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INITIAL-BOUNDARY-VALUE PROBLEM OF HYPERBOLIC EQUATIONS FOR VISCOUS BLOOD FLOW THROUGH A TAPERED VESSEL

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ABSTRACT. In this paper, the effect of viscosity on blood flow through a tapered artery is studied. Approximate solutions of the coupled nonlinear partial differential equations that model the viscous blood a complaint artery are obtained using Adomian decomposition method (ADM). The convergence and parametric study of the solution are presented and discussed including shock development. The result of the computation shows that viscosity has significant influence on blood flow.

Keywords and phrases: Blood flow, ADM, shock waves, conservation law.
2010 Mathematical Subject Classification:

1. INTRODUCTION

Blood is a viscous suspension of cells that is responsible for the circulation of digested food and removal of waste product from the body. Mathematically speaking, it as an incompressible viscous fluid flowing through a cylindrical tube of varying cross-sectional area.

In recent times, various aspects of blood flow problem have been studied in literature based on the assumption that blood is an incompressible inviscid fluid flowing through pre-stressed thin elastic tube (see refs. [1] - [11]). In this article, attention is focused on a recent paper by Ruan et al [12] in which an initial-boundary-value problem of a system of hyperbolic, partial-differential equations that models blood flow in a vessel was investigated. Analysis was performed by neglecting the effect of blood viscosity and approximating the energy quantity. From application point of view, inviscid blood flow is only valid as a first approximation that cannot give useful information on blood profile. A more reliable result that is useful in many real life situations can be obtained if the effect of blood viscosity is accounted for.

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In many cases of interest, viscosity of blood plays a dominant role in the diagnosis and treatment of some diseases or physiological disorders like stenosis [13]-[15], Sickle cell disease [16] and many more. Several interesting results on the effect of viscosity on blood flows under different conditions can also be found in [17]-[19].

Motivated by the above discussions, the specific objective of this paper is to investigate the effect of this all-important flow property on blood flow through a compliant artery which has not been accounted for in previous model used in [12]. To achieve this, the nonlinear equations of motion for both the tube and the fluid are formulated and solved analytically by Adomain decomposition method [20-24].

The rest of the paper is organized as follows; in section 2 of the paper, the problem is formulated and approximate solution of the problem is obtained in section 3. Results are presented and discussed in the section 4 of the paper while section 5 gives some concluding remarks.

2. MATHEMATICAL ANALYSIS

The conservation laws that models blood flow through an artery can be written in dimensionless form as [12]. The continuity equation

$$\frac{\partial A}{\partial t} + \frac{\partial (Au)}{\partial x} = 0, \tag{1}$$

$$\frac{\partial (Au)}{\partial t} + \frac{\partial (\alpha A u^2)}{\partial x} + \frac{A}{\rho}\frac{\partial p}{\partial x} = 2R'\nu \left[\frac{\partial u_x}{\partial r}\right]_0^{R'}, \tag{2}$$

while the pressure-area relationship takes the form

$$p = \kappa(A^\beta(t,x) - a^\beta), \tag{3}$$

where $u_x = u_x(u, r, R')$, $A(t,x) = \pi R^2(t,x)$, $u(t,x)$,$p(t,x)$, are nonnegative functions, t is the time, x is the axial direction, $A(t,x)$ is the cross sectional area, a is the constant cross sectional area at the entrance of the tube, u is the average axial velocity, r is the radial direction, ρ, p,ν are fluid density, fluid pressure and kinematic viscosity respectively.R' is the radius of the cross section, β is a positive constant, u_x is the axial velocity, α is dimensionalized energy quantity, $\kappa = \frac{Eh}{\pi R_0^2}$ is the wall behavior parameter. E is the young modulus of elasticity, h is the arterial thickness, R_0 is the unstressed radius.

The average axial velocity is defined by

$$u = \frac{1}{R^2}\int_0^R 2ru_x dr, \tag{4}$$

while the dimensionalized energy quantity as

$$\alpha = \frac{1}{u^2R^2}\int_0^R 2r(u_x)^2 dr, \tag{5}$$

Following [25]-[27], in order to solve equations (1)-(3) a specific relation for u_x is needed and for laminar flow in slightly tapered vessel the velocity profile is flat. Therefore, we assume

$$u_x = u\left(1 - \frac{r}{R'}\right) \tag{6}$$

then the viscous term gives

$$\left[\frac{\partial u_x}{\partial r}\right]_0^{R'} = \left[\frac{\partial u}{\partial r} - \frac{u}{R'} - \frac{r}{R'}\frac{\partial u}{\partial r}\right]_0^{R'}, \tag{7}$$

