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COEFFICIENT ESTIMATES FOR CERTAIN CLASSES OF ANALYTIC FUNCTIONS

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ABSTRACT. Bounds on early coefficients of analytic functions normalized by $f(0) = f'(0) - 1 = 0$ which satisfy

$$\mathrm{Re}\, \frac{f(z)^{\alpha-1} f'(z)}{z^{\alpha-1}} \left(1 + \frac{zf''(z)}{f'(z)}\right) > 0$$

in the unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ are obtained using known properties of functions with positive real part.

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1. INTRODUCTION

Let A denote the class of functions

$$f(z) = z + a_2 z^2 + a_3 z^3 + ... \tag{1.1}$$

which are analytic in U. We denote by S the subclass of A which consist of univalent functions only. Let R, S^* and K be the usual subclasses of S consisting of functions which are, respectively, of bounded turning, starlike, and convex in the unit disk U; and have the following geometric conditions: Re $f'(z) > 0$, Re $zf'(z)/f(z) > 0$ and Re $(1 + zf''(z)/f'(z)) > 0$. In [10], Singh studied another subclass of S denoted by $B_1(\alpha)$ which consists of functions which are a special case of Bazilevic functions with geometric condition:

$$\mathrm{Re}\, \frac{f(z)^{\alpha-1} f'(z)}{z^{\alpha-1}} > 0, \ z \in U$$

for non negative real number α. It is not difficult to see, as Singh noted in his work, that the cases $\alpha = 0$ and $\alpha = 1$ correspond to S^* and R respectively. Numerous results have appeared in print on these subclasses including their coefficient estimates. For detail, see [1, 3].

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In this work, we denote by $\mathcal{G}_\alpha$, the subclass of A which consists of normalized analytic functions satisfying the geometric condition

$$\operatorname{Re} \frac{f(z)^{\alpha-1} f'(z)}{z^{\alpha-1}} \left(1 + \frac{zf''(z)}{f'(z)}\right) > 0 \tag{1.2}$$

for non negative real number α.

Let P denote the class of Caratheodory functions $p(z) = 1+c_1z+c_2z^2+\cdots$ which are analytic and satisfy $p(0) = 1$, Re $p(z) > 0$ in open unit disk U. The condition (1.2) implies that

$$\frac{f(z)^{\alpha-1} f'(z)}{z^{\alpha-1}} \left(1 + \frac{zf''(z)}{f'(z)}\right)$$

belongs to the class P of Caratheodory functions. We determine best possible upper bounds on some of the coefficients of $\mathcal{G}_\alpha$ using the well known method of classical analysis based on the close association between $\mathcal{G}_\alpha$ and the P.

We note that the classes of functions, $\mathcal{G}_\alpha$, contain two interesting special cases. These are (i) the case $\alpha = 0$, we have $\mathcal{G}_0$, which consists of functions in A which satisfy

$$\operatorname{Re} \frac{zf'(z)}{f(z)} \left(1 + \frac{zf''(z)}{f'(z)}\right) > 0,$$

and (ii) the case $\alpha = 1$, that is $\mathcal{G}_1$, which consists of functions in A which satisfy

$$\operatorname{Re} f'(z) \left(1 + \frac{zf''(z)}{f'(z)}\right) > 0.$$

Both of these special cases of $\mathcal{G}_\alpha$ consist of univalent functions only, being subclasses of S^* and R respectively [5].

In section 2, we outline some known results that we shall employ in our computations and in section 3, we state and prove our main results.

2. PRELIMINARY

To prove the main result in the next section, we need the following lemmas.

Lemma 1:[3, 4, 7, 9] Let $p \in P$. Then $|c_k| \leq 2, k = 1, 2, 3....$ Equality is attained by the moebius function

$$L_0(z) = \frac{1+z}{1-z}.$$

Lemma 2:[7] Let $p \in P$. Then

$$\left|c_2 - \frac{c_1^2}{2}\right| \leq 2 - \frac{|c_1|^2}{2}.$$

The result is sharp and equality holds for the function

$$p(z) = \frac{1 + \frac{1}{2}(c_1 + \epsilon \bar{c_1})z + \epsilon z^2}{1 - \frac{1}{2}(c_1 - \epsilon \bar{c_1})z - \epsilon z^2}, \quad |\epsilon| = 1.$$

