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SYMMETRIC TWO-STEP RUNGE-KUTTA COLLOCATION METHODS FOR STIFF SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS

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ABSTRACT. Symmetric two-step Runge-Kutta collocation methods have been derived for solution of stiff and oscillatory differential equations. The methods are of orders six and eight with five and seven stages respectively and hence substantial improvements in efficiency and flexibility are achieved when using them. They are shown to be A-stable, self-starting, convergent and cope effectively with stable systems of initial value problems with large Lipschitz constants. These methods as compared, for example, with some other recently derived methods of the same order, provide approximations of high accuracy to solutions of systems of initial value problems in ordinary differential equations over the entire interval of integration. The analytic discussion is confirmed by numerical examples.

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1. Introduction

Generally, Runge-Kutta methods have acceptable computational properties for a variety of problems in ordinary differential equations. For example, the family of the Gauss-Legendre methods, are self-adjoint, meaning that they provide the same solutions when integrating forward and backward, in time. Runge-Kutta methods compute the first derivatives, f several times per step. They have both advantages and disadvantages. They are stable and easy to adapt for variable stepsize and order. However, they have difficulties in achieving high accuracy at reasonable cost. The general form of the Runge-Kutta method with s-stage is defined by

$$Y_i = y_{n-1} + h\sum_{j=1}^{s} a_{ij} f(x_{n-1} + hc_j, Y_j), \qquad i = 1, 2, 3, \cdots, s \quad (1)$$

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$$y_n = y_{n-1} + h \sum_{i=1}^{s} b_i f(x_{n-1} + hc_j, Y_j), \tag{2}$$

where the quantities $Y_1, Y_2, Y_3, \cdots, Y_s$ are called internal stage values and y_n is the update at the n^{th} step, that is, the numerical approximation to the solution $y(x)$ at $x = x_n$. They are approximations to the solution values $y(x_{n-1}+c_i h)$ at the points $x_{n-1}+c_i h$. The integer s is the number of stages of the method. The c_i represents the position of the internal stages within one-step and h denotes the step size $x_n - x_{n-1}$, which is sometimes constant or varied during integration. Also a_{ij}, b_j and c_j are the constant coefficients which can be constructed so that y_n is a good approximation to the solution $y(x_n) = y(x_{n-1}+h)$. For convenience, the stage derivatives $f(x_{n-1}+c_j h, Y_j)$ are often written as F_j. For Runge-Kutta method to be consistent such that it will be suitable for solving differential equations the following consistency condition is required:

$$\sum_{j=1}^{s} b_j = 1. \tag{3}$$

Further, another condition,

$$\sum_{i=1}^{s} a_{ij} = c_j, \qquad j = 1, 2, 3, \cdots, s \tag{4}$$

is necessary to guarantee that the correct value is obtained at each of the stages. Runge-Kutta method can be characterized using the $s \times s$ matrix A and $s \times 1$ vectors b and c,

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1s} \\ \vdots & \ddots & \vdots \\ a_{s1} & \cdots & a_{ss} \end{bmatrix}, \qquad b = \begin{bmatrix} b_1 \\ \vdots \\ b_s \end{bmatrix}, \qquad c = \begin{bmatrix} c_1 \\ \vdots \\ c_s \end{bmatrix}.$$

The Runge-Kutta method can be conveniently represented by the following Butcher Tableau,

$$\begin{array}{c|c} \text{c} & \text{A} \\ \hline & b^T \end{array}$$

where $A = \{a_{ij}\}, b^T = \{b_i\}$ and $c = \{c_i\}$. The set of numbers a_{ij} are coefficients used to find the internal stages using linear combinations of the stage derivatives. The components of the vector b^T are coefficients which represent how the numerical solution at this step depends on the derivatives of the internal stages. The vector $c = [c_1, c_2, c_3, \cdots, c_s]$ is called the abscissae. If the matrix A in the Butcher's Tableau, is strictly lower triangular, that is, the internal stages can be calculated without depending on later stages, then the method is called an explicit Runge-Kutta method. Otherwise the internal stages depend not only on the previous stages but also on the current stage and later stages, and then the method is called implicit Runge-Kutta method. Implicit Runge-Kutta method involves Newton's iterations to evaluate the stage values. The implicit methods have an algebraic nicety not possessed by the explicit methods in that the set of implicit methods under a very natural operation is homomorphic to certain group whereas the subset corresponding to explicit methods is only a semigroup[1].

2. Derivation of the STRK Collocation Methods

We describe the construction of the Symmetric Two-Step Runge-Kutta (STRK) Collocation Methods by considering the following form of a multistep collocation approach of [11] which was an extension of [10] of the form

$$y(x) = \sum_{j=0}^{t-1} \phi_j(x) y_{n+j} + h \sum_{j=0}^{m-1} \psi_j(x) f(\overline{x}_j, y(\overline{x}_j)) \tag{5}$$

where t denotes the number of interpolation points $x_{n+j}, j = 0, 1, 2$ $, ..., t-1$ taken from $\{x_n, x_{n+1}\}$; and m denotes the distinct collocation points $\overline{x}_j, j = 0, 1, 2, ..., m-1$ chosen from the interval (x_n, x_{n+1}). Here $\phi_j(x)$ and $\psi_j(x)$ are parameters of the methods which are to be determined. They are assumed polynomials of the form

$$\phi_j(x) = \sum_{i=0}^{t+m-1} \phi_{j,i+1} x^i, \qquad h\psi_j(x) = \sum_{i=0}^{t+m-1} h\psi_{j,i+1} x^i. \tag{6}$$

The order of the collocation method(5)-(6) is given by $p = t+m$-*1.* The proof of this result follows from [10]. When a collocation point $\overline{x}_j \in (x_n, x_{n+1})$ is a Gaussian point then it is counted twice for the order of the method. Thus, if all the m collocation points are Gaussian points we have that $p = t+2m\text{-}1$, which gives superconvergence method of [10], see also [12, 14, 15, 16]. The numerical constant coefficients $\phi_{j,i+1}$ and $\psi_{j,i+1}$ in (6) are to be determined. They are selected so that accurate approximations of well behaved problems are obtained efficiently.