Introducing the limits and applying the condition $u(0)$ and the axisymetric condition $\frac{\partial u(0)}{\partial r} = 0$, equation (7) reduces to

$$\left[\frac{\partial u_x}{\partial r}\right]_0^{R'} = -\frac{u}{R'}, \tag{8}$$

substituting (3) and (8) in (2), we obtain

$$\frac{\partial u}{\partial t} + (2\alpha - 1)u\frac{\partial u}{\partial x} + \frac{(\alpha-1)u^2}{A}\frac{\partial A}{\partial x} + \frac{\kappa}{\rho}\beta A^{\beta-1}\frac{\partial A}{\partial x} = -\frac{2\pi\mu u}{A\rho}, \tag{9}$$

subject to initial-boundary conditions

$$A(0,x) = \phi(x) = A(0)e^{-k_0x}, u(0,x) = u_0(x) = e^{\frac{-x^2}{2}}, p(t,0) = p_l, \tag{10}$$

where k_0 is the taper parameter, $A(0) = \pi R_0^2$ is the cross sectional area at the entrance of the vessel in cm^2 and $R_0 = \frac{d}{2}$ is the constant radius of the artery at the entrance, d is arterial diameter at the entrance and $\beta = 1$

3. ADOMIAN DECOMPOSITION METHOD OF SOLUTION

To obtain the solution of the coupled differential equations using Adomian decomposition method, we set

$$L_t = \frac{\partial}{\partial t}, L_x = \frac{\partial}{\partial x}, L_t^{-1} = \int_0^t (.)dt, \tag{11}$$

using (10)-(11) the coupled equation (9) can be written in integral form as

$$A(t,x) = \phi(x) - \int_0^t (A(s,x)L_x u(s,x) + u(s,x)L_x A(s,x))ds, \quad (12)$$

together with

$$u(t,x) = u_0(x) - \int_0^t ((2\alpha - 1)u(s,x)L_x u(s,x))ds$$
$$- \int_0^t \left(\frac{(\alpha-1)u^2(s,x)}{A(s,x)} L_x A(s,x) + \frac{\kappa}{\rho} L_x A(s,x) + \frac{2\pi\mu u(s,x)}{A(s,x)\rho} \right) ds, \quad (13)$$

while the nonlinear terms in (12)-(13) are identified as

$$B_n = u(t,x)\frac{\partial A(t,x)}{\partial x}, \quad (14)$$

$$C_n = A(t,x)\frac{\partial u(t,x)}{\partial x}, \quad (15)$$

$$E_n = u(t,x)\frac{\partial u(t,x)}{\partial x}, \quad (16)$$

$$F_n = \frac{u^2(t,x)}{A(t,x)}\frac{\partial A(t,x)}{\partial x}, \quad (17)$$

$$G_n = \frac{u(t,x)}{A(t,x)}, \quad (18)$$

Adomian decomposition method assumes an infinite series in the form

$$u(t,x) = \sum_{n=0}^{\infty} u_n(t,x), \quad (19)$$

$$A(t,x) = \sum_{n=0}^{\infty} A_n(t,x), \quad (20)$$

substituting (19) and (20) into (14)-(18), we obtain the Adomian polynomials

$$B_0 = u_0(t,x)A_{0x}(t,x), \quad (21)$$

$$B_1 = u_1(t,x)A_{0x}(t,x) + u_0(t,x)A_{1x}(t,x), \quad (22)$$

$$B_2 = u_2(t,x)A_{0x}(t,x) + u_1(t,x)A_{1x}(t,x) + u_0(t,x)A_{2x}(t,x), \quad (23)$$

$$B_3 = u_3(t,x)A_{0x}(t,x) + u_2(t,x)A_{1x}(t,x) + u_1(t,x)A_{2x}(t,x) + u_0(t,x)A_{3x}(t,x), \quad (24)$$

$$C_0 = A_0(t,x)A_{0x}(t,x), \quad (25)$$

$$C_1 = A_1(t,x)A_{0x}(t,x) + A_0(t,x)A_{1x}(t,x), \quad (26)$$

$$C_2 = A_2(t,x)A_{0x}(t,x) + A_1(t,x)A_{1x}(t,x) + A_0(t,x)A_{2x}(t,x), \quad (27)$$

