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DYNAMIC BEHAVIOUR OF NON-PRISMATIC RAYLEIGH BEAM ON PASTERNAK FOUNDATION AND UNDER PARTIALLY DISTRIBUTED MASSES MOVING AT VARYING VELOCITIES

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ABSTRACT: The dynamic analysis of the behaviour of Non-prismatic Rayleigh beam on Pasternak foundation under partially distributed masses moving at varying velocities is investigated in this paper. The solution technique is based on the expansion of Heaviside function in series form, the use of the generalized Galerkin method and a modification of Struble's asymptotic method which reduces the governing fourth order partial differential equation to a coupled second order ordinary differential equation. Closed form solutions are obtained and numerical results in plotted curves are presented. The results show that as the value of rotatory inertia correction factor r^0 increases, the response amplitude of the Rayleigh beam decreases. Similarly, higher values of the foundation stiffness K, shear modulus G and axial force N decrease the transverse deflection of the beam. The results further show that for fixed r^0, K, G and N, the transverse deflection of the non-uniform Rayleigh beam resting on Pasternak foundation and under partially distributed masses moving at varying velocities are higher than those when only the force effects of the moving load are considered indicating clearly that resonance is reached earlier in moving distributed mass problem. This further confirms results in literature stressing the need to always consider the inertia terms when heavy loads traverse any form of structural members.

Keyword and phrases: Pasternak Foundation, Partially distributed load, Moving load, Foundation Stiffness, Shear Modulus, Axial force, Rotatory Inertia, Resonance.
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1. INTRODUCTION

The problems of transverse motions of elastic beams on elastic foundations have been analyzed by many investigators Fryba[l], Oni [2], Oni and Awodola [3], Oni and Omolofe [4]) Oni and Ogunyebi [5], Muscolino and Palmeri [6], Wu ([7] and Isede and Gbadeyan [8]. Practical problems like railroad trucks, highway pavements, navigation locks and structural foundations are examples of such. These studies have however been limited to cases where the distributed

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parameter systems are assumed to be prismatic and the speeds at which the subsystems move to be uniform. Practical problems involving variable cross-section and where the speed of the subsystem is non-uniform are not common in open literature. The class of problems involving variable speeds was first tackled by Lowan [9] who solved the problem of the transverse oscillations of beams under the action of moving variable loads. Much later was the work of Kokhmanyuk and Filippov [10] who treated the dynamic effects on the transverse motion of a uniform beam of a load moving at variable speed. In a more recent development, Gbadeyan and Aiyesimi [11] undertook the analysis of the dynamic response of finite beam continuously supported by a visco-elastic foundation to a load moving at variable speed. Only the force effect of the moving load was considered. Oni [12] later investigated the flexural motions of a prestressed uniform beam resting on Winkler foundation and under the actions of a concentrated mass travelling with variable velocity and considered all the inertia terms. The effects of some structural parameters on the displacement response of the vibrating beam were analysed. Huang and Thambiratnam [13], in a similar manner studied isotropic homogeneous elastic rectangular plate resting on an elastic Winkler foundation under a single concentrated load. Finite strip method was employed. Numerical examples show that when the load moves with zero or a small initial velocity, the dynamic response of the structure is steady and unlike the response due to the sudden application of a load. Very recently, Oni and Omolofe [14] also studied the dynamic response of uniform Rayleigh beam resting on elastic foundation and subjected to concentrated masses travelling at varying velocities. It was concluded that for the same natural frequency of an elastic beam, resonance is reached earlier in the moving mass system than in the moving force system. The authors in most of these various investigations used Winkler foundation model often used in pavement modelling. However, Winkler model idealization is not entirely adequate when applied to real soil and predictions from it exhibit discrepancies with observed in situ behaviour [15]. Also known as Vlasov model, Pasternak foundation offers an attractive alternative to the winkler model in providing a degree of shear interaction between adjacent soil elements and has been named the preferable option for subgrade model [16]. Also, in most of the investigations, moving loads have been idealized as moving concentrated loads which acts at a certain point on the structure and along a single line in space [17]. In practice, it is known that loads are actually distributed over a small segment or over the entire length of the structural member as they traverse the structure [18]. Thus in this work the dynamic behaviour of non-prismatic prestressed Rayleigh beam on Pasternak foundation and under partially distributed masses moving at varying velocities is considered. This work incorporates all the pertinent components of inertia terms of the moving distributed loads in the governing equations.

2. MATHEMATICAL MODEL

The flexural vibrations of finite non-uniform Rayleigh beam under partially distributed loads moving at non-uniform velocities is considered. The corresponding governing equation is the fourth order partial differential equation [1]

$$\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 V(x,t)}{\partial x^2}\right]-N\frac{\partial^2 V(x,t)}{\partial x^2}+\mu(x)\frac{\partial^2 V(x,t)}{\partial t^2}-\mu(x)r^0\frac{\partial^4 V(x,t)}{\partial x^2\partial t^2}+Z_F(x,t)=P(x,t) \tag{1}$$

Where $EI(x)$ is the variable flexural rigidity of the structure, N is the axial force, $\mu(x)$ is the variable mass per unit length of the beam and r^0 is the rotatory inertia correction factor, $Z_F(x,t)$is the foundation reaction, $P(x,t)$ is the transverse distributed load, x is the spatial coordinate and t is the time.

The relationship between the foundation reaction and the lateral deflection $V(x,t)$ is given as

$$Z_F(x,t)=KV(x,t)-G\frac{\partial^2 V(x,t)}{\partial x^2} \tag{2}$$

where G is the shear modulus and K is the foundation stiffness. If the inertia effect of the moving load is considered, the distributed load $P(x,t)$ takes the form

$$P(x,t)=P_f(x,t)\left[1=\frac{1}{g}\frac{\mathrm{d}^2 V(x,t)}{\mathrm{d}t^2}\right] \tag{3}$$

where the continuous moving force $P_f(x,t)$ acting on the beam model is given by

$$P_f(x,t)MgH[x-f(t)] \tag{4}$$

where g is the acceleration due to gravity and $\frac{\mathrm{d}^2}{\mathrm{d}t^2}$ is a convective acceleration operator defined as [1]

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}=\frac{\partial^2}{\partial t^2}+2\frac{\mathrm{d}}{\mathrm{d}t}f(t)\frac{\partial^2}{\partial x\partial t}+\left(\frac{\mathrm{d}f(t)}{\mathrm{d}t}\right)^2\frac{\partial^2}{\partial x^2}+\frac{\mathrm{d}^2}{\mathrm{d}t^2}f(t)\frac{\partial}{\partial x} \tag{5}$$

and the distance covered by the load on the same structure at any given instance of time is given as

$$f(t) = x_0 + ct + \frac{1}{2}at^2 \tag{6}$$

where x_0 is the point of application of force $P(x,t)$ at the instance $t = 0$, c is the initial velocity and a is the constant acceleration of motion. Using equations (2) to (6) in equation (1), one obtains

$$\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 V(x,t)}{\partial x^2}\right] - N\frac{\partial^2 V(x,t)}{\partial x^2} + \mu(x)\frac{\partial^2 V(x,t)}{\partial t^2} - \mu(x)r^0\frac{\partial^4 V(x,t)}{\partial x^2 \partial t^2} - G\frac{\partial^2 V(x,t)}{\partial x^2}$$
$$+ KV(x,t) + MH\left[x - \left(x_0 + ct + \frac{1}{2}at^2\right)\right]\left[\frac{\partial^2 V(x,t)}{\partial t^2} + 2(c+at)\frac{\partial^2 V(x,t)}{\partial x \partial t}\right.$$
$$\left. +(c+at)^2\frac{\partial^2 V(x,t)}{\partial x^2} + a\frac{\partial V(x,t)}{\partial x}\right] = MgH\left[x - \left(x_0 + ct + \frac{1}{2}at^2\right)\right] \tag{7}$$

Adopting the examples in [7], $I(x)$ and $\mu(x)$ are taken to be of the form

$$I(x) = I_0\left(1 + \sin\frac{\pi x}{L}\right)^3 \quad \text{and} \quad \mu(x) = \mu_0\left(1 + \sin\frac{\pi x}{L}\right) \tag{8}$$

where $I(x)$ is the variable moment of inertia.
Substituting equation (8) into equation (7) after some simplifications and rearrangements one obtains

$$\frac{\partial^2}{\partial x^2}\left[EI_0\left(1 + \sin\frac{\pi x}{L}\right)^3 \frac{\partial^2 V(x,t)}{\partial x^2}\right] - N\frac{\partial^2 V(x,t)}{\partial x^2} - G\frac{\partial^2 V(x,t)}{\partial x^2} + \mu_0\left(1 + \sin\frac{\pi x}{L}\right)$$
$$\frac{\partial^2 V(x,t)}{\partial t^2} - \frac{\partial}{\partial x}\left[\mu_0\left(1 + \sin\frac{\pi x}{L}\right)\frac{\partial^3 V(x,t)}{\partial x \partial t^2}\right] + KV(x,t) + MH\left[x - \left(x_0 + ct + \frac{1}{2}at^2\right)\right]$$
$$\left[\frac{\partial^2 V(x,t)}{\partial t^2} + 2(c+at)\frac{\partial^2 V(x,t)}{\partial x \partial t} + \left(c + at^2\right)\frac{\partial^2 V(x,t)}{\partial x^2} + a\frac{\partial V(x,t)}{\partial x}\right]$$
$$= MgH\left[x - \left(x_0 + ct + \frac{1}{2}at^2\right)\right] \tag{9}$$