The following corollary to the above was given by Babalola in [1].
corollary 1:[1] Let $p \in P$. Then we have sharp inequalities

$$\left|c_2 - \sigma\frac{c_1^2}{2}\right| \leq \begin{cases} 2(1-\sigma), & \text{if } \sigma \leq 0, \\ 2, & \text{if } 0 \leq \sigma \leq 2, \\ 2(\sigma - 1), & \text{if } \sigma \geq 2. \end{cases}$$

Lemma 3:[6] Let

$$p(z) = c_0 + c_1 z + c_2 z^2 + \cdots, \ (|z| < 1)$$

be analytic and satisfy the condition $\mathrm{Re}[p(z)] > 0$ in U, then for $n \geq 2$ and $s \geq 1$,

$$\left|\frac{c_n}{c_0} - \frac{c_s c_{n-s}}{c_0^2}\right| \leq 2\left|\frac{\mathrm{Re} c_0}{c_0}\right| \leq 2.$$

These inequalities are sharp for all n and for all s, equality being attained for each n and s by the function:

$$p(z) = (\mathrm{Re}\ c_0)\left(\frac{1+z}{1-z}\right) + i\mathrm{Im}\ c_0, \quad Re\ c_0 > 0.$$

corollary 2: Let $p \in P$. Then

$$|c_n - \sigma c_s c_{n-s}| \leq \begin{cases} 2(3 - 2\sigma), & \text{if } \sigma \leq 1, \\ 2, & \text{if } \sigma = 1, \\ 2(2\sigma - 1), & \text{if } \sigma \geq 1. \end{cases}$$

Proof: We write

$$\begin{aligned} |c_n - \sigma c_s c_{n-s}| &= |c_n - c_s c_{n-s} + c_s c_{n-s} - \sigma c_s c_{n-s}| \\ &\leq |c_n - c_s c_{n-s}| + |c_s c_{n-s} - \sigma c_s c_{n-s}| \\ &\leq 2 + |c_s||c_{n-s}||1 - \sigma| \\ &\leq 2 + 4|1 - \sigma|. \end{aligned}$$

Using Lemmas 1 and 3, we have the result.
corollary 3: Let $p \in P$. Then

$$|c_3 - \sigma\frac{c_1^3}{2}| \leq \begin{cases} 2(3 - 2\sigma), & \sigma \leq 0, \\ 6, & 0 \leq \sigma \leq 2, \\ 4\sigma - 2, & \sigma \geq 2. \end{cases}$$

Proof: We write

$$\begin{aligned}|c_3 - \sigma\frac{c_1^3}{2}| &= \left|c_3 - c_1c_2 + c_1c_2 - \sigma\frac{c_1^3}{2}\right| \\ &\leq |c_3 - c_1c_2| + \left|c_1c_2 - \sigma\frac{c_1^3}{2}\right| \\ &\leq |c_3 - c_1c_2| + |c_1|\left|c_2 - \sigma\frac{c_1^2}{2}\right|.\end{aligned}$$

Applying Corollaries 1 and 2, we have the inequalities.

3. MAIN RESULT

Next we proof the main theorem:
Theorem 1: Let $f \in \mathcal{G}_\alpha$. Then

$$|a_2| \leq \frac{2}{\alpha+3} \tag{3.1}$$

$$|a_3| \leq \begin{cases} \frac{2(\alpha+15)}{(\alpha+3)^2(\alpha+8)}, & \text{if } 0 \leq \alpha \leq 1 \\ \frac{2}{\alpha+8}, & \text{if } \alpha \geq 1 \end{cases} \tag{3.2}$$

$$|a_4| \leq \begin{cases} \frac{26\alpha^4+418\alpha^3+1594\alpha^2+1094\alpha+324}{3(\alpha+3)^3(\alpha+8)(\alpha+15)}, & 0 \leq \alpha \leq 1, \\ \frac{2(7\alpha^2+83\alpha+378)}{3(\alpha+3)(\alpha+8)(\alpha+15)}, & \alpha \geq 1 \end{cases} \tag{3.3}$$

$$|a_5| \leq \begin{cases} \frac{16\alpha^5+598\alpha^4+8510\alpha^3+57842\alpha^2+173898\alpha+132384}{(\alpha+3)^2(\alpha+8)^2(\alpha+15)(\alpha+24)}, & 0 \leq \alpha \leq 1, \\ \frac{20\alpha^5+762\alpha^4+11818\alpha^3+91038\alpha^2+323850\alpha+360480}{(\alpha+3)^2(\alpha+8)^2(\alpha+15)(\alpha+24)}, & \alpha \geq 1 \end{cases} \tag{3.4}$$