$$C_3 = A_3(t,x)A_{0x}(t,x) + A_2(t,x)A_{1x}(t,x) + A_1(t,x)A_{2x}(t,x) + A_0(t,x)A_{3x}(t,x), \quad (28)$$

$$E_0 = u_0(t,x)u_{0x}(t,x), \quad (29)$$

$$E_1 = u_1(t,x)u_{0x}(t,x) + u_0(t,x)u_{1x}(t,x), \quad (30)$$

$$E_2 = u_2(t,x)u_{0x}(t,x) + u_1(t,x)u_{1x}(t,x) + u_0(t,x)u_{2x}(t,x), \quad (31)$$

$$E_3 = u_3(t,x)u_{0x}(t,x) + u_2(t,x)u_{1x}(t,x) + u_1(t,x)u_{2x}(t,x) + u_0(t,x)u_{3x}(t,x), \quad (32)$$

$$F_0 = \frac{u_0^2(t,x)A_{0x}(t,x)}{A_0(t,x)}, \quad (33)$$

$$F_1 = \frac{A_0(t,x)A_1(t,x)u_0^2(t,x)A_{0x}(t,x) + 2A_0(t,x)u_0(t,x)u_1(t,x)A_{0x}(t,x)}{A_0^2(t,x)}, \quad (34)$$

$$G_0 = \frac{u_0(t,x)}{A0(t,x)}, \quad (35)$$

$$G_1 = \frac{A_0(t,x)u_1(t,x) - A_1(t,x)u_0(t,x)}{A_0^2(t,x)}, \quad (36)$$

$$G_2 = \frac{A_1^2u_0(t,x) - A_0A_2u_0(t,x) - A_0A_1u_1 + A_0^2u_2(t,x)}{A_0^3(t,x)}, \quad (37)$$

using (21)-(37) in (12), we get

$$\sum_{n=0}^{\infty} A_n(t,x) = \phi(x) - \int_0^t (\sum_{n=0}^{\infty} B_n(s,x) - \sum_{n=0}^{\infty} C_n(s,x))ds, \quad (38)$$

together with

$$\sum_{n=0}^{\infty} u_n(t,x) = u_0(x) - \int_0^t ((2\alpha-1)\sum_{n=0}^{\infty} E_n(s,x)ds$$

$$-\int_0^t ((\alpha-1)\sum_{n=0}^{\infty} F_n(t,x) + \frac{\kappa}{\rho}\sum_{n=0}^{\infty} L_x(A_n(t,x)) + \frac{2\pi}{\mu}\sum_{n=0}^{\infty} G_n(t,x)\rho)ds, \quad (39)$$

then the zeroth components of (34) and (39) are

$$A_0(t,x) = \phi(x), \quad (40)$$

$$u_0(t,x) = u_0(x), \quad (41)$$

while the recursive relation is

$$A_{n+1}(t,x) = -\int_0^t (B_n(s,x) + C_n(s,x))ds, \quad (42)$$

and

$$u_{n+1}(t,x) = -\int_0^t ((2\alpha-1)E_n(s,x)ds$$

$$-\int_0^t ((\alpha-1)F_n(t,x) + \frac{\kappa}{\rho}L_x(A_n(t,x)) + \frac{2\pi}{\mu}G_n(t,x)\rho)ds, \quad (43)$$

then the partial sum

$$u(t,x) = \sum_{n=0}^{k} u_n(t,x) \quad (44)$$

$$A(t,x) = \sum_{n=0}^{k} A_n(t,x) \tag{45}$$

are the approximate solutions which satisfies both the initial and boundary condition (10) provided the solutions are convergent, twice differentiable with respect to x and differentiable with respect to t.

4. RESULTS AND DISCUSSION

To show the convergence of the Adomian series solution solution, the following experimental values are used (see refs. [16],[28]) as follows: $\alpha = 0.9645, \rho = 1.055, d = 0.777, \mu = 0.049, \kappa = 0.5, t = 0.0001$, $k = 2, x = 0$. The result is presented in Table 1.

