 3-planes in $\mathbb{R}^n$, $\mathbb{R}F(3,n-3)$. The manifold $\mathbb{R}F(1,1,1,n-3)$ naturally fibers over $\mathbb{R}F(3,n-3)$. The projection

$$\mathbb{R}F(1,1,1,n-3) \longrightarrow \mathbb{R}F(3,n-3) \tag{2}$$

is a smooth fibration with fibre, the parallelizabe flag $\mathbb{R}F(1,1,1)$. A few results on the span of $\mathbb{R}F(3,n-3)$ is known (cf.[3]).

Another manifold related to $\mathbb{R}F(1,1,1,n-3)$ is the projective Stiefel manifold, $X_{n,3}$. The projection map

$$X_{n,3} \longrightarrow \mathbb{R}F(1,1,1,n-3) \tag{3}$$

is a principal fibration [10], Korbas and Zvengrowski [9] gave some results on span$X_{n,3}$.

The only known bound of span$\mathbb{R}F(1,1,1,n-3)$ is the lower bound given for a general flag manifold by Korbaš[7] and the only known value is that of $n = 4$, the only parallelizable flag of length 3, $\mathbb{R}F(1,1,1,1)$, whose span is 6.

The aim of this paper is to give bounds for the span of $\mathbb{R}F(1,1,1,n-3)$ using fibre bundles with $\mathbb{R}F(1,1,1,n-3)$ as total space and $\mathbb{R}F(3,n-3)$ and $\mathbb{R}F(1,1,n-2)$ as base spaces, respectively; and fibre bundles with $X_{n,3}$ as total space and $\mathbb{R}F(1,1,1,n-3)$ as base space. In particular, we determine the span for $n = 5, 6$ using non-vanishing Stiefel- Whitney classes of $\mathbb{R}F(1,1,1,2)$ and $\mathbb{R}F(1,1,1,3)$ respectively. Among others, we give the following bounds:
For n odd:

$$4 \leq \text{span } \mathbb{R}F(1,1,1,2^m-2) \leq 2^{m+1}-3, \quad m \geq 3$$

$$\text{span } \mathbb{R}F(1,1,1,n-3) \geq 3.$$

For n even:

$$\text{span } \mathbb{R}F(1,1,1,2^m-5) \leq 2^{m+1}-6, \quad m \geq 3.$$

We also give the following exact values:

$$\text{span } \mathbb{R}F(1,1,1,2) = 3$$
$$\text{span } \mathbb{R}F(1,1,1,3) = 7$$

2. PROOF OF RESULTS

We need the following results,
Theorem 1:[8] If $F \longrightarrow E \longrightarrow B$ is a smooth fibre bundle, then span $E \geq$ span B.

The smooth fibrations

$$\mathbb{R}F(1,1,1,n-3) \longrightarrow \mathbb{R}F(3,n-3)$$

and

$$\mathbb{R}F(1,1,1,n-3) \longrightarrow \mathbb{R}F(1,1,n-2)$$

yield the following:

Proposition 1:

$$\text{span } \mathbb{R}F(1,1,1,n-3) \geq \text{span } \mathbb{R}F(3,n-3)$$
$$\text{span } \mathbb{R}F(1,1,1,n-3) \geq \text{span } \mathbb{R}F(1,1,n-2).$$

By Korbas [6], we have the following lower bound results:

Proposition 2: span $\mathbb{R}F(1,1,1,n-3) \geq \rho(n)$

Applying Propositions 1 and 2 we have the following lower bounds:

Proposition 3:

$$\text{span } \mathbb{R}F(1,1,1,n-3) \geq \begin{cases} 1 & n \equiv 2 \mod 4; \text{ or } n \textit{ odd} \\ 3 & n \equiv 4 \mod 8 \\ 7 & n \equiv 8 \mod 16 \\ 8 & n \equiv 16 \mod 32 \\ 4 & n = 2^m+1, m \geq 3 \end{cases}$$

Proof: For $n \equiv 2 \mod 4$, $n \equiv 4 \mod 8$ and $n \equiv 8 \mod 16$, the result follows from Ilori and Ajayi's results on span of $\mathbb{R}F(1,1,n-2)$ [5] and Proposition 1. Alternatively, if n is odd then $\rho(n) = 1$;

if $n \equiv 2 \mod 4$ then $n = 2(1+2k)$ implies $a = k, c = 1, d = 0$ and $\rho(n) = 1$;

if $n \equiv 4 \mod 8$ then $n = 2^2(1+2k)$ this implies $a = k, c = 2, d = 0$ and $\rho(n) = 3$;

if $n \equiv 8 \mod 16$ then $n = 2^3(1+2k)$ this implies $a = k, c = 3, d = 0$ and $\rho(n) = 7$;

if $n \equiv 16 \mod 32$ then $n = 2^4(1+2k)$ this implies $a = k, c = 0, d = 1$ and $\rho(n) = 8$.