Proof: Since

$$\text{Re}\left\{\frac{f(z)^{\alpha-1}f'(z)}{z^{\alpha-1}}\left(1+\frac{zf''(z)}{f'(z)}\right)\right\} > 0,$$

then there exists $p \in P$ such that

$$\frac{f(z)^{\alpha-1}f'(z)}{z^{\alpha-1}}\left(1+\frac{zf''(z)}{f'(z)}\right) = p(z)$$

which implies that

$$f(z)^{\alpha-1}[f'(z) + zf''(z)] = z^{\alpha-1}p(z).$$

Using (1.1) the left hand side becomes

$$
\begin{aligned}
f(z)^{\alpha-1}[f'(z)+zf''(z)] &= z^{\alpha-1}+((\alpha+3)a_2)\,z^{\alpha} \\
&+\left[(\alpha+8)a_3+\left(\frac{\alpha^2+5\alpha-6}{2}\right)a_2^2\right]z^{\alpha+1} \\
&+\left[(\alpha+15)a_4+(\alpha^2+10\alpha-11)a_2a_3+\left(\frac{\alpha^3+6\alpha^2-25\alpha+18}{6}\right)a_2^3\right]z^{\alpha+2} \\
&+\left[(\alpha+24)a_5+(\alpha^2+17\alpha-18)a_2a_4+\left(\frac{\alpha^3+11\alpha^2-40\alpha+28}{2}\right)a_2^2a_3\right. \\
&\left.+\left(\frac{\alpha^2+15\alpha-16}{2}\right)a_3^2+\left(\frac{\alpha^4+6\alpha^3-61\alpha^2+126\alpha-72}{24}\right)a_2^4\right]z^{\alpha+3}+\cdots
\end{aligned}
$$

and expanding the right hand side gives

$$z^{\alpha-1}+c_1z^{\alpha}+c_2z^{\alpha+1}+c_3z^{\alpha+2}+c_4z^{\alpha+3}+...$$

Comparing coefficients on both sides of the equation, we obtain

$$(\alpha+3)a_2=c_1 \tag{3.5}$$

$$(\alpha+8)a_3+\left(\frac{\alpha^2+5\alpha-6}{2}\right)a_2^2=c_2 \tag{3.6}$$

$$(\alpha+15)a_4+(\alpha^2+10\alpha-11)a_2a_3+\left(\frac{\alpha^3+6\alpha^2-25\alpha+18}{6}\right)a_2^3=c_3 \tag{3.7}$$

$$
\begin{aligned}
c_4 &= (\alpha+24)a_5+(\alpha^2+17\alpha-18)a_2a_4+\left(\frac{\alpha^2+15\alpha-16}{2}\right)a_3^2 \\
&+\left(\frac{\alpha^3+11\alpha^2-40\alpha+28}{2}\right)a_2^2a_3+\left(\frac{\alpha^4+6\alpha^3-61\alpha^2+126\alpha-72}{24}\right)a_2^4.
\end{aligned} \tag{3.8}
$$

By Lemma 1, we obtain from equation (3.5)

$$|a_2|\le\frac{2}{\alpha+3}$$

Using (3.5) in (3.6), we have

$$(\alpha+8)|a_3|\le\left|c_2-\left(\frac{\alpha^2+5\alpha-6}{\alpha^2+6\alpha+9}\right)\frac{c_1^2}{2}\right|.$$

Now taking

$$\sigma=\frac{\alpha^2+5\alpha-6}{\alpha^2+6\alpha+9},$$

then $\sigma\le 0$ if $\alpha\in[0,1]$ and $0\le\sigma\le 2$ for $\alpha\ge 1$, so applying Corollary 1, we obtain

$$|a_3|\le\begin{cases}\frac{2(\alpha+15)}{(\alpha+3)^2(\alpha+8)}, & \text{if } 0\le\alpha\le 1\\ \frac{2}{\alpha+8}, & \text{if } \alpha\ge 1.\end{cases}$$

From (3.7) we have

$$(\alpha+15)a_4 = c_3 - \frac{\alpha^2+10\alpha-11}{(\alpha+3)(\alpha+8)}c_1c_2 + \frac{2\alpha^4+31\alpha^3+76\alpha^2-163\alpha+54}{6(\alpha^4+17\alpha^3+99\alpha^2+243\alpha+216)}c_1^3$$

for $\alpha \le 1$, $\alpha^2+10\alpha-11$ is negative, so that we have

$$(\alpha+15)a_4 = c_3 + \frac{11-10\alpha-\alpha^2}{(\alpha+3)(\alpha+8)}c_1c_2 + \frac{2\alpha^4+31\alpha^3+76\alpha^2-163\alpha+54}{6(\alpha^4+17\alpha^3+99\alpha^2+243\alpha+216)}c_1^3.$$