Table 1: Computation showing rapid convergence of the solution

n	$A_n(t,x)$	$\sum_{n=0}^{k} A_n(t,x)$	$u_n(t,x)$	$\sum_{n=0}^{k} u_n(t,x)$
0	0.474168	0.474168	1	1
1	1.28025E-06	0.474169	-6.03313E-05	0.99994
2	-4.57007E-09	0.474169	-4.54247E-09	0.99994

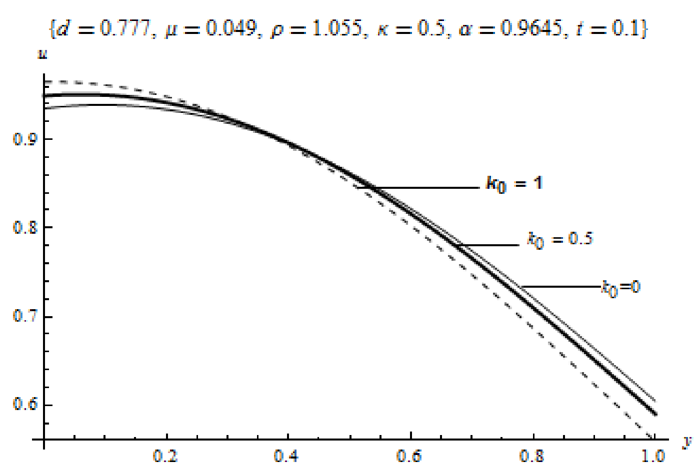

Figure 1. Effect of taper parameter on velocity profile

Figure 1 shows the effect of increase in the taper parameter on the velocity profile. As observed from the plot, an increase in the