Using Proposition 2 we have the results.

The proof of the last inequality follows from span $\mathbb{R}F(1,1,2^m-1) \geq 4$ (cf. [7]) and Proposition 1.

From [9], the span of $X_{n,3}$ is related to the span of $\mathbb{R}F(1,1,1,n-3)$ as follows:

Proposition 4: span $X_{n,3} \geq$ span $\mathbb{R}F(1,1,1,n-3)$.

Results on the span of $X_{n,3}$ (cf.[9]) yield the following upper bounds

Proposition 5:

$$\text{span } \mathbb{R}F(1,1,1,2^m-5) \leq 2^{m+1}-6;\ m \geq 3$$

$$\text{span } \mathbb{R}F(1,1,1,2^m-2) \leq 2^{m+1}-3;\ m \geq 2.$$

To obtain lower bounds for span of manifolds, the knowledge of Stiefel-Whitney classes have proved important because of the following:

Proposition 6: [12] If $w_k(M) \neq 0$, then span $M \leq m-k$ where m is the dimension of M.

Results on Stiefel-Whitney classes of $\mathbb{R}F(1,1,1,n-3)$ in [1] yield,

Proposition 7: If n is even, then $w_3 = \sigma_1\sigma_2 + \sigma_3 \neq 0$; if $n \equiv 2,4,6 \mod 8$ then $w_4 \neq 0$; if $n \equiv 2,6 \mod 8$ then $w_5 = \sigma^2(\sigma_1\sigma_2+\sigma_3) \neq 0$; if $n \equiv 5 \mod 8$ then $w_6 = \sigma_1\sigma_2\sigma_3 \neq 0$; and if $n \equiv 2 \mod 8$ then $w_7 = \sigma_1^4(\sigma_1\sigma_2+\sigma_3) \neq 0$.

Proposition 7 together with Proposition 6 yield:

Corollary 1: If $n \equiv 2 \mod 8$ then span $\mathbb{R}F(1,1,1,n-3) \leq 3n-13$, if $n \equiv 6 \mod 8$ then span $\mathbb{R}F(1,1,1,n-3) \leq 3n-11$, if $n \equiv 5 \mod 8$ then span $\mathbb{R}F(1,1,1,n-3) \leq 3n-12$ if $n \equiv 4 \mod 8$ then span $\mathbb{R}F(1,1,1,n-3) \leq 3n-10$ and if $n \equiv 0 \mod 8$ then span $\mathbb{R}F(1,1,1,n-3) \leq 3n-9$.

Theorem 2:

$$\text{span } \mathbb{R}F(1,1,1,2) = 3$$
$$\text{span } \mathbb{R}F(1,1,1,3) = 7$$

Proof:

$$\text{span } \mathbb{R}F(1,1,3) = 3 \Rightarrow \text{span } \mathbb{R}F(1,1,1,2) \geq 3$$

and

$$\text{span } X_{5,3} = 5 \Rightarrow \text{span } \mathbb{R}F(1,1,1,n-3) \leq 5$$

i.e.

$$3 \leq \text{ span}\mathbb{R}F(1,1,1,2) \leq 5.$$

From [1], for $n = 5$, $w_6(\mathbb{R}F(1,1,1,2)) = \sigma_1\sigma_2\sigma_3 \neq 0$. Since $\dim \mathbb{R}F(1,1,1,2) = 9$, and using Proposition 5 we have span $\mathbb{R}F(1,1,1,2) \leq 3$. Hence the result.

$$\text{span } \mathbb{R}F(3,3) = 7 \text{ implies span } \mathrm{RF}(1,1,1,3) \geq 7$$

and

$$\text{span } X_{6,3} = 10 \Rightarrow \text{span } \mathbb{R}F(1,1,1,3) \leq 10$$

i.e.

$$7 \leq \text{ span}\mathbb{R}F(1,1,1,3) \leq 10.$$

We know by [1], that $w_5 = \sigma_1^2(\sigma_1\sigma_2 + \sigma_3)$ for $n \equiv 6(8)$ and $\sigma_1\sigma_2 + \sigma_3 \neq 0$ and $\sigma_1^2 \neq 0$. Therefore, $w_5 \neq 0$ and $span\mathbb{R}F(1,1,1,3) \leq 12-5 = 7$ gives the result.

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