From (5) and (6) following [14] we have,

$$\begin{aligned} y(x) &= \sum_{j=0}^{t-1} \sum_{i=0}^{t+m-1} \phi_{j,i+1} x^i y_{n+j} + h \sum_{j=0}^{m-1} \sum_{i=0}^{t+m-1} h\psi_{j,i+1} x^i f(\overline{x}_j, y(\overline{x}_j)) \\ &= \sum_{i=0}^{t+m-1} \left\{ \sum_{j=0}^{t-1} \phi_{j,i+1} y_{n+j} + \sum_{j=0}^{m-1} h\psi_{j,i+1} f(\overline{x}_j, y(\overline{x}_j)) \right\} x^i \\ &= \sum_{i=0}^{t+m-1} a_i x^i \end{aligned} \tag{7}$$

where

$$a_i = \left\{ \sum_{j=0}^{t-1} \phi_{j,i+1} y_{n+j} + \sum_{j=0}^{m-1} h\psi_{j,i+1} f(\overline{x}_j, y(\overline{x}_j)) \right\}.$$

From (7) we have

$$\begin{aligned} y(x) = \Bigg\{ & \sum_{j=0}^{t-1} \phi_{j,1} y_{n+j} + \sum_{j=0}^{m-1} h\psi_{j,1} f_{n+j}, \\ & \sum_{j=0}^{t-1} \phi_{j,2} y_{n+j} + \sum_{j=0}^{m-1} h\psi_{j,2} f_{n+j}, \\ & \quad \vdots \qquad\qquad\qquad \vdots, \\ & \sum_{j=0}^{t-1} \phi_{j,t+m-1} y_{n+j} + \sum_{j=0}^{m-1} h\psi_{j,t+m-1} f_{n+j} \Bigg\} \left(1, x, ., x^{t+m-1}\right)^T. \end{aligned} \tag{8}$$

Thus, expanding (8) fully, we have the propose continuous scheme as,

$$y(x) = (y_n, \cdots, y_{n+t-1}, f_n, \cdots, f_{n+m-1}) C^T \left(1, x, \cdots, x^{t+m-1}\right)^T \tag{9}$$

where T denotes transpose of the matrix C and the vector $(1, x, \cdots, x^{t+m-1})$.

Thus,

$$C = \begin{pmatrix} \phi_{0,1} & \cdots & \phi_{t-1,1} & h\psi_{0,1} & \cdots & h\psi_{m-1,1} \\ \phi_{0,2} & \cdots & \phi_{t-1,2} & h\psi_{0,2} & \cdots & h\psi_{m-1,2} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \phi_{0,t} & \cdots & \phi_{t-1,t} & h\psi_{0,t} & \cdots & h\psi_{m-1,t} \\ \phi_{0,t+1} & \cdots & \phi_{t-1,t+1} & h\psi_{0,t+1} & \cdots & h\psi_{m-1,t+1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \phi_{0,t+m} & \cdots & \phi_{t-1,t+m} & h\psi_{0,t+m} & \cdots & h\psi_{m-1,t+m} \end{pmatrix} \equiv D^{-1} \tag{10}$$

and

$$D = \begin{pmatrix} 1 & x_n & x_n^2 & \cdots & x_n^{t+m-1} \\ 1 & x_{n+1} & x_{n+1}^2 & \cdots & x_{n+1}^{t+m-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n+t-1} & x_{n+t-1}^2 & \cdots & x_{n+t-1}^{t+m-1} \\ 0 & 1 & 2\overline{x}_0 & \cdots & (t+m-1)\overline{x}_0^{t+m-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & 2\overline{x}_{n+m} & \cdots & (t+m-1)\overline{x}_{n+m}^{t+m-2} \end{pmatrix}. \tag{11}$$

The matrices D and C are both of dimensions $(t+m) \times (t+m)$. We call D the multistep collocation and interpolation matrix which has a very simple structure. From (10), the columns of C, which give the continuous coefficients $\phi_j(x)$ and $\psi_j(x)$ can be obtain from the corresponding columns of D^{-1}. As can be seen the entries of C are the constant coefficients of the polynomial given in (5) which are to be determined. The matrix C is the solution vector (output)and D is termed the Data (input), which is assumed to be non-singular for the existence of the inverse matrix C. An efficient algorithm for obtaining the elements of the inverse matrix C is found in [11]. We now derive the continuous formulation of the symmetric two-step Runge-Kutta collocation methods following the derivation techniques discussed in this section.

Definition 1.1: *The method in (5) has the order of consistency $p \geq 1$ provided that there exists an error constant C_{p+1} such that the local truncation error (LTE) satisfies*

$$\| LTE \| = C_{p+1} h^{p+1} + \bigcirc(h^{p+2}),$$

where $\| \,.\, \|$ *is the maximum norm.*

3. Symmetric Two-Step Runge-Kutta Collocation Method of Order Six

To obtain better stability and greater flexibility of the Runge-Kutta methods, we consider the construction of the methods based on the nodal points of Chebyshev polynomials .These Chebyshev polynomials were chosen because of their superior convergence rate and stiffly accurate characteristic properties in relation to the approximation of functions [7]. For the symmetric two-step Runge-Kutta collocation methods in this paper, we consider specific methods with high order of accuracy for the numerical integration of systems of initial value problems of the form,

$$\begin{cases} \frac{dy}{dx} = f(x, y), \quad (a \leq x \leq b), \\ y(0) = y(x_0). \end{cases} \tag{12}$$