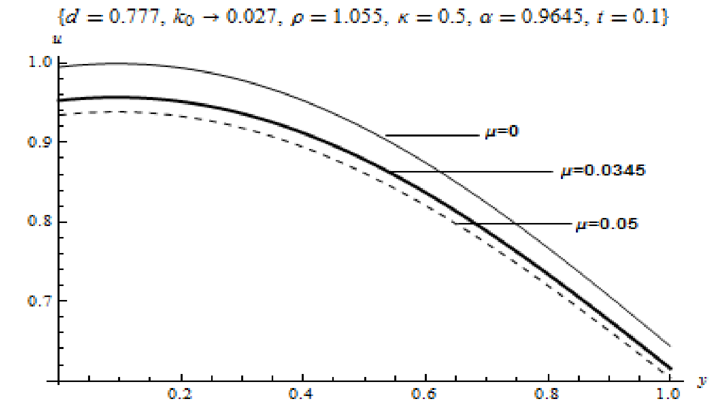

Figure 2. Effect of dynamic viscosity on velocity profile

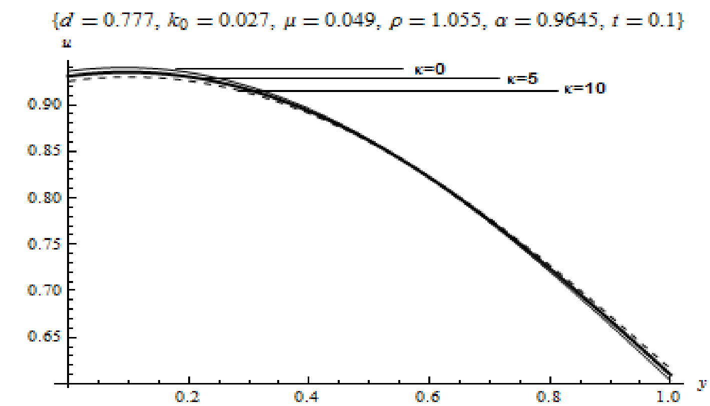

Figure 3. Effect of elasticity parameter on velocity profile

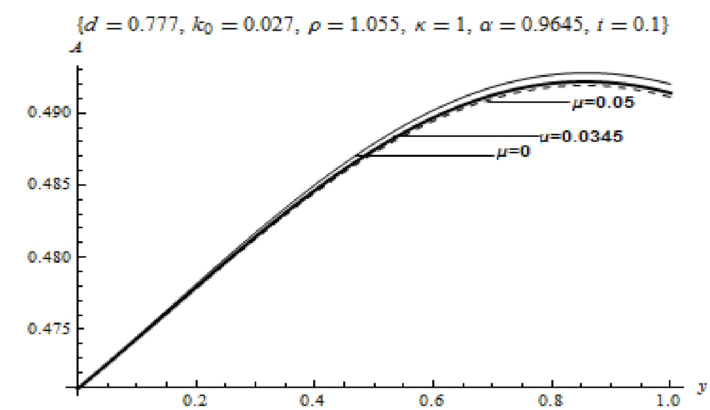

Figure 4. Effect of dynamic viscosity on arterial cross sectional area

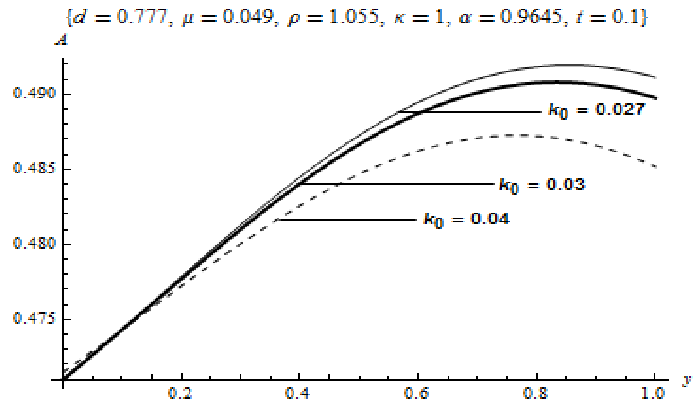

Figure 5. Effect of taper parameter on arterial cross sectional area

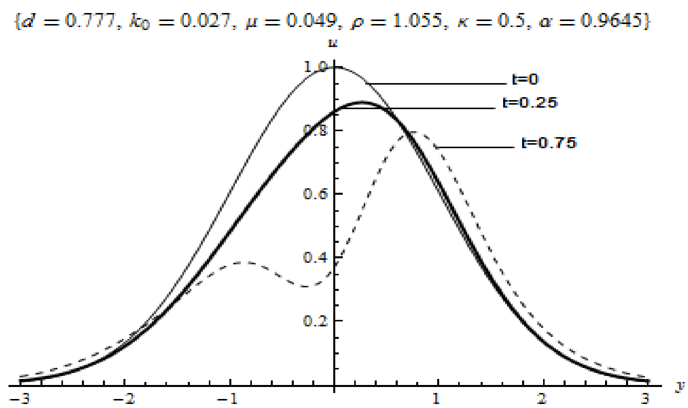

Figure 6. Snap shot of the velocity profile at different times

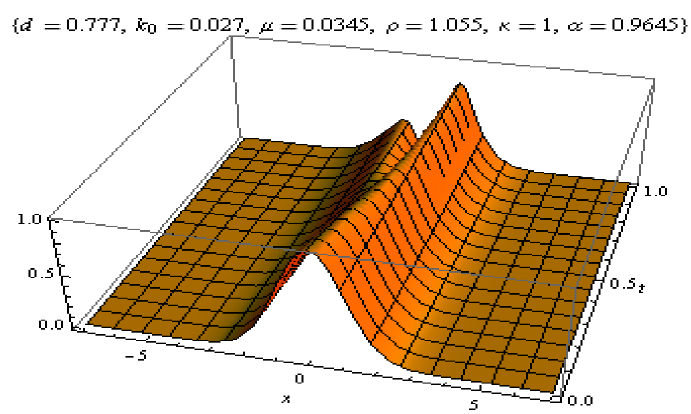

Figure 7. Travelling wave plot of $u(t, x)$

taper parameter leads to an increase in the velocity maximum in the centre of the tube while flow velocity at the wall of the tube decreases. Figure 2 depicts the velocity profile for variations in dynamic viscosity. The result shows that an increase in blood viscosity decreases the flow velocity across the tube due to fluid thickening this eventually results in rise in blood pressure. Figure 3 shows the effect variations in tube elasticity on the velocity profile. Maximum velocity at the center of the tube is observed to decrease with an increase in the wall elasticity. However, at the wall the flow velocity is observed to increase due to radial displacement. The effect of rise in the dynamic viscosity on the arterial cross-sectional area is presented in Figure 4. As observed, an increase in blood viscosity decreases the radial displacement of the artery. This is physically true due to rise in frictional forces between fluid layers. Also, Figure 5 shows the relationship between the taper parameter and the arterial cross sectional area. The result shows that as the taper parameter increases there is decrease in the cross sectional area of the artery. Figure 6 represents the development of shock waves within the tube at time increases. Therefore, the solution is short-lived since it develops discontinuity (spike-formation) as t becomes large. Lastly, Figure 7 represents the smooth traveling wave due to intermittent injection of blood into the arterial tree from the heart.

5. CONCLUSION

In the present study, the main concern is to investigate the effect of viscosity on arterial blood flow through an elastic tube. Approximate solution of the coupled nonlinear partial differential equations are obtained by using Adomian decomposition method. Just like a typical hyperbolic partial differential equation, the solution blows up as t becomes large. The result confirmed that this method is a powerful mathematical technique in obtaining solution of coupled nonlinear hyperbolic differential equations without any need for perturbation, transformations or discretization.

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