By triangle inequality and applying Lemma 1, we obtain the first part of the result. For $\alpha \ge 1$, we have

$$\begin{aligned}(\alpha+15)a_4 &= c_3 - \frac{\alpha^2+10\alpha-11}{(\alpha+3)(\alpha+8)}c_1c_2 + \frac{2\alpha^4+31\alpha^3+76\alpha^2-163\alpha+54}{6(\alpha^4+17\alpha^3+99\alpha^2+243\alpha+216)}c_1^3\\ &= c_3 - c_1c_2 + \frac{\alpha+35}{\alpha^2+11\alpha+24}c_1c_2 + \frac{1}{3}c_1^3 - \frac{3\alpha^3+122\alpha^2+649\alpha+378}{6(\alpha+3)^3(\alpha+8)}c_1^3\end{aligned}$$

so that

$$(\alpha+15)|a_4| \le |c_3-c_1c_2| + \frac{1}{3}|c_1^3| + \frac{\alpha+35}{(\alpha+3)(\alpha+8)}|c_1|\left|c_2 - \frac{3\alpha^3+122\alpha^2+649\alpha+378}{3(\alpha+3)^2(\alpha+35)}\frac{c_1^2}{2}\right|.$$

By Lemma 1 and 3, and Corollary 1, with

$$\sigma = \frac{3\alpha^3+122\alpha^2+649\alpha+378}{3(\alpha^3+41\alpha^2+219\alpha+315)} \le 1$$

we have

$$(\alpha+15)|a_4| \le 2 + \frac{8}{3} + \frac{4(\alpha+35)}{(\alpha+3)(\alpha+8)}$$

so that

$$|a_4| \le \frac{2(7\alpha^2+83\alpha+378)}{3(\alpha+3)(\alpha+8)(\alpha+15)}$$

From equation (3.8), we have

$$\begin{aligned}(\alpha+24)a_5 = c_4 &- \frac{\alpha^2+15\alpha-16}{2(\alpha+8)^2}c_2^2 - \frac{\alpha^2+17\alpha-18}{(\alpha+3)(\alpha+15)}c_1c_3\\ &+ \frac{2\alpha^5+71\alpha^4+734\alpha^3+1719\alpha^2-37741\alpha+1248}{2(\alpha+3)^2(\alpha+8)^2(\alpha+15)}c_1^2c_2\\ &- \frac{6\alpha^7+247\alpha^6+3361\alpha^5+16653\alpha^4+12725\alpha^3-61612\alpha^2+33804\alpha-5184}{24(\alpha+3)^4(\alpha+8)^2(\alpha+15)}c_1^4.\end{aligned}$$

For $\alpha \le 1$, $\alpha^2+15\alpha-16$ and $\alpha^2+17\alpha-18$ are negative, so we have

$$\begin{aligned}(\alpha+24)a_5 = c_4 &+ \frac{16-\alpha^2-15\alpha}{2(\alpha+8)^2}c_2^2 + \frac{18-\alpha^2-17\alpha}{(\alpha+3)(\alpha+15)}c_1c_3\\ &+ \frac{2\alpha^5+71\alpha^4+734\alpha^3+1719\alpha^2-37741\alpha+1248}{2(\alpha+3)^2(\alpha+8)^2(\alpha+15)}c_1^2c_2\\ &- \frac{6\alpha^7+247\alpha^6+3361\alpha^5+16653\alpha^4+12725\alpha^3-61612\alpha^2+33804\alpha-5184}{24(\alpha+3)^4(\alpha+8)^2(\alpha+15)}c_1^4\end{aligned}$$

which we rewrite as

$$(\alpha+24)a_5 = c_4 + \frac{16-\alpha^2-15\alpha}{2(\alpha+8)^2}c_2^2 + \frac{18-\alpha^2-17\alpha}{(\alpha+3)(\alpha+15)}c_1c_3$$
$$- \frac{3\alpha^4+264\alpha^3+4407\alpha^2+20766\alpha+16032}{2(\alpha+3)^2(\alpha+8)^2(\alpha+15)}c_1^2c_2 + c_1^2c_2 - \frac{1}{2}c_1^4$$
$$+ \frac{11\alpha^6+1019\alpha^5+21687\alpha^4+175465\alpha^3+584710\alpha^2+736020\alpha+471744}{24(\alpha+3)^4(\alpha+8)^2(\alpha+15)}c_1^4.$$

