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OPTIMIZATION OF INVESTMENT RETURNS WITH N-STEP UTILITY FUNCTIONS

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ABSTRACT. In this paper, we examine different ways of allocating investments, maximizing and generating optimal wealth of investment returns with N-step utility functions; in an N period setting where the investor maximizes the expected utility of the terminal wealth in a stochastic market with different utility functions. The specific utility functions considered are negative exponential, logarithm, square root and power structures as the market state changes according to a Markov chain. The states of the market describe the prevailing economic, financial, social and other conditions that affect the deterministic parameters of the models using martingale approach to obtain the optimal solution. Thus, we determine the optimization strategies for investment returns in situations where investors at different utility functions could end up doubling or halving their stake. The performance of any utility function is determined by the ratio $q : q^{'}$ of the probability of rising to falling as well as the ratio $p : p^{'}$ of the risk neutral probability measure of rising to the falling.

Keywords and phrases: Markov Chain, Negative exponential, Logarithm, Square-root, Power utility functions.

1. INTRODUCTION

Portfolio management is a fundamental activity in our day-to-day life. It is an important activity in our society for households, pension fund managers, as well as for government debt managers. One has got a certain amount of money and tries to use it in such a way that one can draw the maximum possible utility from the results of the corresponding activities.

Thus, in simple mathematical formula can be put in any of the following equivalent form

$$a_i(1+R_i) = a_i x_i \tag{1}$$

$$a_i = \frac{a_i x_i}{1+R_i}$$

$$a_i R_i = a_i(x_i - 1)$$

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$$R_i = x_i - 1$$

(i= 1, 2, ..., n) where;
a_i = amount invested in security i
$a_i x_i$= investment returns
x_i= a non-negative random variable
R_i= the rate of return from investment i
Here a payment a_i returns an amount $a_i x_i$ after one period. The rate of return is that value R_i that makes the present value of the return equal to the initial payment.

2. PRELIMINARY

Definition 1: Utility function is a function that measures investor's preferences for wealth and the amount of risk they are willing to undertake in the hope of attaining greater wealth.
Thus, a utility function is a twice-differentiable function of wealth U(w) defined for $w > 0$ the first derivative $U'(w) > 0$ and the second derivative $U''(w) < 0$.
Markowitz H. (1952) is the pioneered of the mean-variance approach in a one-period decision model. It still has great importance in real life applications, and is widely applied in the risk management departments of banks. Merton (1971) considered as a pioneering point for the continuous-time portfolio management. He used stochastic control method to the asset allocation problem, and expressed optimal portfolio rule in terms of the solution of a second-order partial differential equation (PDE). He was to obtain explicit solution for special examples with the growing application of stochastic calculus to finance from the eighties, an alternative approach, the martingale method to portfolio optimization was developed by Pliska (1986), Karatzas eta l. (1987) and Cox and Huang (1989) based on martingale theory and convex optimization.

3. Martingale methods for N-step utility functions

Prices of assets depend crucially on their risk as investors typically demand more profit for bearing more uncertainty. Therefore, today's price of a claim on a risky amount realized tomorrow will generally differ from its expected value. Most commonly, investors are risk-averse and today's price is below the expectation, remunerating those who bear the risk.
In a financial market where investors are facing uncertainty, the return of an investment in assets is in general not known. A stock

yield depends on the resale price and the dividends. How to choose between several possible investments? In order to determine desirable strategies in an uncertain context, the preferences of the investor should be made explicit, and this is usually done in terms of expected utility criterion.
The pay-off for N period model, Y_i at time step i is given by

$$Y_i = 2^{2(n-i)}$$

with probability $\binom{2n}{i}(1-p)^{2n-i}p^i$; $i = 0, 1, \ldots, 2n$ for an even step or

$$Y_i = 2^{2(n-i)-1}$$

with probability $\binom{2n-1}{i}(1-p)^{2n-i-1}p^i$; $i = 0, \ldots, 2n-1$ for an odd step and for initial capital x with probability of increase and decrease q and p respectively and risk-neutral probability measure $q^{'}$.
The dynamic optimization problem above can be represented as a static optimization problem over terminal wealth:

$$V_0(x) = \sup_H E\left[U(H)\right] \tag{2}$$

subject to

$$E[H]^Q = x \tag{3}$$

where H denotes state, U the utility function and Q risk neutral probability.