Here the unknown y is a mapping $[a, b] \to R^d$, the right-hand side function f maps $[a, b] \times R^d \to R^d$ and the initial vector y_0 is given in R^d. It is assumed that f satisfies some conditions that guarantee the existence and uniqueness of a solution $y(x) \in C^1[a, b]$ as an Initial Value Problem (IVP) or as a Boundary Value Problem (BVP). For the solution of equation (12) we shall derive single symmetric continuous scheme for accurate solution of the problem everywhere in the interval of integration, based on Chebyshev nodal points. We consider for the first method the case $m = 5, t = 1$ and $\xi = (x - x_n)$ and we let the collocation points m be $x_n, x_{n+u}, x_{n+1}, x_{n+v}$ and x_{n+2} in (5) where u and v are zeros of $p_2(x) = 0$ Chebyshev polynomial of degree *2* in the standard symmetric interval [-1, 1], which were transformed into the interval $[x_n, x_{n+2}]$ by means of the following linear transformation $x \in [-1, 1] \to [x_n, x_{n+2}]$. Hence we have,

$$\left.\begin{aligned} \overline{x}_0 = x_{n+u}, u = \left(\frac{2 - \sqrt{2}}{2}\right) \\ \overline{x}_1 = x_{n+v}, v = \left(\frac{2 + \sqrt{2}}{2}\right) \end{aligned}\right\} \tag{13}$$

and are valid in the interval $[x_n, x_{n+2}]$. The proposed continuous scheme (9) takes the following form,

$$\begin{aligned} y(x) &= \phi_0(x)y_n + [\psi_0(x)f_n + \psi_1(x)f_{n+u} + \psi_2(x)f_{n+1} \\ &\quad +\psi_3(x)f_{n+v} + \psi_4(x)f_{n+2}], \end{aligned} \tag{14}$$

where

$$\phi_0(x) = -1,$$

$$\begin{aligned} \psi_0(x) &= [(408 - 288\sqrt{2})\xi^5 - (2550 - 1800\sqrt{2})h\xi^4 \\ &\quad +(5780 - 4080\sqrt{2})h^2\xi^3 - (5610 - 3960\sqrt{2})h^3\xi^2 \\ &\quad +(2040 - 1440\sqrt{2})h^4\xi]/(h^4(2040 - 1440\sqrt{2}) \end{aligned}$$

$$\begin{aligned} \psi_1(x) &= [-(96 - 72\sqrt{2})\xi^5 + (390 - 300\sqrt{2})h\xi^4 - (440 - \\ &\quad 360\sqrt{2})h^2\xi^3 + (120 - 120\sqrt{2})h^3\xi^2]/(h^4(240 - 180\sqrt{2}) \end{aligned}$$

$$\psi_2(x) = [24\xi^5 - 120h\xi^4 + 180h^2\xi^3 - 60h^3\xi^2]/60h^4$$

$$\begin{aligned} \psi_3(x) &= [-(96 - 72\sqrt{2})\xi^5 + (570 - 420\sqrt{2})h\xi^4 - (1160 \\ &\quad -840\sqrt{2})h^2\xi^3 + (840 - 600\sqrt{2})h^3\xi^2]/[h^4(240 \\ &\quad -180\sqrt{2})] \end{aligned}$$

$$\psi_4(x) = [12\xi^5 - 45h\xi^4 + 50h^2\xi^3 - 15h^3\xi^2]/60h^4$$

Evaluating the continuous scheme $y(x)$ in (14) at the points $x = x_{n+2}, x_{n+u}, x_{n+1}$ and x_{n+v} to obtain symmetric two-step block hybrid scheme,

$$y_{n+2} = y_n + \frac{h}{15}[f_n + 8f_{n+u} + 12f_{n+1} + 8f_{n+v} + f_{n+2}], \tag{15}$$

$$\begin{aligned} y_{n+u} &= y_n + \frac{h}{240}[(23 + 4\sqrt{2})f_n + (64 - 13\sqrt{2})f_{n+u} + (96 \\ &\quad -72\sqrt{2})f_{n+1} + (64 - 43\sqrt{2})f_{n+v} + (-7 + 4\sqrt{2})f_{n+2}], \end{aligned}$$

$$\begin{aligned} y_{n+1} &= y_n + \frac{h}{60}[2f_n + (16 + 15\sqrt{2})f_{n+u} + 24f_{n+1} + (16 \\ &\quad -15\sqrt{2})f_{n+v} + 2f_{n+2}], \end{aligned}$$

$$\begin{aligned} y_{n+v} &= y_n + \frac{h}{240}[(23 - 4\sqrt{2})f_n + (64 + 43\sqrt{2})f_{n+u} + (96 \\ &\quad +72\sqrt{2})f_{n+1} + (64 + 13\sqrt{2})f_{n+v} - (7 + 4\sqrt{2})f_{n+2}]. \end{aligned}$$

Converting the block hybrid scheme (15), into symmetric two-step Runge-Kutta collocation method,

$$\begin{aligned} y_n &= y_{n-2} + h\left(\frac{1}{15}\right)F_1 + h\left(\frac{8}{15}\right)F_2 \\ &\quad + h\left(\frac{12}{15}\right)F_3 + h\left(\frac{8}{15}\right)F_4 + h\left(\frac{1}{15}\right)F_5 \end{aligned} \tag{16}$$

where the stage values at the n^{th} step are computed as,

$$Y_1 = y_{n-2},$$

$$\begin{aligned} Y_2 &= y_{n-2} + h\left(\frac{23}{240} + \frac{\sqrt{2}}{60}\right)F_1 + h\left(\frac{4}{15} - \frac{13\sqrt{2}}{240}\right)F_2 \\ &\quad + h\left(\frac{2}{5} - \frac{3\sqrt{2}}{10}\right)F_3 + h\left(\frac{4}{15} - \frac{43\sqrt{2}}{240}\right)F_4 - h\left(\frac{7}{240} - \frac{\sqrt{2}}{60}\right)F_5, \end{aligned}$$

$$\begin{aligned} Y_3 &= y_{n-2} + h\left(\frac{1}{30}\right)F_1 + h\left(\frac{4}{15} + \frac{\sqrt{2}}{4}\right)F_2 \\ &\quad + h\left(\frac{2}{5}\right)F_3 + h\left(\frac{4}{15} - \frac{\sqrt{2}}{4}\right)F_4 + h\left(\frac{1}{30}\right)F_5, \end{aligned}$$

$$\begin{aligned} Y_4 &= y_{n-2} + h\left(\frac{23}{240} - \frac{\sqrt{2}}{60}\right)F_1 + h\left(\frac{4}{15} + \frac{43\sqrt{2}}{240}\right)F_2 \\ &\quad + h\left(\frac{2}{5} + \frac{3\sqrt{2}}{10}\right)F_3 + h\left(\frac{4}{15} + \frac{13\sqrt{2}}{240}\right)F_4 - h\left(\frac{7}{240} + \frac{\sqrt{2}}{60}\right)F_5, \end{aligned}$$