Further simplifications and rearrangements yields

$$\frac{EI_0}{\mu_0}\left\{\frac{\partial^4 V(x,t)}{\partial x^4}\left\{\frac{5}{2} + \frac{15}{4}\sin\frac{\pi x}{L} - \frac{1}{4}\sin\frac{3\pi x}{L} - \frac{3}{2}\cos\frac{2\pi x}{L}\right\}\right.$$
$$\left. + \frac{\partial^2 V(x,t)}{\partial x^2}\left\{\frac{9\pi^2}{4L^2}\sin\frac{3\pi x}{L} - \frac{15\pi^2}{4L^2}\sin\frac{\pi x}{L} + \frac{6\pi^2}{L^2}\cos\frac{2\pi x}{L}\right\}\right\} - \left(\frac{N+G}{\mu_0}\right)\frac{\partial^2 V(x,t)}{\partial x^2}$$

$$+\left(1+\sin\frac{\pi x}{L}\right)\frac{\partial^2 V(x,t)}{\partial t^2}-r^0\left[\left(1+\sin\frac{\pi x}{L}\right)\frac{\partial^4 V(x,t)}{\partial x^2\partial t^2}+\frac{\pi}{L}\cos\frac{\pi x}{L}\frac{\partial^3 V(x,t)}{\partial x\partial t^2}\right]$$

$$+\frac{K}{\mu_0}V(x,t)+\frac{MH}{\mu_0}\left[x-\left(x_0+ct+\frac{1}{2}at^2\right)\right]\left[\frac{\partial^2 V(x,t)}{\partial t^2}\right.$$

$$\left.+(c+at)\frac{\partial^2 V(x,t)}{\partial x\partial t}+(c+at)^2\frac{\partial^2 V(x,t)}{\partial x^2}+a\frac{\partial V(x,t)}{\partial x}\right]=\frac{MH}{\mu_0}\left[x-\left(x_0+ct+\frac{1}{2}at^2\right)\right] \tag{10}$$

For our analysis, the Rayleigh beam under consideration is to be taken to be simply supported. Thus, the boundary conditions are given by

$$V(0,t)=V(L,t)=0;\quad \frac{\partial^2 V(0,t)}{\partial x^2}=\frac{\partial^2 V(L,t)}{\partial x^2} \tag{11}$$

The initial conditions, without any loss of generality, is taken to be

$$V(x,0)=0=\frac{\partial V(x,0)}{\partial t} \tag{12}$$

Equation (10) is the fourth order partial differential equation with variable coefficients of the non-uniform Rayleigh beam under the action of uniform partially distributed loads travelling at non-uniform velocity. The beam properties such as the moment of inertia and the mass per unit length of the beam are considered as varying along the length L of the beam. Evidently, the method of separation of variables is inapplicable as a difficulty arises in getting separate equations whose functions are functions of a single variable. In fact, an exact closed form solution of the above equation does not exist.

3. ANALYTICAL APPROXIMATE SOLUTION

As a result of the foregoing difficulty, an approximate analytical solution is sought. One of the approximate methods best suited for solving diverse problems in dynamics of structures is the Generalized Galerkin method discussed in [2]. This method requires that the solution of equation (10) takes the form

$$V_n(x,t)=\sum_{m=1}^{\infty}Z_m(t)U_m(x) \tag{13}$$

where $U_m(x)$ is chosen such that the pertinent boundary conditions are satisfied. Equation (13) is substituted into equation (10) and after some simplifications and rearrangements one obtains

$$\sum_{m=1}^{n}\Bigg\{\Bigg[\frac{EI_0}{\mu_0}\Bigg(\frac{5}{2}U_m^{iv}(x)+\frac{15}{4}\sin\frac{\pi x}{L}-\frac{1}{4}\sin\frac{3\pi x}{L}U_m^{iv}(x)-\frac{3}{2}\cos\frac{2\pi x}{L}U_m^{iv}(x)\Bigg)+$$

$$\Bigg(\frac{9\pi^2}{4L^2}\sin\frac{3\pi x}{L}U_m^{n}(x)-\frac{15\pi^2}{4L^2}\sin\frac{\pi x}{L}U_m^{n}(x)+\frac{6\pi^2}{L^2}\cos\frac{2\pi x}{L}U_m^{n}(x)\Bigg)\Bigg]Z_m(t)$$

$$-\left(\frac{N+G}{\mu_0}\right)U_m^{n}(x)Z_m(t)+\left(1+\sin\frac{\pi x}{L}\right)U_m(x)\ddot{Z}_m(t)$$

$$-r^0\left[\left(1+\sin\frac{\pi x}{L}\right)U_m^{n}(x)\ddot{Z}_m(t)+\frac{\pi}{L}\cos\frac{\pi x}{L}U_m^{n}(x)\ddot{Z}_m(t)\right]+\frac{K}{\mu_0}U_m(x)Z_m(t)$$

$$+\frac{MH}{\mu_0}\left[x-\left(x_0+ct+\frac{1}{2}at^2\right)\right]\Big[U_m(x)\ddot{Z}_m(t)+(c+at)U_m'(x)\dot{Z}_m(t)$$

$$+(c+at)^2U_m^{n}(x)Z_m(t)\Big]\Bigg\}=\frac{MgH}{\mu_0}\left[x-\left(x_0+ct+\frac{1}{2}at^2\right)\right] \tag{14}$$

where $U_m'(x)$ and $\dot{Z}_m(x)$ are first derivatives of $U_m(x)$ and $Z_m(x)$with respect to x and t respectively. In order to determine $Z_m(x)$, it is required that the expression on the right hand side of equation (14) be orthogonal to function $U_k(k)$. Hence

$$\sum_{m}^{n}\Bigg\{\left[D_1(m,k)-r^0(D_2(m,k)+D_3(m,k))\right]\ddot{Z}_m(t)+[\alpha_1(T_0+T_1)+\alpha_2D_4(m,k)$$

$$+\alpha_3D_5(m,k)\ Z_m+\frac{M}{\mu_0}\left[D_1(t)\ddot{Z}_m(t)+D_2(t)\dot{Z}_m(t)+D_3(t)Z_m+D_4(t)Z_m(t)\right]\Bigg\}$$

$$=\frac{Mg}{\mu_0}D_5(t) \tag{15}$$

where

$$T_0=D_6+D_7-(D_8+D_9);\qquad T_1=D_{10}-D_{11}+D_{12} \tag{16}$$

$$\alpha_1=\frac{EI_0}{\mu_0};\qquad \alpha_2=\frac{G+N}{\mu_0};\qquad \alpha_3=\frac{K}{\mu_0} \tag{17}$$

$$D_1(t)=\int_0^L H\left[x-\left(x_0+ct+\frac{1}{2}at^2\right)\right]U_m(x)U_k(m)\mathrm{d}x$$

$$D_2(t)=\int_0^L 2(c+at)H\left[x-\left(x_0+ct+\frac{1}{2}at^2\right)\right]U_m'(x)U_k(x)\mathrm{d}x$$

$$D_3(t) = \int_0^L (c+at)^2 H\left[x - \left(x_0 + ct + \frac{1}{2}at^2\right)\right] U_m^n(x)U_k(x)\mathrm{d}x$$

$$D_4(t) = \int_0^L aH\left[x - \left(x_0 + ct + \frac{1}{2}at^2\right)\right] U_m'(x)U_k(x)\mathrm{d}x$$

$$D_5(t) = \int_0^L H\left[x - \left(x_0 + ct + \frac{1}{2}at^2\right)\right] U_k(x)\mathrm{d}x$$

$$D_1(m,k) = \int_0^L \left(1 + \sin\frac{\pi x}{L}\right) U_m(x)U_k(x)\mathrm{d}x$$

$$D_2(m,k) = \int_0^L \left(1 + \sin\frac{\pi x}{L}\right) U_m^n(x)U_k(x)\mathrm{d}x$$

$$D_3(m,k) = \int_0^L \cos\frac{\pi x}{L} U_m'(x)U_k(x)\mathrm{d}x$$

$$D_4(m,k) = \int_0^L U_m^n(x)U_k(x)\mathrm{d}x$$

$$D_5(m,k) = \int_0^L U_m(x)U_k(x)\mathrm{d}x$$

$$D_6(m,k) = \frac{5}{2}\int_0^L U_m^{iv}(x)U_k(x)\mathrm{d}x$$

$$D_7(m,k) = \frac{15}{4}\int_0^L \sin\frac{\pi x}{L} U_m^{iv}(x)U_k(x)\mathrm{d}x$$

$$D_6(m,k) = \frac{5}{2}\int_0^L U_m^{iv}(x)U_k(x)\mathrm{d}x$$

$$D_7(m,k) = \frac{15}{4}\int_0^L \sin\frac{\pi x}{L} U_m^{iv}(x)U_k(x)\mathrm{d}x$$

$$D_8(m,k) = \frac{1}{4}\int_0^L \sin\frac{3\pi x}{L} U_m^{iv}(x)U_k(x)\mathrm{d}x$$

$$D_9(m,k) = \frac{3}{2}\int_0^L \cos\frac{2\pi x}{L} U_m^{iv}(x)U_k(x)\mathrm{d}x$$

$$D_{10}(m,k) = \frac{9\pi^2}{4L^2}\int_0^L \sin\frac{3\pi x}{L} U_m^n(x)U_k(x)\mathrm{d}x$$

$$D_9(m,k) = \frac{3}{2}\int_0^L \cos\frac{2\pi x}{L} U_m^{iv}(x)U_k(x)\mathrm{d}x$$

$$D_{10}(m,k) = \frac{9\pi^2}{4L^2}\int_0^L \sin\frac{3\pi x}{L} U_m^n(x)U_k(x)\mathrm{d}x$$

$$D_{11}(m,k) = \frac{15\pi^2}{4L^2}\int_0^L \sin\frac{\pi x}{L} U_m^n(x)U_k(x)\mathrm{d}x$$

$$D_{12}(m,k) = \frac{6\pi^2}{L^2}\int_0^L \cos\frac{2\pi x}{L} U_m^n(x)U_k(x)\mathrm{d}x \tag{18}$$