Therefore,

$$(\alpha+24)|a_5| \leq |c_4| + \frac{16-\alpha^2-15\alpha}{2(\alpha+8)^2}|c_2^2|$$
$$+ \frac{18-\alpha^2-17\alpha}{(\alpha+3)(\alpha+15)}|c_1||c_3| + |c_1^2|\left|c_2 - \frac{1}{2}\frac{c_1^2}{2}\right|$$
$$+ \frac{3\alpha^4+264\alpha^3+4407\alpha^2+20766\alpha+16032}{2(\alpha+3)^2(\alpha+8)^2(\alpha+15)}|c_1^2|\left|\sigma\frac{c_1^2}{2} - c_2\right|$$

where

$$\sigma = \frac{11\alpha^6+1019\alpha^5+21687\alpha^4+175465\alpha^3+584710\alpha^2+736020\alpha+471744}{6(\alpha+3)^2(3\alpha^4+264\alpha^3+4407\alpha^2+20766\alpha+16032)}$$

which on applying Lemma 1 and corollary 1, gives the first bound on a_5.
For $\alpha \geq 1$ we have,

$$(\alpha+24)a_5 = c_4 - c_1c_3 + \frac{\alpha+63}{(\alpha+3)(\alpha+15)}c_1c_3 - \frac{c_2^2}{2} + \frac{\alpha+80}{2(\alpha+8)^2}c_2^2 + c_1^2c_2$$
$$- \frac{3\alpha^4+264\alpha^3+4407\alpha^2+20766\alpha+16032}{2(\alpha+3)^2(\alpha+8)^2(\alpha+15)}c_1^2c_2 - \frac{c_1^4}{4}$$
$$+ \frac{11\alpha^6+1019\alpha^5+21687\alpha^4+175465\alpha^3+584710\alpha^2+736020\alpha+471744}{24(\alpha+3)^4(\alpha+8)^2(\alpha+15)}c_1^4$$

so that

$$(\alpha+24)a_5 \leq |c_4| + |c_1|\left|c_3 - \sigma_1\frac{c_1^3}{2}\right|$$
$$+ \frac{\alpha+63}{(\alpha+3)(\alpha+15)}|c_1|\left|c_3 - \sigma_2\frac{c_1^3}{2}\right| + \frac{|c_2|}{2}\left|c_2 - 4\frac{c_1^2}{2}\right|$$
$$+ \frac{\alpha+80}{2(\alpha+8)^2}|c_2|\left|c_2 - \sigma_3\frac{c_1^2}{2}\right|.$$

where

$$\sigma_1 = \frac{11\alpha^6+1019\alpha^5+21687\alpha^4+175465\alpha^3+584710\alpha^2+736020\alpha+471744}{12(\alpha+3)^4(\alpha+8)^2(\alpha+15)},$$

$$\sigma_2 = \frac{\alpha^2+18\alpha+45}{2(\alpha+63)}$$

and

$$\sigma_3 = \frac{2(3\alpha^4 + 264\alpha^3 + 4407\alpha^2 + 20766\alpha - 16032)}{(\alpha+3)^2(\alpha+15)(\alpha+80)}.$$

By Lemmas 1 and Corollaries 1 and 2, noting that σ_1 and σ_2 lie in the closed interval $[0, 2]$ while $\sigma_3 \geq 2$, we obtain

$$|a_5| \leq \frac{20\alpha^5 + 762\alpha^4 + 11818\alpha^3 + 91038\alpha^2 + 323850\alpha + 360480}{(\alpha+3)^2(\alpha+8)^2(\alpha+15)(\alpha+24)}$$

as required, and this completes the proof.

corollary : Let $f \in \mathcal{G}_0$. Then

$$|a_2| \leq \frac{2}{3},\ a_3| \leq \frac{5}{12},\ |a_4| \leq \frac{7}{10},\ |a_5| \leq \frac{751}{432}.$$

corollary : Let $f \in \mathcal{G}_1$. Then

$$|a_2| \leq \frac{1}{2},\ |a_3| \leq \frac{2}{9},\ |a_4| \leq \frac{13}{24},\ |a_5| \leq \frac{1024}{675}.$$

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