$$\begin{aligned} Y_5 &= y_{n-2} + h\left(\frac{1}{15}\right)F_1 + h\left(\frac{8}{15}\right)F_2 + h\left(\frac{4}{5}\right)F_3 \\ &\quad + h\left(\frac{8}{15}\right)F_4 + h\left(\frac{1}{15}\right)F_5, \end{aligned}$$

with the stage derivatives as follows:

$$\begin{aligned} F_1 &= f\left(x_{n-2} + h(0), Y_1\right), \\ F_2 &= f\left(x_{n-2} + h\left(\frac{2-\sqrt{2}}{2}\right), Y_2\right), \\ F_3 &= f\left(x_{n-2} + h(1), Y_3\right), \\ F_4 &= f\left(x_{n-2} + h\left(\frac{2+\sqrt{2}}{2}\right), Y_4\right), \\ F_5 &= f(x_{n-2} + h(2), Y_5). \end{aligned}$$

The symmetric two-step Runge-Kutta collocation method (16) has order $p = 6$ and error constant:

$$C_7 = \frac{1}{37800} h^7 y^{(7)}(x_0).$$

4. Symmetric Two-Step Runge–Kutta Collocation Method of Order Eight

For the higher order method we consider the proposed continuous scheme (9) where u, q and v are zeros of Chebyshev polynomial of the form $p_3(x) = 0$ obtained in the same manner as in (13), given as,

$$\left.\begin{aligned} \overline{x}_0 &= x_{n+u}, u = \left(\frac{2-\sqrt{3}}{2}\right) \\ \overline{x}_1 &= x_{n+q}, q = 1 \\ \overline{x}_2 &= x_{n+v}, v = \left(\frac{2+\sqrt{3}}{2}\right) \end{aligned}\right\} \tag{17}$$

and are valid in the interval $[x_n, x_{n+2}]$. With $w = \frac{1}{2}$ and $r = \frac{3}{2}$, we obtain the following symmetric continuous scheme,

$$\begin{aligned} y(x) = {} & \phi_0(x) y_n + [\psi_0(x) f_n + \psi_1(x) f_{n+u} + \psi_2(x) f_{n+w} \\ & + \psi_3(x) f_{n+1} + \psi_4(x) f_{n+r} + \psi_5(x) f_{n+v} \\ & + \psi_6(x) f_{n+2}] \end{aligned} \tag{18}$$

where

$$\phi_0(x) = -1,$$

$$\begin{aligned} \psi_0(x) = {} & [(119016000 - 68713920\sqrt{3})\xi^7 \\ & -(971964000 - 561163680\sqrt{3})h\xi^6 \\ & +(3165825600 - 1827790272\sqrt{3})h^2\xi^5 \\ & +(5206950000 - 3006234000\sqrt{3})h^3\xi^4 \\ & +(4495333500 - 2595382020\sqrt{3})h^4\xi^3 \\ & -(1900536750 - 1097275410\sqrt{3})h^5\xi^2 \\ & +(312417000 - 180374040\sqrt{3})h^6\xi]/ \\ & 1260(247950 - 143154\sqrt{3})h^6 \end{aligned}$$

$$\begin{aligned}\psi_1(x) &= [-(2085120-1203840\sqrt{3})\xi^7\\ &\quad +(12489120-7210560\sqrt{3})h\xi^6\\ &\quad -(27498240-15876000\sqrt{3})h^2\xi^5\\ &\quad +(27083700-15636600\sqrt{3})h^3\xi^4\\ &\quad -(11371920-6565440\sqrt{3})h^4\xi^3\\ &\quad +(1466640-846720\sqrt{3})h^5\xi^2]/\\ &\quad 1260(2172-1254\sqrt{3})h^6\end{aligned}$$

$$\begin{aligned}\psi_2(x) &= [960\xi^7-7280h\xi^6+21168h^2\xi^5\\ &\quad -28770h^3\xi^4+17080h^4\xi^3-2520h^5\xi^2]/1260h^6\end{aligned}$$

$$\begin{aligned}\psi_3(x) &= [-960\xi^7+6720h\xi^6-17472h^2\xi^5\\ &\quad +20160h^3\xi^4-9380h^4\xi^3+1260h^5\xi^2]/1260h^6\end{aligned}$$

$$\begin{aligned}\psi_4(x) &= [960\xi^7-6160h\xi^6+14448h^2\xi^5\\ &\quad -14910h^3\xi^4+6440h^4\xi^3-840h^5\xi^2]/1260h^6\end{aligned}$$

$$\begin{aligned}\psi_5(x) &= [-(2085120-1203840\sqrt{3})\xi^7\\ &\quad +(16702560-9643200\sqrt{3})h\xi^6\\ &\quad -(52778880-30471840\sqrt{3})h^2\xi^5\\ &\quad +(82385100-47565000\sqrt{3})h^3\xi^4\\ &\quad -(64039920-36973440\sqrt{3})h^4\xi^3\\ &\quad +(20427120-11793600\sqrt{3})h^5\xi^2]\\ &\quad /1260(2172-1254\sqrt{3})h^6\end{aligned}$$

$$\begin{aligned}\psi_6(x) &= [480\xi^7-2800h\xi^6+6048h^2\xi^5\\ &\quad -5880h^3\xi^4+2450h^4\xi^3-315h^5\xi^2]/1260h^6.\end{aligned}$$