Since our elastic system has simple supports at edges $x = 0$ and $x = L$, we choose

$$U_m(x) = \sin\frac{m\pi x}{L} \qquad \text{and} \qquad U_k(x) = \sin\frac{k\pi x}{L} \tag{19}$$

Substitution of expressions for $U_m(x)$ and $U_k(x)$ into equation (15) and the use of the Fourier series representation of the Heaviside unit step function namely;

$$H = \frac{1}{4} + \frac{1}{\pi}\sum_{n=0}^{\infty}\frac{\sin(2n+1)\pi\left[x - \left(x_0 + ct + \frac{1}{2}at^2\right)\right]}{2n+1}, \quad 0 < x < L, \tag{20}$$

after some simplifications and rearrangements give

$$\left[\left(1 - r^0\frac{m^2\pi^2}{L^2}\right)I_{49}(m,k) - r^0\frac{m^2\pi^2}{L^2}I_{65}(m,k)\right]\frac{\mathrm{d}^2 Z_m(t)}{\mathrm{d}t^2}\left\{\alpha_1\frac{m^2\pi^4}{L^4}\left[\frac{5}{2}I_1(m,k)\right.\right.$$

$$\left.+\left(\frac{30m^2+15}{4}\right)I_{81}(m,k)\right] - \left(\frac{4m^2+9}{4}\right)I_{97}(m,k) - \left(\frac{3m^2+12}{2}\right)I_{113}(m,k)$$

$$\left.+\left[\alpha_2\frac{m^2\pi^2}{L^2} + \alpha_3\right]I_1(m,k)\right\}\dot{Z}_m(t) + \Gamma_0 L\left[\left(\frac{1}{4}I_1(m,k) + \frac{1}{\pi}\sum_{n=0}^{\infty}\frac{\cos(2n+1)\pi}{2n+1}\right.\right.$$

$$\left.\left(x_0 + ct + \frac{1}{2}at^2\right)I_{17}(n,m,k) - \frac{1}{\pi}\sum_{n=0}^{\infty}\frac{\sin(2n+1)\pi}{2n+1}\left(x_0 + ct + \frac{1}{2}at^2\right)I_{33}(n,m,k)\right)$$

$$\ddot{Z}_m(t) + 2(c+at)\left(\frac{m\pi}{4L}I_5(m,k) + \frac{1}{\pi}\sum_{n=0}^{\infty}\frac{\cos(2n+1)\pi}{2n+1}\left(x_0 + ct + \frac{1}{2}at^2\right)\frac{m\pi}{L}\ I_{21}(n,m,k)\right.$$

$$\left.-\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{\sin(2n+1)\pi}{2n+1}\left(x_0 + ct + \frac{1}{2}at^2\right)\frac{m\pi}{L}I_{37}(n,m,k)\right)\dot{Z}_m(t) - (c+at)^2$$

$$\left(\frac{m^2\pi^2}{4L^2}I_1(m,k) + \frac{1}{\pi}\sum_{n=0}^{\infty}\frac{\cos(2n+1)\pi}{2n+1}\left(x_0 + ct + \frac{1}{2}at^2\right)I_{17}(n,m,k)\right.$$

$$-\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{\sin(2n+1)\pi}{2n+1}\left(x_0+ct+\frac{1}{2}at^2\right)\frac{m^2\pi^2}{L^2}I_{33}(n,m,k)\Bigg)Z_m(t)+a\left(\frac{m\pi}{4L}I_5(m,k)\right.$$

$$+\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{\cos(2n+1)\pi}{2n+1}\left(x_0+ct+\frac{1}{2}at^2\right)\frac{m\pi}{L}I_{21}(n,m,k)-\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{\sin(2n+1)\pi}{2n+1}$$

$$\left.\left(x_0+ct+\frac{1}{2}at^2\right)\frac{m\pi}{L}I_{37}(n,m,k)\right)Z_m(t)\Bigg]=\frac{PL}{\mu_0 k\pi}\left[-(-1)^k+\cos\frac{k\pi x}{L}\left(x_0+ct+\frac{1}{2}at^2\right)\right] \tag{21}$$

where

$$I_1(m,k)=\int_0^L \sin\frac{m\pi x}{L}\sin\frac{k\pi x}{L}\mathrm{d}x$$

$$I_5(m,k)=\int_0^L \cos\frac{m\pi x}{L}\sin\frac{k\pi x}{L}\mathrm{d}x$$

$$I_{17}(n,m,k)=\int_0^L \sin(2n+1)\pi x\sin\frac{m\pi x}{L}\sin\frac{k\pi x}{L}\mathrm{d}x$$

$$I_{21}(n,m,k)=\int_0^L \sin(2n+1)\pi x\cos\frac{m\pi x}{L}\sin\frac{k\pi x}{L}\mathrm{d}x$$

$$I_{33}(n,m,k)=\int_0^L \cos(2n+1)\pi x\sin\frac{m\pi x}{L}\sin\frac{k\pi x}{L}\mathrm{d}x$$

$$I_{37}(n,m,k)=\int_0^L \cos(2n+1)\pi x\cos\frac{m\pi x}{L}\sin\frac{k\pi x}{L}\mathrm{d}x$$

$$I_{49}(m,k)=\int_0^L \left(1+\sin\frac{\pi x}{L}\right)\sin\frac{m\pi x}{L}\sin\frac{k\pi x}{L}\mathrm{d}x$$

$$I_{65}(m,k)=\int_0^L \cos\frac{\pi x}{L}\cos\frac{m\pi x}{L}\sin\frac{k\pi x}{L}\mathrm{d}x$$

$$I_{81}(m,k)=\int_0^L \sin\frac{\pi x}{L}\cos\frac{m\pi x}{L}\sin\frac{k\pi x}{L}\mathrm{d}x$$

$$I_{97}(m,k)=\int_0^L \sin\frac{3\pi x}{L}\cos\frac{m\pi x}{L}\sin\frac{k\pi x}{L}\mathrm{d}x$$

$$I_{113}(m,k)=\int_0^L \cos\frac{2\pi x}{L}\sin\frac{m\pi x}{L}\sin\frac{k\pi x}{L}\mathrm{d}x \tag{22}$$

Equation (21) is the transformed equation governing the problem of a non-uniform Rayleigh beam on a Pasternak foundation being traversed by uniform partially distributed masses.

Solving integrals in (22) and substituting into (21) after some simplifications and rearrangements, yield

$$\left[\frac{L}{2} + \frac{L}{4\pi}HH1 - r^0\left(\frac{m\pi}{4L}HH2 - \frac{m^2\pi^2}{2L} - \frac{m^2\pi}{4L}HH1\right)\right]\ddot{Z}_m(t)$$

$$+\left[\alpha_1\left(\frac{5m^4\pi^4}{4L^3} + \frac{15m^2\pi^3}{16L^3}\left(2m^2+1\right)HH1\right) - \frac{m^2\pi^4}{4L^4}(4m^2+9)HH3 +\right.$$

$$\left.\alpha_2\frac{m^2\pi^2}{2L} + \alpha_3\frac{L}{2}\right]Z_m(t) + \Gamma_0 L\left\{\frac{L}{8} + \sum_{n=0}^{\infty}\frac{(2n+1)L^2}{2\pi^2}\left(\frac{(-1)^{m+k}\cos(2n+1)\pi L - 1}{[(2n+1)L]^2-(m+k)^2}\right.\right.$$

$$\left.-\frac{(-1)^{m-k}\cos(2n+1)\pi L - 1}{[(2n+1)L]^2-(m-k)^2}\right)\frac{\cos(2n+1)\pi\left(x_0+ct+\frac{1}{2}at^2\right)}{2n+1}$$

$$-\sum_{n=0}^{\infty}\frac{(2n+1)L^2}{2\pi^2}\left(\frac{(-1)^{m-k}\sin(2n+1)\pi L}{[(2n+1)L]^2-(m-k)^2} - \frac{(-1)^{m+k}\sin(2n+1)\pi L}{[(2n+1)L]^2-(m+k)^2}\right)$$

$$\left.\frac{\sin(2n+1)\pi\left(x_0+ct+\frac{1}{2}at^2\right)}{2n+1}\right]\dot{Z}_m(t) + \left[2(c+at)\left(\frac{m}{2\pi}\sum_{n=0}^{\infty}\frac{\cos(2n+1)\pi}{2n+1}\left(x_0+ct+\frac{1}{2}at^2\right)\right.\right.$$

$$\left((m+k)\frac{(-1)^{m+k}\sin(2n+1)\pi L}{[(2n+1)L]^2-(m+k)^2} - \frac{(m-k)(-1)^{m-k}\sin(2n+1)\pi L}{[(2n+1)L]^2-(m-k)^2}\right)$$