Suppose we adapt a utility function of the Negative exponential utility function such that

$$U\left(h_i\right) = -e^{-\gamma h_i} \tag{4}$$

The utility function compares to a constant absolute risk aversion situation where γ is the error term.
The expected value $E[U(H)]$ for even time step L_e is given as

$$L_e = -\sum_{i=0}^{2n} \binom{2n}{i} (1-p)^{2n-i} p^i e^{-\gamma h_i} \tag{5}$$

subject to

$$\sum_{i=0}^{2n} \binom{2n}{i} \left(1-p^{'}\right)^{2n-i} p^{'i} h_i = x \tag{6}$$

$i = 0, \ldots, 2n$; and $p + q = 1$ and $p^{'} + q^{'} = 1$; where $q^{'}$ is the risk-neutral probability given as

$$q^{'} = \frac{(1+R)S - S^d}{S^u - S^d} \tag{7}$$

The initial stock price S can go either up to S^u or down to S^d. If the interest rate is $R > (0)$, we note that $S^d \leq (1+R)S \leq S^u$
The solution to our problem lies in maximizing L_e, the present wealth subject to the constraint, the terminal wealth. Adopting the Lagrangian method strategy, we differentiate L where

$$L = -\sum_{i=0}^{2n} \binom{2n}{i} (1-p)^{2n-i} p^i e^{-\gamma h_i} - \lambda \left(\sum_{i=0}^{2n} \binom{2n}{i} \left(1 - p^{'}\right)^{2n-i} p^{'i} h_i - x \right) \tag{8}$$

and equating to zero, we have

$$\frac{\delta L}{\delta h_i} = \gamma \binom{2n}{i} (1-p)^{2n-i} p^i e^{-\gamma h_i} - \lambda \binom{2n}{i} \left(1 - p^{'}\right)^{2n-i} p^{'i} = 0 \tag{9}$$

$$e^{-\gamma h_i} = \frac{\lambda}{\gamma} \left(\frac{1-p^{'}}{1-p}\right)^{2n-i} \left(\frac{p^{'}}{p}\right)^i$$

$$\lambda = \gamma e^{-\gamma h_i} \left(\frac{1-p}{1-p^{'}}\right)^{2n-i} \left(\frac{p}{p^{'}}\right)^i$$

$$\lambda = \gamma e^{-\gamma h_{i+1}} \left(\frac{1-p}{1-p^{'}}\right)^{2n-i-1} \left(\frac{p}{p^{'}}\right)^{i+1}$$

$$e^{-\gamma h_{i+1}} = e^{-\gamma h_i} \left(\frac{1-p}{1-p^{'}}\right) \left(\frac{p^{'}}{p}\right)$$

$$e^{-\gamma h_1} = e^{-\gamma h_0} \left(\frac{1-p}{1-p^{'}}\right) \left(\frac{p^{'}}{p}\right)$$

$$e^{-\gamma h_2} = e^{-\gamma h_0} \left(\frac{1-p}{1-p^{'}}\right)^2 \left(\frac{p^{'}}{p}\right)^2$$

$$e^{-\gamma h_{2n}} = e^{-\gamma h_0} \left(\frac{1-p}{1-p^{'}}\right)^{2n} \left(\frac{p^{'}}{p}\right)^{2n}.$$

Thus,

$$e^{-\gamma h_i} = \left[\frac{p^{'}(1-p)}{p(1-p^{'})}\right]^i e^{-\gamma h_0}$$

$$e^{-\gamma h_i + \gamma h_0} = \left[\frac{p^{'}(1-p)}{p(1-p^{'})}\right]^i.$$

Taking the logarithm of both sides and solving for h_i, we have

$$h_i = h_0 + \frac{1}{\gamma} log \left[\frac{p(1-p^{'})}{p^{'}(1-p)} \right]^i . \tag{10}$$

But differentiating L w.r.t. λ and equating to zero, we have

$$\frac{\delta L}{\delta \lambda} = \sum_{i=0}^{2n} \binom{2n}{i} (1-p^{'})^{2n-i} p^{'i} h_i - x = 0. \tag{11}$$