We evaluate the continuous scheme *y(x)* in (18) at the points $x = x_{n+2}$, x_{n+u}, x_{n+w}, x_{n+1}, x_{n+r} and x_{n+v} to obtain the symmetric block hybrid scheme,

$$\begin{aligned}y_{n+2} &= y_n+h[9f_n+80f_{n+u}+144f_{n+w}+164f_{n+1}+144f_{n+r}\\ &\quad +80f_{n+v}+9f_{n+2}]/315 \qquad (19)\end{aligned}$$

$$\begin{aligned} y_{n+u} &= y_n + h[(424-72\sqrt{3})f_n + (1280-235\sqrt{3})f_{n+u} \\ &+(2269-1368\sqrt{3})f_{n+w} + (2624-1488\sqrt{3})f_{n+1} \\ &+(2339-1368\sqrt{3})f_{n+r} + (1280-725\sqrt{3})f_{n+v} \\ &-(136-72\sqrt{3})f_{n+2}]/10080 \end{aligned}$$

$$\begin{aligned} y_{n+w} &= y_n + h[504f_n + (10360+6615\sqrt{3})f_{n+u} + 14679f_{n+w} \\ &-2576f_{n+1} + 1449f_{n+r} + (10360-6615\sqrt{3})f_{n+v} \\ &+504f_{n+2}]/70560 \end{aligned}$$

$$\begin{aligned} y_{n+1} &= y_n + h[53f_n + (160+70\sqrt{3})f_{n+u} + 638f_{n+w} + 328f_{n+1} \\ &-62f_{n+r} + (160-70\sqrt{3})f_{n+v} - 17f_{n+2}]/1260 \end{aligned}$$

$$\begin{aligned} y_{n+r} &= y_n + h[24f_n + (120+105\sqrt{3})f_{n+u} + 489f_{n+w} \\ &+624f_{n+1} + 279f_{n+r} + (120-105\sqrt{3})f_{n+v} \\ &+24f_{n+2}]/1120 \end{aligned}$$

$$\begin{aligned} y_{n+v} &= y_n + h[(424-72\sqrt{3})f_n + (1280+725\sqrt{3})f_{n+u} + (2269 \\ &+1368\sqrt{3})f_{n+w} + (2624+1488\sqrt{3})f_{n+1} + (2339 \\ &+1368\sqrt{3})f_{n+r} + (1280+235\sqrt{3})f_{n+v} - (136 \\ &+72\sqrt{3})f_{n+2}]/10080. \end{aligned}$$

As usual, converting the block hybrid scheme to symmetric two-step Runge-Kutta collocation method:

$$\begin{aligned} y_n &= y_{n-2} + h\left(\frac{9}{315}\right)F_1 + h\left(\frac{80}{315}\right)F_2 + h\left(\frac{144}{315}\right)F_3 + h\left(\frac{164}{315}\right)F_4 \\ &+h\left(\frac{144}{315}\right)F_5 + h\left(\frac{80}{315}\right)F_6 + h\left(\frac{9}{315}\right)F_7. \end{aligned} \tag{20}$$

The stage values at the n^{th} step are calculated as,

$$Y_1 = y_{n-2},$$

$$\begin{aligned} Y_2 &= y_{n-2} + h\left(\frac{53}{1260} - \frac{\sqrt{3}}{140}\right)F_1 + h\left(\frac{8}{63} - \frac{47\sqrt{3}}{2016}\right)F_2 + h\left(\frac{2269}{10080} - \frac{19\sqrt{3}}{140}\right)F_3 \\ &+h\left(\frac{82}{315} - \frac{31\sqrt{3}}{210}\right)F_4 + h\left(\frac{2339}{10080} - \frac{19\sqrt{3}}{140}\right)F_5 + h\left(\frac{8}{63} - \frac{145\sqrt{3}}{2016}\right)F_6 \\ &-h\left(\frac{17}{1260} - \frac{\sqrt{3}}{140}\right)F_7, \end{aligned}$$

$$\begin{aligned}
Y_3 &= y_{n-2} + h\left(\frac{1}{140}\right)F_1 + h\left(\frac{259}{1764} - \frac{147\sqrt{3}}{1568}\right)F_2 + h\left(\frac{233}{1120}\right)F_3 \\
&\quad -h\left(\frac{23}{630}\right)F_4 + h\left(\frac{23}{1120}\right)F_5 + h\left(\frac{259}{1764} - \frac{147\sqrt{3}}{1568}\right)F_6 + h\left(\frac{1}{140}\right)F_7,
\end{aligned}$$

$$\begin{aligned}
Y_4 &= y_{n-2} + h\left(\frac{53}{1260}\right)F_1 + h\left(\frac{8}{63} - \frac{\sqrt{3}}{18}\right)F_2 + h\left(\frac{319}{630}\right)F_3 + h\left(\frac{82}{315}\right)F_4 \\
&\quad -h\left(\frac{31}{630}\right)F_5 + h\left(\frac{8}{63} - \frac{\sqrt{3}}{18}\right)F_6 - h\left(\frac{17}{1260}\right)F_7,
\end{aligned}$$

$$\begin{aligned}
Y_5 &= y_{n-2} + h\left(\frac{3}{140}\right)F_1 + h\left(\frac{3}{28} + \frac{3\sqrt{3}}{32}\right)F_2 + h\left(\frac{489}{1120}\right)F_3 + h\left(\frac{39}{70}\right)F_4 \\
&\quad +h\left(\frac{279}{1120}\right)F_5 + h\left(\frac{3}{28} - \frac{3\sqrt{3}}{32}\right)F_6 + h\left(\frac{3}{140}\right)F_7,
\end{aligned}$$

$$\begin{aligned}
Y_6 &= y_{n-2} + h\left(\frac{53}{1260} - \frac{\sqrt{3}}{140}\right)F_1 + h\left(\frac{8}{63} + \frac{145\sqrt{3}}{2016}\right)F_2 + h\left(\frac{2269}{10080}\right. \\
&\quad \left.+\frac{19\sqrt{3}}{140}\right)F_3 + h\left(\frac{82}{315} + \frac{31\sqrt{3}}{210}\right)F_4 + h\left(\frac{2339}{10080} + \frac{19\sqrt{3}}{140}\right)F_5 \\
&\quad +h\left(\frac{8}{63} + \frac{47\sqrt{3}}{2016}\right)F_6 - h\left(\frac{17}{1260} + \frac{\sqrt{3}}{140}\right)F_7,
\end{aligned}$$