$$-\frac{m}{2\pi}\sum_{n=0}^{\infty}\left(\frac{(m+k)(-1)^{m+k}\cos(2n+1)\pi L - 1}{[(2n+1)L]^2-(m+k)^2} - \frac{(m-k)(-1)^{m-k}\sin(2n+1)\pi L - 1}{[(2n+1)L]^2-(m-k)^2}\right)$$

$$\left.\left.\left.\frac{\sin(2n+1)\pi\left(x_0+ct+\frac{1}{2}at^2\right)}{2n+1}\right)\right)\right]\dot{Z}_m(t)\left[\frac{(c+at)^2m^2\pi^2}{L^2}\left(\frac{L}{8} + \sum_{n=0}^{\infty}\frac{(2n+1)L^2}{2\pi^2}\right.\right.$$

$$\left.\left(\frac{(-1)^{m+k}\cos(2n+1)\pi L - 1}{[(2n+1)L]^2-(m+k)^2} - \frac{(-1)^{m-k}\sin(2n+1)\pi L - 1}{[(2n+1)L]^2-(m-k)^2}\right)\frac{\cos(2n+1)\pi\left(x_0+ct+\frac{1}{2}at^2\right)}{2n+1}\right)$$

$$-\sum_{n=0}^{\infty}\frac{(2n+1)L^2}{2\pi^2}\left(\frac{(-1)^{m-k}\sin(2n+1)\pi L}{[(2n+1)L]^2-(m-k)^2} - \frac{(-1)^{m+k}\sin(2n+1)\pi L}{[(2n+1)L]^2-(m+k)^2}\right)$$

$$\frac{\sin(2n+1)\pi\left(x_0+ct+\frac{1}{2}at^2\right)}{2n+1}+\frac{m}{2\pi}\sum_{n=0}^{\infty}\left(\frac{(m+k)(-1)^{m+k}\sin(2n+1)\pi L}{[(2n+1)L]^2-(m-k)^2}\right.$$

$$\left.-\frac{(m-k)(-1)^{m-k}\sin(2n+1)\pi L}{[(2n+1)L]^2-(m+k)^2}\right)\frac{\cos(2n+1)\pi}{2n+1}\left(x_0+ct+\frac{1}{2}at^2\right)$$

$$-\frac{m}{2\pi}\sum_{n=0}^{\infty}\left(\frac{(m-k)(-1)^{m-k}\cos(2n+1)\pi L}{[(2n+1)L]^2-(m-k)^2}-\frac{(m-k)(-1)^{m-k}\sin(2n+1)\pi L}{[(2n+1)L]^2-(m+k)^2}\right)$$

$$\left.\left.\frac{\sin(2n+1)\pi}{2n+1}\left(x_0+ct+\frac{1}{2}at^2\right)\right]Z_m(t)\right\}=\frac{PL}{\mu_0 k\pi}\left[-(-1)^k+\cos\frac{k\pi x}{L}\left(x_0+ct+\frac{1}{2}at^2\right)\right] \tag{23}$$

where

$$HH1=\frac{\cos(1+2m)\pi-1}{1+2m}+\frac{\cos(1-2m)\pi-1}{1-2m}+4 \tag{24}$$

$$HH2=\frac{\cos(1-2m)\pi-1}{1-2m}+\frac{\cos(1+2m)\pi-1}{1+2m} \tag{25}$$

$$HH3=\frac{L}{3\pi}+\frac{L}{4\pi}\left[\frac{\cos(3+2m)\pi-1}{3+2m}+\frac{\cos(3-2m)\pi-1}{3-2m}\right] \tag{26}$$

Equation (23) is the fundamental transformed equation of the simply supported non-uniform Rayleigh beam resting on Pasternak foundation and under partially distributed masses travelling at varying velocities. In what follows, we shall discuss two cases of the equation.

CASE I: SIMPLY SUPPORTED NON-PRISMATIC RAYLEIGH BEAM TRAVERSED BY MOVING DISTRIBUTED FORCE

If we neglect the inertia term, we have the classical case of a moving force problem. Under this assumption $\Gamma_0 = 0$ and equation (23) after some simplifications and rearrangements becomes

$$\left[\frac{L}{2}+\frac{L}{4\pi}HH1-r^0\left(\frac{m\pi}{4L}HH2-\frac{m^2\pi^2}{2L}-\frac{m^2\pi}{4L}HH1\right)\right]\ddot{Z}_m(t)$$

$$+\left[\alpha_1\left(\frac{5m^4\pi^4}{4L^3}+\frac{15m^2\pi^3}{16L^3}(2m^2+1)HH1\right)-\frac{m^2\pi^4}{4L^4}(4m^2+9)HH3+\alpha_2\frac{m^2\pi^2}{2L}\right.$$

$$\left.+\alpha_3\frac{L}{2}\right]Z_m(t)=\frac{PL}{\mu_0 k\pi}\left[-(-1^k)+\cos\frac{k\pi}{L}\left(x_0+ct+\frac{1}{2}at^2\right)\right] \tag{27}$$

where

$$\psi_{11} = \frac{L}{2} + \frac{L}{4\pi}\left[\frac{\cos(1-2m)\pi - 1}{1+2m} + \frac{\cos(1-2m)\pi - 1}{1-2m} + 4\right]$$

$$- r^0\left\{\frac{m\pi}{4L}\left(\frac{\cos(1-2m)\pi - 1}{1-2m} + \frac{\cos(1+2m)\pi - 1}{1+2m}\right) - \frac{m^2\pi^2}{2L} -\right.$$

$$\left.\frac{m^2\pi}{4L}\left(\frac{\cos(1+2m)\pi - 1}{1+2m} + \frac{\cos(1-2m)\pi - 1}{1-2m} + 4\right)\right\}$$

$$\psi_{22} = \alpha_1\left\{\frac{5m^4\pi^4}{4L^3} + \frac{15m^2\pi^3}{16L^3}\left(m^3 - 1\right)\left(\frac{\cos(1+2m)\pi - 1}{1+2m} + \frac{\cos(1+2m)\pi - 1}{1-2m} + 4\right)\right.$$

$$- \frac{m^2\pi^4}{4L^4}(9 + 4m^2)\left[\frac{L}{3\pi} + \frac{L}{4\pi}\left(\frac{\cos(3+2m)\pi - 1}{3+2m} + \frac{\cos(3-2m)\pi - 1}{3-2m}\right)\alpha_2\right.$$

$$\left.\frac{m^2\pi^2}{2L} + \alpha_3\frac{L}{2}\right] \tag{28}$$

Equation (27) can further be rearranged to give

$$\ddot{Z}_m(t) + \Omega_{pp}^2 Z_m(t) = P_{mm}\left[-(-1)^k + \cos\frac{k\pi}{L}\left(x_0 + ct + \frac{1}{2}at^2\right)\right] \tag{29}$$

$$\text{where} \quad \Omega_{pp}^2 = \frac{\psi_{22}}{\psi_{11}} \quad \text{and} \quad P_{mm} = \frac{\Gamma_0 L^2 g}{k\pi\psi_{11}} \tag{30}$$

In order to solve equation (29), variation of parameters technique is resorted to. Firstly, it is straight forward to show that the general solution of the homogeneous part of (29) is given by

$$Z_c(t) = C_1\cos\Omega_{pp}t + C_2\sin\Omega_{pp}t \tag{31}$$

where C_1 and C_2 are constants. Thus a particular solution to equation (29) takes the form

$$Z_p(t) = P_1\cos\Omega_{pp}t + P_2\sin\Omega_{pp}t \tag{32}$$

where $P_1(t)$ and $P_2(t)$ are functions to be determined. From equation (33), it is straight forward to show that

$$P_1(t) = -\frac{P_{mm}}{\Omega_{pp}}\int\left[-\cos\lambda_k + \cos\frac{\lambda_k}{L}\left(x_0 + ct + \frac{1}{2}at^2\right)\right]\sin\Omega_{pp}t \tag{33}$$

$$P_2(t) = -\frac{P_{mm}}{\Omega_{pp}}\int\left[-\cos\lambda_k + \cos\frac{\lambda_k}{L}\left(x_0 + ct + \frac{1}{2}at^2\right)\right]\cos\Omega_{pp}t \tag{34}$$