Substituting for h_i in Equation 11, we have

$$\sum_{i=0}^{2n} \binom{2n}{i} (1-p^{'})^{2n-i} p^{'i} \left[h_0 + \frac{1}{\gamma} log \left[\frac{p(1-p^{'})}{p^{'}(1-p)} \right]^i \right] - x = 0.$$

On simplifying, it is easy to see that

$$h_0 = x + \frac{2np^{'}}{\gamma} log \left[\frac{p^{'}(1-p)}{p(1-p^{'})} \right] \tag{12}$$

Clearly, we have

$$h_i^* = x + \frac{2np^{'}}{\gamma} log \left[\frac{p^{'}(1-p)}{p(1-p^{'})} \right] + \frac{i}{\gamma} log \left[\frac{p(1-p^{'})}{p^{'}(1-p)} \right] \tag{13}$$

Thus, the return on investment is given by

$$\frac{e^{-\gamma h_i^*}}{e^{-\gamma x}} = \left[\frac{p(1-p^{'})}{p^{'}(1-p)} \right]^{2np^{'}-i} \tag{14}$$

The Equation (13) gives the h_i corresponding to the exponential utility function for particular wealth x, and probabilities p and $p^{'}$ for a specific security. Thus, $\gamma = \frac{-U''(i)}{U'(i)}$ is guarantee to produce the maximal terminal wealth if the utility function is assumed to be negative exponential.

For the choice of a Logarithm utility function $U(h_i) = \ln h_i$, on application of the martingale method result in an optimal wealth given by

$$h_i^* = x \left[\frac{p}{p^{'}} \right]^i \left[\frac{1-p}{1-p^{'}} \right]^{2n-i} . \tag{15}$$

Hence, the ratio of h_i^* to x for an even N-step is given as

$$\left(\frac{h_i^*}{x} \right)^{\frac{1}{2}} = \left[\frac{p}{p^{'}} \right]^i \left[\frac{1-p}{1-p^{'}} \right]^{2n-i} \tag{16}$$

and for odd N-step

$$\left(\frac{h_i^*}{x}\right)^{\frac{1}{2}} = \left[\frac{p}{p'}\right]^i \left[\frac{1-p}{1-p'}\right]^{2n-i-1}. \tag{17}$$

A market with a square root utility function with even step structure given as

$$U(h_i) = h_i^{\frac{1}{2}}.$$

Let

$$L_e = \sum_{i=0}^{2n} \binom{2n}{i} p^i (1-p)^{2n-i} h_i^{\frac{1}{2}}$$

subject to

$$\sum_{i=0}^{2n} \binom{2n}{i} p'^i \left(1-p'\right)^{2n-i} h_i = x.$$

Thus, let

$$L = \sum_{i=0}^{2n} \binom{2n}{i} p^i (1-p)^{2n-i} h_i^{\frac{1}{2}} - \lambda \left(\sum_{i=0}^{2n} \binom{2n}{i} p'^i \left(1-p'\right)^{2n-i} h_i - x \right). \tag{18}$$

Hence, differentiating Equation 18 w.r.t. h_i and λ using Lagrangian multiplier and equating to zero, we have

$$\frac{\delta L}{\delta h_i} = \frac{1}{2} \binom{2n}{i} p^i (1-p)^{2n-i} h_i^{\frac{-1}{2}} - \lambda \binom{2n}{i} p'^i \left(1-p'\right)^{2n-i} = 0.$$

Thus,

$$h_i = \left[\frac{1}{2\lambda} (\frac{p}{p'})^i (\frac{1-p}{1-p'})^{2n-i}\right]^2 \tag{19}$$

and

$$h_{i+1} = h_i \left[\frac{p(1-p')}{p'(1-p)}\right]^2$$

Hence,

$$h_i = h_0 \left[\frac{p(1-p')}{p'(1-p)}\right]^{2i}. \tag{20}$$

Also,

$$\frac{\delta L}{\delta \lambda} = \sum_{i=0}^{2n} \binom{2n}{i} p'^i \left(1-p'\right)^{2n-i} h_i = x \tag{21}$$

substituting for h_i in in Equation 21, we have

$$\sum_{i=0}^{2n} \binom{2n}{i} p'^i (1-p')^{2n-i} h_0 \left[\frac{p(1-p')}{p'(1-p)}\right]^{2i} = x. \tag{22}$$