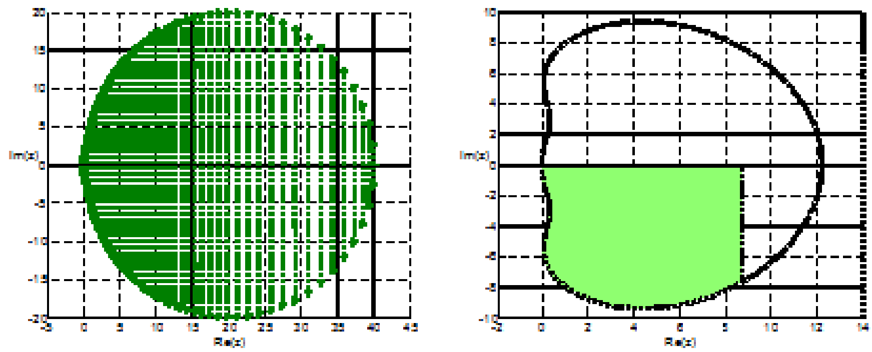
$$\begin{aligned}
Y_7 &= y_{n-2} + h\left(\frac{1}{35}\right)F_1 + h\left(\frac{16}{63}\right)F_2 + h\left(\frac{16}{35}\right)F_3 + h\left(\frac{164}{315}\right)F_4 + h\left(\frac{16}{35}\right)F_5 \\
&\quad +h\left(\frac{16}{63}\right)F_6 + h\left(\frac{1}{35}\right)F_7,
\end{aligned}$$

with the stage derivatives as follows,

$$\begin{aligned}
F_1 &= f\left(x_{n-2} + h(0), Y_1\right), \\
F_2 &= f\left(x_{n-2} + h\left(\frac{2-\sqrt{3}}{2}\right), Y_2\right), \\
F_3 &= f\left(x_{n-2} + h\left(\frac{1}{2}\right), Y_3\right), \\
F_4 &= f\left(x_{n-2} + h(1), Y_4\right), \\
F_5 &= f\left(x_{n-2} + h\left(\frac{3}{2}\right), Y_5\right), \\
F_6 &= f\left(x_{n-2} + h\left(\frac{2+\sqrt{3}}{2}\right), Y_6\right), \\
F_7 &= f(x_{n-2} + h(2), Y_7).
\end{aligned}$$

The order of the method is $p = 8$ and error constant:

$$C_9 = \frac{1}{50803200} h^9 y^{(9)}(x_0).$$



Method (16) **Method (20)**

Fig. 1. Regions of Absolute Stability of the STRK Collocation Methods

We plotted the regions of absolute stability of the symmetric two-step Runge-Kutta collocation methods using the method used in [4] as shown in Figure 1. **Remark 1.1**
It is obvious from equations (16) and (20) that the methods derived using Chebyshev nodal-points have smaller error constants compared to the one in [8] of the same order of convergence. Further, the coefficients of the methods obtained are convenient to use (in programs) and analyze.

5. Numerical Experiments

We have applied the newly derived methods in sections 3 and 4 of the paper to a variety of well-known problems which have appeared in the literature. All the numerical experiments were done in Matlab and carried out at the Mathematical Sciences Research and Development Laboratory, Abubakar Tafawa Balewa University, Bauchi, Nigeria. The test examples are systems of ordinary differential equations written as first order initial value problems. We plotted these solutions and compared the graphs with the graphs of the exact solutions and graphs from some existing methods of the same order and similar derivation, for example, the Block Adams-Moulton's Methods (BAMMs) of [8] and [13]. We also compared the computed solutions side by side in Tables with the solutions from some existing methods [8]and [13]. In each of the computation the number of function evaluations *(nfe)* is indicated.

Example 1: For the first test example, we consider a well-known classical system,

$$\begin{aligned} y_1' &= 998y_1 + 1998y_2, \qquad y_1(0) = 1, \\ y_2' &= -999y_1 - 1999y_2, \qquad y_2(0) = 1. \end{aligned}$$

It is a mildly stiff problem composed of two first order equations and the exact solutions given by the sum of two decaying exponential components,

$$\begin{cases} y_1(x) = 4e^{-x} - 3e^{-1000x}, \\ y_2(x) = -2e^{-x} + 3e^{-1000x}. \end{cases}$$

The stiffness ratio is R = 1000 and the problem is solved numerically on the interval [0,100].

Table 1. Absolute errors of numerical solutions of example 1

Mesh values	BAMM[13] y_1 *component*	Method (16) y_1 *component*	BAMM[8] y_1 *component*	Method (20) y_1 *component*
10.0	2.8418×10^{-1}	1.0999×10^{-1}	1.0532×10^{-1}	4.8945×10^{-2}
20.0	1.4474×10^{-2}	1.8266×10^{-3}	1.8513×10^{-3}	1.0130×10^{-5}
30.0	7.5062×10^{-4}	3.1753×10^{-5}	3.2542×10^{-5}	2.0969×10^{-7}
40.0	3.8937×10^{-5}	5.5334×10^{-7}	5.7202×10^{-7}	4.3403×10^{-9}
50.0	2.0198×10^{-6}	9.6439×10^{-9}	1.0054×10^{-8}	8.9836×10^{-11}
60.0	1.0477×10^{-7}	1.6808×10^{-10}	1.7674×10^{-10}	1.8594×10^{-14}
70.0	5.4353×10^{-9}	2.9294×10^{-12}	3.1067×10^{-12}	3.8487×10^{-16}
80.0	2.8195×10^{-10}	5.1056×10^{-14}	5.4609×10^{-14}	7.9663×10^{-18}
90.0	1.4626×10^{-11}	3.9588×10^{-16}	9.5991×10^{-16}	1.6488×10^{-21}
100.0	7.5871×10^{-13}	1.5508×10^{-17}	1.6873×10^{-17}	3.4129×10^{-23}

Example 2:

$$\begin{aligned} y_1' &= 3y_1 - 0.002y_1y_2, \qquad y_1(0) = 1000 \\ y_2' &= 0.0006y_1y_2 - 0.5y_2, \qquad y_2(0) = 200. \end{aligned}$$

In the second example we consider a real life problem of mathematical models for predicting the population dynamics of competing species [3]. The problem is concerned with two species competing for food supply. The number of the species alive at time t are denoted by $y_1(x)$ and $y_2(x)$. It is often assumed that, while the birth rate of each of the species is simply proportional to the number of the species alive at that time, the death rate of each species depends upon the population of both species. In Figure 3, we displayed the graphical outputs of the problem.