Equations (33) and (34) can be simplified by using Fresnel functions which are integrals that involve quadratic expressions in the sine and cosine functions. They are defined as follows;

$$S(x) = \int \sin(az^2)\mathrm{d}z = \frac{1}{\sqrt{a}}\sqrt{\frac{\pi}{2}}S\left(\sqrt{a}\sqrt{\frac{2}{\pi}}z\right)$$

and

$$C(x) = \int \cos(az^2)\mathrm{d}z = \frac{1}{\sqrt{a}}\sqrt{\frac{\pi}{2}}C\left(\sqrt{a}\sqrt{\frac{2}{\pi}}z\right) \tag{35}$$

where $S(x)$ is the Fresnel Sine function and $C(x)$ is the Fresnel Cosine function. Using (35) in equations (33) and (34) after some simplifications and rearrangements one obtains

$$P_1(t) = -\frac{P_{mm}\sqrt{\pi}}{\sqrt{2a}}\left\{\cos\left(\frac{b_1^2}{4a} - C_0\right)S\left(\frac{b_1 + 2at}{\sqrt{2\pi a}}\right) - \sin\left(\frac{b_1^2}{4a} - C_0\right)C\left(\frac{b_1 + 2at}{\sqrt{2\pi a}}\right)\right.$$
$$- \cos\left(\frac{b_2^2}{4a} - C_0\right)S\left(\frac{b_2 + 2at}{\sqrt{2\pi a}}\right) + \sin\left(\frac{b_2^2}{4a} - C_0\right)C\left(\frac{b_2 + 2at}{\sqrt{2\pi a}}\right)$$
$$\left.+\frac{1}{2\Omega_{pp}}\left[\cos\left(\lambda_k - \Omega_{pp}t\right) + \cos\left(\lambda_k + \Omega_{pp}t\right)\right]\right\} \tag{36}$$

and

$$P_2(t) = -\frac{P_{mm}\sqrt{\pi}}{\sqrt{2a}}\left\{\cos\left(\frac{b_1^2}{4a} - C_0\right)C\left(\frac{b_1 + 2at}{\sqrt{2\pi a}}\right) + \sin\left(\frac{b_1^2}{4a} - C_0\right)S\left(\frac{b_1 + 2at}{\sqrt{2\pi a}}\right)\right.$$
$$+ \cos\left(\frac{b_2^2}{4a} - C_0\right)C\left(\frac{b_2 + 2at}{\sqrt{2\pi a}}\right) + \sin\left(\frac{b_2^2}{4a} - C_0\right)S\left(\frac{b_2 + 2at}{\sqrt{2\pi a}}\right)$$
$$\left.+\frac{1}{2\Omega_{pp}}\left[\sin\left(\lambda_k + \Omega_{pp}t\right) - \sin\left(\lambda_k - \Omega_{pp}t\right)\right]\right\} \tag{37}$$

Using (36) and (37), the particular solution of the non-homogeneous second order differential equation takes the form

$$Z_p(t) = \frac{P_{mm}\sqrt{\pi}}{\sqrt{2a}}\left\{\sin\left(\frac{b_1^2}{4a} - C_0\right)C\left(\frac{b_1 + 2at}{\sqrt{2\pi a}}\right) - cos\left(\frac{b_1^2}{4a} - C_0\right)S\left(\frac{b_1 + 2at}{\sqrt{2\pi a}}\right)\right.$$
$$+ \cos\left(\frac{b_2^2}{4a} - C_0\right)S\left(\frac{b_2 + 2at}{\sqrt{2\pi a}}\right) - \sin\left(\frac{b_2^2}{4a} - C_0\right)C\left(\frac{b_2 + 2at}{\sqrt{2\pi a}}\right) + \frac{\cos\lambda_k}{\Omega_{pp}}$$
$$- \frac{1}{2\Omega_{pp}}\left[\cos\left(\lambda_k - \Omega_{pp}t\right) + \cos\left(\lambda_k + \Omega_{pp}t\right)\right] + \cos\left(\frac{b_1^2}{4a} - C_0\right)C\left(\frac{b_1 + 2at}{\sqrt{2\pi a}}\right)$$
$$+ \sin\left(\frac{b_1^2}{4a} - C_0\right)S\left(\frac{b_1 + 2at}{\sqrt{2\pi a}}\right) + \cos\left(\frac{b_2^2}{4a} - C_0\right)C\left(\frac{b_2 + 2at}{\sqrt{2\pi a}}\right)$$
$$+ \sin\left(\frac{b_2^2}{4a} - C_0\right)S\left(\frac{b_2 + 2at}{\sqrt{2\pi a}}\right) - \frac{1}{2\Omega_{pp}}\left[\sin\left(\lambda_k + \Omega_{pp}t\right) - \sin\left(\lambda_k - \Omega_{pp}t\right)\right] \tag{38}$$

Consequently,

$$Z_G(t) = Z_c(t) + Z_p(t) \tag{39}$$

Applying the initial conditional (11) to (39), the constants are found to be

$$C_1 = \frac{P_{mm}\sqrt{\pi}}{\sqrt{2a}}\left\{\cos\left(\frac{b_1^2}{4a} - C_0\right) S\left(\frac{b_1}{\sqrt{2\pi a}}\right) - \sin\left(\frac{b_1^2}{4a} - C_0\right) C\left(\frac{b_1}{\sqrt{2\pi a}}\right) - \cos\left(\frac{b_2^2}{4a} - C_0\right)\right.$$

$$\left. S\left(\frac{b_2}{\sqrt{2\pi a}}\right) + \sin\left(\frac{b_2^2}{4a} - C_0\right) + C\left(\frac{b_2}{\sqrt{2\pi a}}\right) + \frac{\cos\lambda_k}{\Omega_{pp}}\right\} \tag{40}$$

and

$$C_2 = -\frac{P_{mm}\sqrt{\pi}}{\sqrt{2a}}\left\{\cos\left(\frac{b_1^2}{4a} - C_0\right) C\left(\frac{b_1}{\sqrt{2\pi a}}\right) + \sin\left(\frac{b_1^2}{4a} - C_0\right) S\left(\frac{b_1}{\sqrt{2\pi a}}\right) + \cos\left(\frac{b_2^2}{4a} - C_0\right)\right.$$

$$\left. C\left(\frac{b_2}{\sqrt{2\pi a}}\right) + \sin\left(\frac{b_2^2}{4a} - C_0\right) + S\left(\frac{b_2}{\sqrt{2\pi a}}\right)\right\} \tag{41}$$

Substituting (40) and (41) into (39) and inverting after some simplifications and rearrangements yield

$$V_n(x,t) = \sum_{m=1}^{\infty} \frac{2P_{mm}\sqrt{\pi}}{L\sqrt{2a}\tau(x)}\left\{\frac{\sin\Omega_{pp}t}{\Omega_{pp}}\left[\cos\left(\frac{b_1^2}{4a} - C_0\right) C\left(\frac{b_1 + 2at}{\sqrt{2\pi a}}\right) + \sin\left(\frac{b_1^2}{4a} - C_0\right)\right.\right.$$

$$S\left(\frac{b_1 + 2at}{\sqrt{2\pi a}}\right) + \cos\left(\frac{b_2^2}{4a} - C_0\right) C\left(\frac{b_2 + 2at}{\sqrt{2\pi a}}\right) + \sin\left(\frac{b_2^2}{4a} - C_0\right) S\left(\frac{b_2 + 2at}{\sqrt{2\pi a}}\right)$$

$$-\cos\left(\frac{b_1^2}{4a} - C_0\right) C\left(\frac{b_1}{\sqrt{2\pi a}}\right) - \sin\left(\frac{b_1^2}{4a} - C_0\right) S\left(\frac{b_1}{\sqrt{2\pi a}}\right) - \cos\left(\frac{b_2^2}{4a} - C_0\right)$$

$$C\left(\frac{b_2}{\sqrt{2\pi a}}\right) - \sin\left(\frac{b_2^2}{4a} - C_0\right) S\left(\frac{b_2}{\sqrt{2\pi a}}\right) - \frac{1}{2\Omega_{pp}}\left[\sin\left(\lambda_k + \Omega_{pp}t\right) - \sin\left(\lambda_k - \Omega_{pp}t\right)\right]$$

$$-\frac{\cos\Omega_{pp}t}{\Omega pp}\left[\cos\left(\frac{b_1^2}{4a} - C_0\right) S\left(\frac{b_1 + 2at}{\sqrt{2\pi a}}\right) - \sin\left(\frac{b_1^2}{4a} - C_0\right) C\left(\frac{b_1 + 2at}{\sqrt{2\pi a}}\right)\right.$$

$$-\cos\left(\frac{b_2^2}{4a} - C_0\right) S\left(\frac{b_2 + 2at}{\sqrt{2\pi a}}\right) \sin\left(\frac{b_2^2}{4a} - C_0\right) C\left(\frac{b_2 + 2at}{\sqrt{2\pi a}}\right) - \cos\left(\frac{b_1^2}{\sqrt{2\pi a}}\right)$$

$$-\cos\left(\frac{b_1^2}{4a} - C_0\right) S\left(\frac{b_1}{\sqrt{2\pi a}}\right) + \sin\left(\frac{b_1^2}{4a} - C_0\right) C\left(\frac{b_1}{\sqrt{2\pi a}}\right) + \cos\left(\frac{b_2^2}{4a} - C_0\right)$$

$$S\left(\frac{b_2}{\sqrt{2\pi a}}\right) - \sin\left(\frac{b_2^2}{4a} - C_0\right) C\left(\frac{b_2}{\sqrt{2\pi a}}\right) - \frac{\cos\lambda_k}{\Omega_{pp}}$$

$$\left.\left. + \frac{1}{2\Omega_{pp}}\left[\cos(\lambda_k - \Omega_{pp}t) + \cos(\lambda_k + \Omega_{pp}t)\right]\right]\right\}\left(\sin\frac{m\pi x}{L}\right) \tag{42}$$

as the transverse displacement response to forces moving at variable velocities of a prestressed non-uniform Rayleigh beam resting on Pasternak elastic foundation.

CASE II: SIMPLY SUPPORTED NON-PRISMATIC RAYLEIGH BEAM TRAVERSED BY MOVING DISTRIBUTED MASS

If the moving load is heavy, the inertia effect of the moving load is not negligible and must be considered. In this case $\Gamma_0 \neq 0$ and we are required to solve the entire equation (23). This is termed the moving mass problem. Evidently, an exact closed form solution of the equation is not possible. Thus, we resort to the approximate analytical solution technique which is a modification of the asymptotic method of Struble extensively discussed in [10]. To this end, we rearrange equation (27) to take the form

$$\ddot{Z}_m(t) + \frac{\Gamma_0(c+at)\left[\begin{array}{l} R_d(k,m) + \sum_{n=0}^{\infty} R_e(k,m,n)\frac{\cos(2n+1)\pi}{2n+1}\left(x_0+ct+\frac{1}{2}at^2\right) \\ -\sum_{n=0}^{\infty} R_f(k,m,n)\frac{\sin(2n+1)\pi}{2n+1}\left(x_0+ct+\frac{1}{2}at^2\right)\end{array}\right]}{\left[1+\Gamma_0\left\{\begin{array}{l} R_a(k,m) + \sum_{n=0}^{\infty} R_b(k,m,n)\frac{\cos(2n+1)\pi}{2n+1}\left(x_0+ct+\frac{1}{2}at^2\right) \\ -\sum_{n=0}^{\infty} R_c(k,m,n)\frac{\sin(2n+1)\pi}{2n+1}\left(x_0+ct+\frac{1}{2}at^2\right)\end{array}\right\}\right]}\dot{Z}_m(t)$$

$$+\frac{\Omega_{pp}^2 + \Gamma_0(c+at)^2\left\{\begin{array}{l} R_g(k,m) + \sum_{n=0}^{\infty} R_h(k,m,n)\frac{\cos(2n+1)\pi}{2n+1}\left(x_0+ct+\frac{1}{2}at^2\right) \\ -\sum_{n=0}^{\infty} R_i(k,m,n)\frac{\sin(2n+1)\pi}{2n+1}\left(x_0+ct+\frac{1}{2}at^2\right)\end{array}\right\}}{\left[1+\Gamma_0\left\{\begin{array}{l} R_a(k,m) + \sum_{n=0}^{\infty} R_b(k,m,n)\frac{\cos(2n+1)\pi}{2n+1}\left(x_0+ct+\frac{1}{2}at^2\right) \\ -\sum_{n=0}^{\infty} R_c(k,m,n)\frac{\sin(2n+1)\pi}{2n+1}\left(x_0+ct+\frac{1}{2}at^2\right)\end{array}\right\}\right]}Z_m(t)$$

$$+\frac{\Gamma_0\left[\begin{array}{l} aR_d(k,m) + a\sum_{n=0}^{\infty} R_e(k,m,n)\frac{\cos(2n+1)\pi}{2n+1}\left(x_0+ct+\frac{1}{2}at^2\right) \\ -\sum_{n=0}^{\infty} R_f(k,m,n)\frac{\sin(2n+1)\pi}{2n+1}\left(x_0+ct+\frac{1}{2}at^2\right)\end{array}\right]}{\left[1+\Gamma_0\left\{\begin{array}{l} R_a(k,m) + \sum_{n=0}^{\infty} R_b(k,m,n)\frac{\cos(2n+1)\pi}{2n+1}\left(x_0+ct+\frac{1}{2}at^2\right) \\ -\sum_{n=0}^{\infty} R_c(k,m,n)\frac{\sin(2n+1)\pi}{2n+1}\left(x_0+ct+\frac{1}{2}at^2\right)\end{array}\right\}\right]}Z_m(t)$$

$$= \frac{\frac{\Gamma_0 L^2 g}{k\pi\psi_{11}(k,m)}\left[-(-1)^k + \cos\frac{k\pi}{L}\left(x_0 + ct + \frac{1}{2}at^2\right)\right]}{\left[1 + \Gamma_0 + \left\{\begin{array}{l} R_a(k,m) + \sum_{n=0}^{\infty} R_b(k,m,n)\frac{\cos(2n+1)\pi}{2n+1}\left(x_0 + ct + \frac{1}{2}at^2\right) \\ -\sum_{n=0}^{\infty} R_c(k,m,n)\frac{\sin(2n+1)\pi}{2n+1}\left(x_0 + ct + \frac{1}{2}at^2\right)\end{array}\right\}\right]} \quad (43)$$

where

$$R_a(k,m) = \frac{L^2}{8\psi_{11}(k,m)}$$

$$R_b(k,m,n) = \frac{(2n+1)L^3}{2\psi_{11}(k,m)\pi}\left(\frac{(-1)^{m+k}\cos(2n+1)\pi L - 1}{[(2n+1)L]^2 - (m+k)^2} - \frac{(-1)^{m-k}\cos(2n+1)\pi L - 1}{[(2n+1)L]^2 - (m-k)^2}\right)$$

$$R_c(k,m,n) = \frac{(2n+1)L^3}{2\psi_{11}(k,m)\pi}\left(\frac{(-1)^{m-k}\sin(2n+1)\pi L}{[(2n+1)L]^2 - (m-k)^2} - \frac{(-1)^{m+k}\sin(2n+1)\pi L}{[(2n+1)L]^2 - (m+k)^2}\right)$$

$$R_d(k,m) = \frac{mkL}{\psi_{11}(k,m)(k^2 - m^2)}$$

$$R_e(k,m,n) = \frac{mL}{2\psi_{11}(k,m)\pi}\left(\frac{(m+k)(-1)^{m+k}\sin(2n+1)\pi L}{[(2n+1)L]^2 - (m+k)^2} - \frac{(m-k)(-1)^{m-k}\sin(2n+1)\pi L}{[(2n+1)L]^2 - (m-k)^2}\right)$$

$$R_f(k,m,n) = \frac{mL}{2\psi_{11}(k,m)\pi}\left((m+k)\frac{(-1)^{m+k}\cos(2n+1)\pi L - 1}{[(2n+1)L]^2 - (m+k)^2} - \frac{(m-k)(-1)^{m-k}\cos(2n+1)\pi L - 1}{[(2n+1)L]^2 - (m-k)^2}\right)$$

$$R_g = \frac{m^2\pi^2}{L^2}R_a(k,m); \quad R_h(k,m) = \frac{m^2\pi^2}{L^2}R_b(k,m,n)$$

$$R_i(k,m) = \frac{m^2\pi^2}{L^2}R_c(k,m,n) \quad (44)$$

Specifically, by means of this technique, one seeks the modified frequency corresponding to the frequency of the free system due to the presence of the moving mass. An equivalent free system operator defined by the modified frequency then replaces equation (43). Thus, we set the right hand side of (43) to zero and consider a parameter $\Gamma_1 < 1$ for any arbitrary mass ratio Γ_0 defined as

$$\Gamma_1 = \frac{\Gamma_0}{1 + \Gamma_0} \quad (45)$$

which implies that

$$\Gamma_0 = \Gamma_1 + O(\Gamma_1^2) \quad (46)$$

and

$$\frac{1}{1+\Gamma_0\left\{\begin{array}{l} R_a(k,m)+\sum_{n=0}^{\infty}R_b(k,m,n)\frac{\cos(2n+1)\pi}{2n+1}\left(x_0+ct+\frac{1}{2}at^2\right) \\ -\sum_{n=0}^{\infty}R_c(k,m,n)\frac{\sin(2n+1)\pi}{2n+1}\left(x_0+ct+\frac{1}{2}at^2\right)\end{array}\right\}}$$

$$= 1-\Gamma_0\left\{R_a(k,m)+\sum_{n=0}^{\infty}R_b(k,m,n)\frac{\cos(2n+1)\pi}{2n+1}\left(x_0+ct+\frac{1}{2}at^2\right)\right.$$

$$\left.-\sum_{n=0}^{\infty}R_c(k,m,n)\frac{\sin(2n+1)\pi}{2n+1}\left(x_0+ct+\frac{1}{2}at^2\right)+O(\Gamma_0^2)\right\} \quad (47)$$

$$\left|\Gamma_0\left\{\begin{array}{l} R_a(k,m)+\sum_{n=0}^{\infty}R_b(k,m,n)\frac{\cos(2n+1)\pi}{2n+1}\left(x_0+ct+\frac{1}{2}at^2\right) \\ -\sum_{n=0}^{\infty}R_c(k,m,n)\frac{\sin(2n+1)\pi}{2n+1}\left(x_0+ct+\frac{1}{2}at^2\right)+O(\Gamma_0^2)\end{array}\right\}\right|<1 \quad (48)$$

When Γ_1 is set to zero in equation (43), a situation corresponding to the case in which the inertia effect of the mass of the system is regarded as negligible is obtained. In such case, the solution is of the form

$$Z_m(t) = C_{mm}\cos\lfloor\Omega_{pp}t-\Phi_{mm}\rfloor \quad (49)$$

where C_{mm}, Ω_{mm} and Φ_{mm} are constants. Furthermore, as $\Gamma_1 < 1$, the Struble's technique requires that the solution of equation (43) be of the form

$$Z_m(t) = A(m,t)\cos[\Omega_{pp}t-\Phi_{pp}t]+\Gamma_1 Z_1(t)+O(\Gamma_1^2) \quad (50)$$

Where $A(m,t)$ and $\Phi(m,t)$ are slowly varying functions of time.