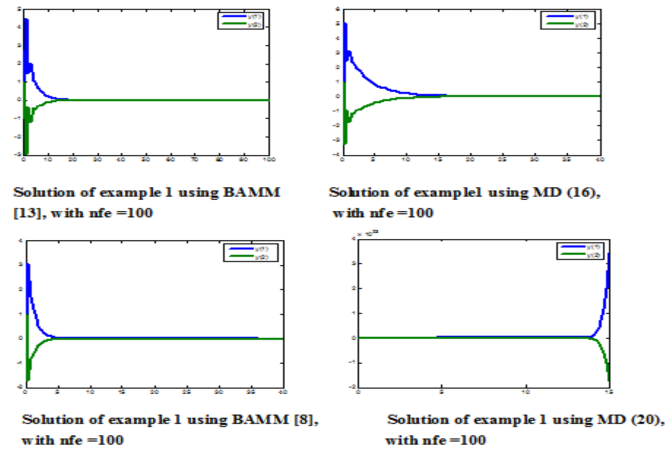


Fig. 2. Graphical Plots of Example 1 Using BAMMs and STRK Collocation Methods

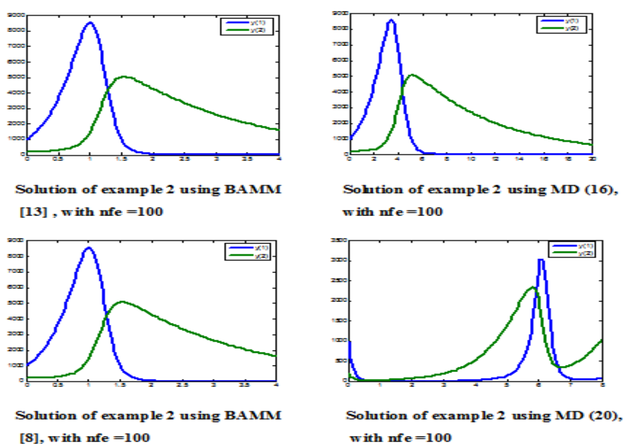


Fig. 3. Graphical Plots of Example 2 Using BAMMs and STRK Collocation Methods

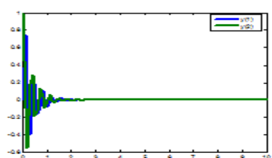
Example 3:

$$y_1' = (-1 - y_2^2)y_1 + 20y_2, \qquad y_1(0) = 0$$
$$y_2' = (-1 - y_1^2)y_2 - 20y_1, \qquad y_2(0) = 1.$$

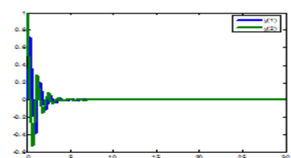
This example is solved using the newly derived methods and the block Adams-Moulton methods of the same order of convergence, we found out that the solutions have highly oscillatory component and asymptotic values of zeros [9].

Table 2. Absolute errors of numerical solutions of example 3

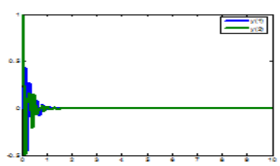
Mesh values	BAMM[13] y_1 *component*	Method (16) y_1 *component*	BAMM[8] y_1 *component*	Method (20) y_1 *component*
10.0	1.7570×10^{-2}	6.2137×10^{-2}	1.1713×10^{-2}	2.9821×10^{-3}
20.0	7.1850×10^{-3}	2.9715×10^{-3}	2.1916×10^{-4}	8.2112×10^{-5}
30.0	8.6465×10^{-5}	1.4011×10^{-4}	3.1866×10^{-6}	1.9070×10^{-7}
40.0	4.2600×10^{-5}	6.5083×10^{-6}	9.9200×10^{-9}	3.3966×10^{-9}
50.0	4.0594×10^{-7}	2.9741×10^{-7}	1.8300×10^{-9}	2.3927×10^{-11}
60.0	2.5232×10^{-7}	1.3344×10^{-8}	1.0659×10^{-10}	7.3506×10^{-14}
70.0	1.7779×10^{-9}	5.8618×10^{-10}	4.4197×10^{-12}	1.5290×10^{-15}
80.0	1.4929×10^{-9}	2.5094×10^{-11}	1.6042×10^{-13}	2.6332×10^{-18}
90.0	6.8384×10^{-12}	1.0393×10^{-12}	5.4139×10^{-15}	6.7082×10^{-20}
100.0	8.8245×10^{-12}	4.1147×10^{-14}	1.7432×10^{-16}	4.1469×10^{-22}



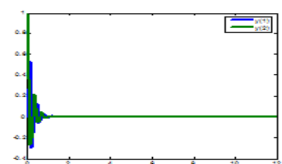
Solution of example 3 using BAMM [13], with nfe =100



Solution of example 3 using MD (16), with nfe =100



Solution of example 3 using BAMM [8], with nfe =100



Solution of example 3 using MD (20), with nfe =100

Fig. 4. Graphical Plots of Example 3 Using BAMMs and STRK Collocation Methods

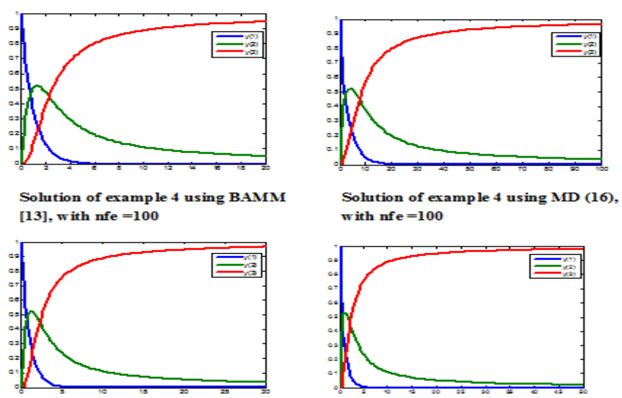
Example 4:

$$
\begin{aligned}
y_1' &= -y_1, & y_1(0) &= 1 \\
y_2' &= y_1 - y_2^2, & y_2(0) &= 0 \\
y_3' &= y_2^2, & y_3(0) &= 0.
\end{aligned}
$$

The fourth experiment displayed in Figure 5, compares the graphical outputs of the new methods and the block Adams Moulton's methods. The problem is a system of nonlinear chemical reaction equations. These results show that the newly derived methods are very promising and certainly have the potential to be competitive solvers of odes.