In order to obtain the modified frequency, equation (50) and its derivatives are substituted into the homogeneous part of equation (43). Thereafter, we extract only the variational part of the equation describing the behaviour of $A(m,t)$ and $\Phi(m,t)$ during the motion of the mass. Thus, making this substitution and neglecting terms that do not contribute to the variational equations we obtain

$$-2A(m,t)\Omega_{pp}\sin[\Omega_{pp}t-\phi(m,t)]+2A(m,t)\phi(m,t)\Omega_{pp}\cos[\Omega_{pp}t-\phi(m,t)]-c\Gamma_1 A(m,t)$$
$$\Omega_{pp}tR_d(k,m)\sin[\Omega_{pp}t-\phi(m,t)]-c\Gamma_1 A(m,t)\Omega_{pp}^2 R_a(k,m)\cos[\Omega_{pp}t-\phi(m,t)]$$
$$+c^2\Gamma_1 A(m,t)R_g(k,m)\cos[\Omega_{pp}t-\phi(m,t)]+a\Gamma_1 A(m,t)R_d(k,m)$$
$$\cos[\Omega_{pp}t-\phi(m,t)] \text{ to } O(\Gamma_1) \text{ only.} \quad (51)$$

The variational equations of our problem are obtained by setting coefficients of $\sin[\Omega_{pp}t - \phi(m,t)]$ and $\cos[\Omega_{pp}t - \phi(m,t)]$ in equation (51) to zero respectively. Thus, we have

$$-2A(m,t)\Omega_{pp} - c\Gamma_1 A(m,t)\Omega_{pp} R_d(k,m) = 0 \tag{52}$$

and

$$2A(m,t)\phi(m,t) - c\Gamma_1 A(m,t)\Omega_{pp}^2 R_a(k,m) + c^2\Gamma_1 A(m,t) R_g(k,m) + a\Gamma_1 A(m,t) R_d(k,m) = 0 \tag{53}$$

Solving equations (52) and (53) respectively, one obtains

$$A(m,t) = A_0 \mathrm{e}^{-\Gamma_1 c R_d(k,m)t} \tag{54}$$

and

$$\phi(m,t) = \frac{\Gamma_1}{2A}\left\{\Omega_{pp} R_a(k,m) - \left[\frac{c^2 R_g(k,m) + aR_d(k,m)}{\Omega_{pp}}\right]\right\} t + \Psi_m \tag{55}$$

where A_0 and Ψ_m are constants.
Therefore when the effect of the mass of the moving load is considered, the first approximation to the homogeneous system is

$$Z_m(t) = C_{mm}\cos[\Omega_{mm}t - \Phi_{mm}] \tag{56}$$

where

$$\Omega_{mm} = \Omega_{pp}\left\{1 - \frac{\Gamma_1}{2}\left[R_a(k,m) - \frac{\left[c^2 R_g(k,m) + aR_d(k,m)\right]}{\Omega_{pp}^2}\right]\right\} \tag{57}$$

To solve the non-homogeneous equation (43), the differential operator which acts on $z_m(t)$ is replaced by the equivalent free system operator defined by the modified frequency Ω_{mm}. That is

$$\ddot{Z}_m(t) + \Omega_{mm}^2 Z_m(t) = \frac{\Gamma_1 L^2 g}{k\pi\varphi_{11}}\left[-(-1)^k + \cos\frac{k\pi}{L}\left(x_0 + ct + \frac{1}{2}at^2\right)\right] \tag{58}$$

Clearly, equation (58) is analogous to equation (29). Thus, using similar argument as in case I, $Z_m(t)$ can be obtained and on inversion yields

$$V_n(x,t) = \sum_{m=1}^{\infty}\frac{\Gamma_1 L g\sqrt{\pi}}{\lambda_k\varphi_{11}\sqrt{2a}\tau(x)}\left\{\frac{\sin\Omega_{mm}t}{\Omega_{mm}}\left[\cos\left(\frac{b_1^2}{4a} - C_0\right)C\left(\frac{b_1 + 2at}{\sqrt{2\pi a}}\right) + \sin\left(\frac{b_1^2}{4a} - C_0\right)\right.\right.$$

$$S\left(\frac{b_1 + 2at}{\sqrt{2\pi a}}\right) + \cos\left(\frac{b_2^2}{4a} - C_0\right)C\left(\frac{b_2 + 2at}{\sqrt{2\pi a}}\right) + \sin\left(\frac{b_2^2}{4a} - C_0\right)S\left(\frac{b_2 + 2at}{\sqrt{2\pi a}}\right)$$

$$-\cos\left(\frac{b_1^2}{4a} - C_0\right)C\left(\frac{b_1}{\sqrt{2\pi a}}\right) - \sin\left(\frac{b_1^2}{4a} - C_0\right)S\left(\frac{b_1}{\sqrt{2\pi a}}\right) - \cos\left(\frac{b_2^2}{4a} - C_0\right)$$

$$C\left(\frac{b_2}{\sqrt{2\pi a}}\right) - \sin\left(\frac{b_2^2}{4a} - C_0\right) S\left(\frac{b_2}{\sqrt{2\pi a}}\right) - \frac{1}{2\Omega_{mm}}\left[\sin(\lambda_k + \Omega_{mm}t) - \sin(\lambda_k - \Omega_{mm}t)\right]$$

$$-\frac{\cos\Omega_{mm}t}{\Omega_{mm}}\left[\cos\left(\frac{b_1^2}{4a} - C_0\right) S\left(\frac{b_1 + 2at}{\sqrt{2\pi a}}\right) - \sin\left[\frac{b_1^2}{4a} - C_0\right] C\left(\frac{b_1 + 2at}{\sqrt{2\pi a}}\right)\right.$$

$$-\cos\left(\frac{b_2^2}{4a} - C_0\right) S\left(\frac{b_2 + 2at}{\sqrt{2\pi a}}\right) + \sin\left(\frac{b_2^2}{4a} - C_0\right) C\left(\frac{b_2 + 2at}{\sqrt{2\pi a}}\right) - \cos\left(\frac{b_1^2}{4a} - C_0\right)$$

$$S\left(\frac{b_1}{\sqrt{2\pi a}}\right) + \sin\left(\frac{b_1^2}{4a} - C_0\right) C\left(\frac{b_1}{\sqrt{2\pi a}}\right) + \cos\left(\frac{b_2^2}{4a} - C_0\right) S\left(\frac{b_2}{\sqrt{2\pi a}}\right)$$

$$-\sin\left(\frac{b_2^2}{4a} - C_0\right) C\left(\frac{b_2}{\sqrt{2\pi a}}\right) - \frac{\cos\lambda_k}{\Omega_{mm}} + \frac{1}{2\Omega_{mm}}\left[\cos(\lambda_k - \Omega_{mm}t)\right.$$
$$\left.\left.+\cos(\lambda_k\Omega_{mm}t)\right]\right\}\left(\sin\frac{m\pi x}{L}\right) \tag{59}$$

4. DISCUSSION OF THE ANALYTICAL SOLUTION

In studying undamped system such as this, the deflection of the beam may increase without bound. Equation (43) reveals clearly that the simply supported non-uniform Rayleigh beam on a bi-parametric subgrade and under a partially distributed moving force encounters a resonance effect when

$$\Omega pp = \frac{m\pi c_c}{L} \tag{60}$$

while (59) shows that the same beam under the action of moving mass experiences resonance when

$$\Omega_{mm} = \frac{m\pi c_c}{L} \tag{61}$$

$$\text{where} \quad \Omega_{mm} = \Omega_{pp}\left\{1 - \frac{\Gamma_1}{2}\left[R_a(k,m) - \frac{\left[c^2 R_g(k,m) + aR_d(k,m)\right]}{\Omega_{pp}^2}\right]\right\} \tag{62}$$

equating (61) and (62) imply that

$$\Omega_{pp} = \frac{\dfrac{m\pi c_c}{L}}{\left[1 - \dfrac{\Gamma_1}{2}\left(R_a(k,m) - \dfrac{\left[c^2 R_g(k,m) + aR_d(k,m)\right]}{\Omega_{pp}^2}\right)\right]} \tag{63}$$

It can be deduce from (60) and (63) that, for the same natural frequency, the critical velocity of a non-uniform simply supported Rayleigh beam traversed by partially distributed mass moving with variable velocity is smaller than that traversed by distributed moving force. Thus, resonance is reached earlier in moving distributed mass system than in the moving distributed force system.

5. NUMERICAL CALCULATIONS AND DISCUSSIONS

In order to illustrate the foregoing analysis, the non-uniform beam of length 12.129m is considered. Furthermore, $EI = 6.068 \times 10^6 m^3/s^2$ and $M/\mu = 0.25$. The values of foundation stiffness K between $0N/m^3$ and $4000000N/m^3$ are used. Axial force N is varied between $0N$ and $2 \times 10^6 N$ and shear modulus is varied between $0N/m$ and $3 \times 10^5 N/m$. The transverse deflections of the simply supported non-uniform Rayleigh beam are calculated and plotted against time for various values of foundation stiffness K, axial force N, shear modulus G and rotatory inertia correction factor r^0.

Figure 5.1 displays the deflection profile of the simply supported Rayleigh beam under the action of partially distributed moving force for various values of foundation stiffness K and fixed values of axial force N, shear modulus G and rotatory inertia correction factor r^0. The figure shows that as K increases the deflection of the non-uniform Rayleigh beam decreases. The same results is obtained when the simply supported Rayleigh beam is traversed by a partially distributed mass moving at variable speed as shown in figure 5.5. Also for various time t, the displacement of the beam for various values of N and fixed values of K, G and r^0 are shown in figure 5.2. It is shown that higher values of axial force reduce the displacement amplitudes of the beam. The same behaviour characterizes the deflection profile of the simply supported Rayleigh beam under partially distributed masses moving at variable velocity as shown in figure 5.6.

The effect of shear modulus on the transverse deflection of the

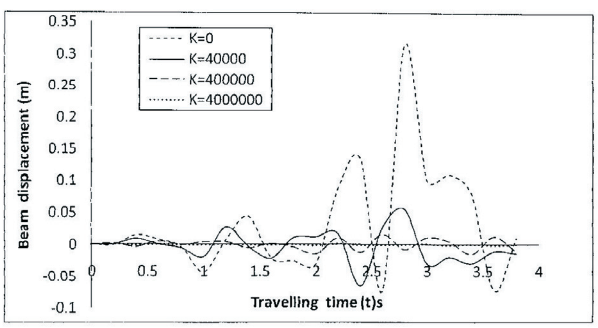

Fig.5.1: Deflection of moving force for simply supported non-uniform beam on Pasternak foundation for various values of foundation moduli K

Rayleigh beam under partially distributed moving force for fixed values of K, N and r^0 is displayed in figures 5.3. It is observed

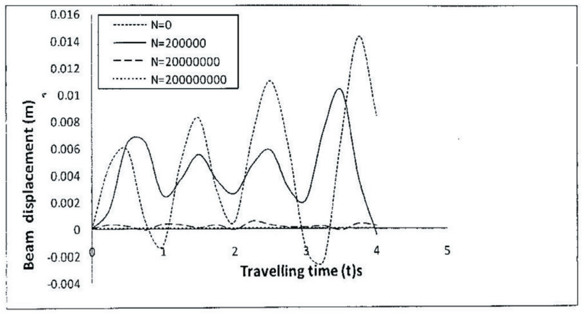

Fig.5.2: Deflection profile of moving force for simply supported non-uniform beam on Pasternak foundation for various values of Axial force N

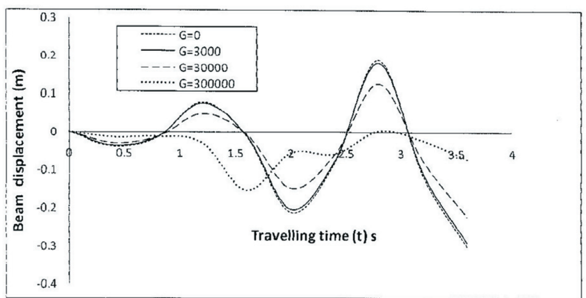

Fig.5.3: Deflection profile of moving force for simply supported non-uniform beam on Pasternak foundation for various values shear moduli G

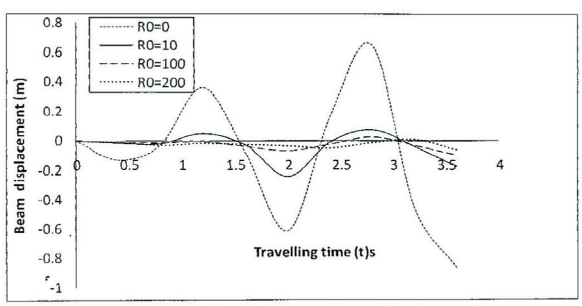

Fig.5.4: Deflection of moving force for simply supported non-uniform beam on Pasternak foundation for various values of rotatory inertia correction factor r^0

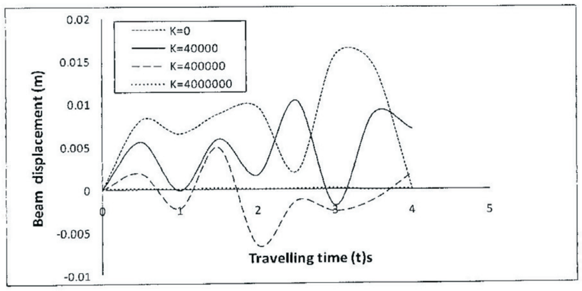

Fig.5.5: Deflection profile of moving mass for simply supported non-uniform beam on Pasternak foundation for various values of foundation moduli K

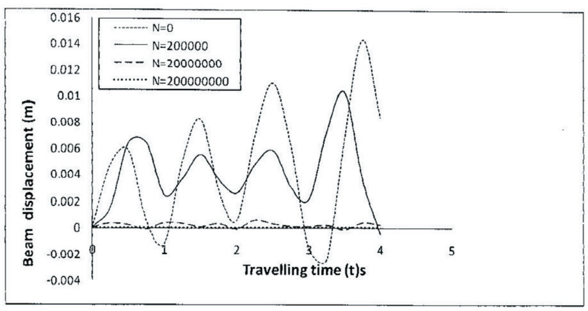

Fig.5.6: Deflection profile of moving mass for simply supported non-uniform beam on Pasternak foundation for various values of Axial force N

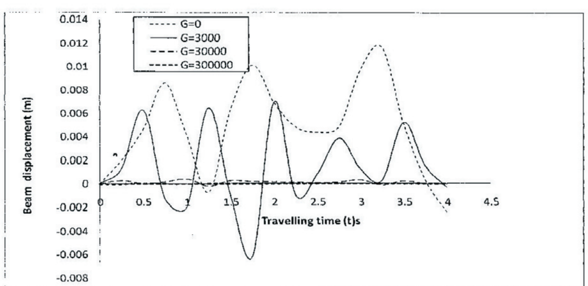

Fig.5.7: Deflection profile of moving mass for simply supported non-uniform beam on Pasternak foundation for various values of shear moduli G

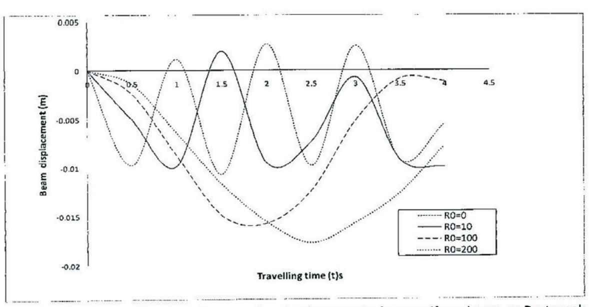

Fig.5.8: Deflection profile of moving mass for simply supported non-uniform beam on Pasternak foundation for various values of rotatory inertia correction factor r^0

that higher values of the shear modulus reduce the deflection of the dynamical system. Figure 5.7 depicts similar behaviour for the transverse displacement of the simply supported Rayleigh beam under the action of partially distributed masses. Figure 5.4 shows the

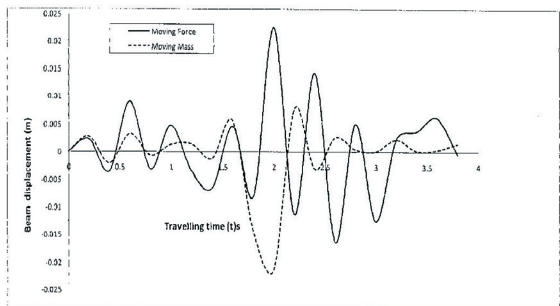

Fig.5.9: Comparison of the deflections of moving force and moving mass for simply supported non-uniform beam on Pasternak foundation K=40000, N=20000, G=30000 and r^0=0.5

deflection profile of the non-uniform Rayleigh beam traversed by a partially distributed moving force for various values of rotatory inertia correction factor r^0 and for fixed values of K, N and G. It is observed that higher values of rotatory inertia correction factor r^0 decrease the deflection of the beam. Clearly, figure 5.8 shows that as the values of rotatory inertia correction factor increases, the response amplitude of the non-uniform Rayleigh beam under the action of partially distributed masses travelling at variable velocity decreases. Finally, figure 5.9 compares the displacement curves of the partially distributed moving force and partially distributed moving mass for fixed $K = 40000, N = 200000, G = 100000$ and $r^0 = 0.5$. Clearly the response amplitude of the moving distributed force is greater than that of the moving distributed mass problem.

6. CONCLUDING REMARKS

This paper presents an analytical solution for the transverse displacement of a non-uniform Rayleigh beam on a bi-parametric subgrade and under partially distributed masses moving at varying velocities. The versatile method of Galerkin has been used to reduce the governing fourth order singular partial differential equation with variable coefficients to a sequence of second order ordinary differential equations with variable coefficients. This equation is treated using a modification of Struble's asymptotic techniques. The resulting second order ordinary differential equation is solved using the method of integral transformations. Numerical analyses are carried out and the results show the following:

(i) For the moving distributed force and moving distributed mass problems the response amplitudes of the beam traversed by distributed load moving with variable velocity decrease with an increase in the values of foundation stiffness K for fixed values of N, G and r^0.

(ii) Higher values of axial force N reduce the response amplitudes for both the moving force and moving mass problems.

(iii) The response amplitudes of the Rayleigh beam decrease with an increase in the values of shear modulus G for fixed values of K, N and r^0.

(iv) Greater values of the sub grade's shear modulus G and rotatory inertia r^0 for fixed values of foundation stiffness K, axial force N and shear modulus G are required for a noticeable effect on the response amplitudes due to moving force and moving mass in the vibrating system.

(v) As K increases, the response amplitude of the non-uniform Rayleigh beams decreases. However the effect of K is more noticeable than that of G.

(vi) For the problem of a non-uniform beam under the actions of a partially distributed load moving with variable velocity, the transverse displacement of the moving force is greater than that of moving mass. This result is at variance with the result in [5]. In particular, the reverse was the case. Hence inertia of the moving load must always be taken into consideration for accurate and safe assessment of the response to moving distributed load of elastic structural members.

Finally, for this dynamical system, for the same natural frequency, the critical speed for moving mass problem is smaller than that of the moving force problem. Hence, resonance is reached earlier in moving mass problem.

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