Table 3. Absolute errors of numerical solutions of example 4

Mesh values	BAMM[13] y_1 *component*	Method (16) y_1 *component*	BAMM[8] y_1 *component*	Method (20) y_1 *component*
10.0	1.6222×10^{-1}	6.9735×10^{-2}	6.5367×10^{-2}	1.0600×10^{-2}
20.0	2.1502×10^{-2}	3.6174×10^{-3}	3.1557×10^{-3}	6.7927×10^{-5}
30.0	2.8500×10^{-3}	1.8765×10^{-4}	1.5234×10^{-4}	4.3489×10^{-7}
40.0	3.7776×10^{-4}	9.7344×10^{-6}	7.3549×10^{-6}	2.7843×10^{-9}
50.0	5.0071×10^{-5}	5.0496×10^{-7}	3.5507×10^{-7}	1.7826×10^{-11}
60.0	6.6364×10^{-6}	2.6194×10^{-8}	1.7141×10^{-8}	1.1413×10^{-13}
70.0	8.7966×10^{-7}	1.3599×10^{-9}	8.2755×10^{-10}	7.3070×10^{-16}
80.0	1.1659×10^{-7}	7.0488×10^{-11}	3.9951×10^{-11}	4.6782×10^{-18}
90.0	1.5454×10^{-8}	3.6565×10^{-12}	1.9287×10^{-12}	2.9951×10^{-20}
100.0	2.0483×10^{-9}	1.8967×10^{-13}	9.3113×10^{-14}	1.9175×10^{-22}



Solution of example 4 using BAMM [13], with nfe =100

Solution of example 4 using MD (16), with nfe =100

Solution of example 4 using BAMM [8], with nfe =100

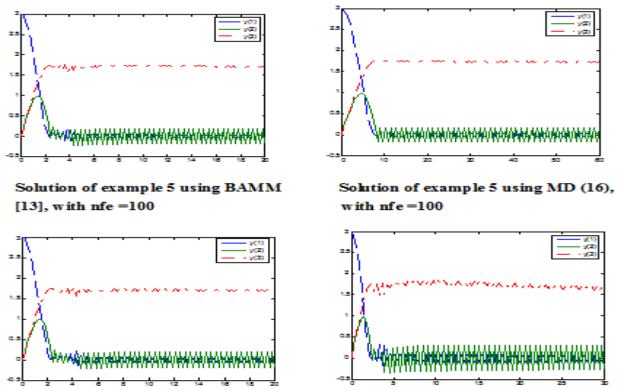
Solution of example 4 using MD (20), with nfe =100

Fig. 5. Graphical Plots of Example 4 Using BAMMs and STRK Collocation Methods

Example 5: The integral surface of a TORUS,

$$\begin{aligned} y_1' &= -y_2 - \frac{y_1 y_3}{\sqrt{y_1^2 + y_2^2}}, \qquad & y_1(0) = 3 \\ y_2' &= \frac{y_1 - y_2 y_3}{\sqrt{y_1^2 + y_2^2}}, \qquad & y_2(0) = 0 \\ y_3' &= \frac{y_1}{\sqrt{y_1^2 + y_2^2}}, \qquad & y_3(0) = 0. \end{aligned}$$

The fifth experiment is a very good standard test example of the integral surface of a **TORUS** which is one of the **DETEST** problem set of [6]. Graphs of the plots exhibit excellent computations of the newly derived methods displayed in Figure 6.



Solution of example 5 using BAMM [13], with nfe =100

Solution of example 5 using MD (16), with nfe =100

Solution of example 5 using BAMM [8], with nfe =100

Solution of example 5 using MD (20), with nfe =100

Fig. 6. Graphical Plots of Example 5 Using BAMMs and STRK Collocation Methods

Example 6: Almost periodic initial value problem.
We consider the almost periodic initial value problem which was earlier studied by some eminent authors, for example [5] and the references therein,

$$y'' + y' = 0.001e^{ix}, \qquad y(0) = 0,\ y'(0) = 0.9995i,\ y \in R^1 \qquad (a)$$

whose theoretical solution is

$$\begin{cases} y(x) = u(x) + iv(x), & u, v \in R^1, \\ u(x) = cosx + 0.0005xsinx, \\ v(x) = sinx - 0.0005xcosx. \end{cases} \qquad (b)$$

The initial value problem in (a) can be expressed as

$$\begin{cases} y_1' = -y_2, & y_1(0) = 1 \\ y_2' = -y_1 + 0.001cos(x), & y_2(0) = 0 \\ y_3' = y_4, & y_3(0) = 0 \\ y_4' = -y_3 + 0.001sin(x), & y_4(0) = 0.9995. \end{cases} \qquad (c)$$

We solved the system in (c) and all the solutions generated by our new methods trace the paths of the theoretical or exact solutions, see Figure 7.

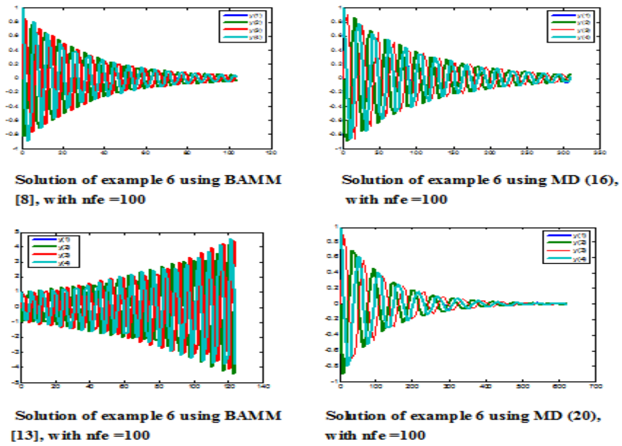


Fig. 7. Graphical Plots of Example 6 Using BAMMs and STRK Collocation Methods

6. Concluding Remarks

For the symmetric two-step Runge-Kutta collocation methods we have evaluated their performance on a set of challenging systems of first order initial value problems and compared their performance with some existing codes or theoretical solutions. The numerical results are quite satisfactory and suggest that these methods may have a useful role in the solution of systems of first order initial value problems in ordinary differential equations.

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