Equation 22 follows a moment generating function of a binomial distribution. Thus,

$$h_0 = x\left[\frac{(1-p)^2}{1-p^{'}}\right]^{2n}\left[\frac{p^{'}}{p^{'}(1-p)^2 + p^2(1-p^{'})}\right]^{2n} \tag{23}$$

Clearly,

$$h_i^* = x\left[\frac{p}{p^{'}}\right]^{2i}\left[\frac{1-p^{'}}{1-p}\right]^{2i}\left[\frac{(1-p)^2}{1-p^{'}}\right]^{2n}\left[\frac{p^{'}}{p^{'}(1-p)^2 + p^2(1-p^{'})}\right]^{2n} \tag{24}$$

$i = 0, \ldots, 2n$, $p + q = 1$ and $p^{'} + q^{'} = 1$.
Thus, an investor with intension of square root utility return with the initial capital x of probability of increase p and decrease q and $q^{'}$ being the risk-neutral probability measure; the return on investment with an even step utility function is given by

$$\frac{U(h)}{U(x)} = \left[\frac{h_i^*}{x}\right]^{1/2} = \left[\frac{p}{p^{'}}\right]^{i}\left[\frac{1-p^{'}}{1-p}\right]^{i}\left[\frac{(1-p)^2}{1-p^{'}}\right]^{n}\left[\frac{p^{'}}{p^{'}(1-p)^2 + p^2(1-p^{'})}\right]^{n} \tag{25}$$

The power utility function is a generalization of the square root utility function. Thus, for an investor with power utility function return we have,

$$U(h_i) = h_i^{\gamma}$$

for $0 < \gamma < 1$
The expected value $E[U(H)]$ for even L_e given by

$$L_e = \sum_{i=0}^{2n}\binom{2n}{i}(1-p)^{2n-i}p^i h_i^{\gamma}$$

subject to

$$\sum_{i=0}^{2n}\binom{2n}{i}(1-p^{'})^{2n-i}p^{'i}h_i = x \tag{26}$$

$i = 0, \ldots, 2n$ $p + q = 1$ and $p^{'} + q^{'} = 1$; where

$$q^{'} = \frac{(1+R)S - S^d}{S^u - S^d}.$$

Let

$$L = \sum_{i=0}^{2n}\binom{2n}{i}(1-p)^{2n-i}p^i h_i^{\gamma} - \lambda\left(\sum_{i}^{2n}\binom{2n}{i}(1-p^{'})^{2n-i}p^{'i}h_i - x\right). \tag{27}$$

Thus, solving by Lagrangian method and differentiating Equation 27 *w.r.t.* to h_i and equating to zero we have,

$$\frac{\delta L}{\delta h_i} = \gamma\binom{2n}{i}(1-p)^{2n-i}p^i h_i^{\gamma-1} - \lambda\binom{2n}{i}(1-p^{'})^{2n-i}p^{'i} = 0 \tag{28}$$

$$h_i = \left[\frac{\gamma}{\lambda}(\frac{p}{p'})^i(\frac{1-p}{1-p'})^{2n-i}\right]^{\frac{1}{1-\gamma}}$$

and

$$h_{i+1} = h_i\left[\frac{p(1-p^{'})}{p^{'}(1-p})\right]^{\frac{1}{1-\gamma}}$$

$$h_i = h_0\left[\frac{p(1-p^{'})}{p^{'}(1-p)}\right]^{\frac{i}{1-\gamma}} \tag{29}$$

and differentiating Equation 27 *w.r.t.* to λ and equating to zero we have

$$\frac{\delta L}{\delta\lambda} = \sum_{i=0}^{2n}\binom{2n}{i}(1-p^{'})^{2n-i}p^{'i}h_i = x. \tag{30}$$

Substituting for h_i in Equation 30 and solving for h_0, it is easy to see that

$$h_0 = x\left[\frac{(1-p)^{\frac{1}{1-\gamma}}}{1-p^{'}}\right]^{2n}\left[\frac{p^{\frac{'\gamma}{1-\gamma}}}{p^{\frac{'\gamma}{1-\gamma}}(1-p)^{\frac{1}{1-\gamma}}+p^{\frac{1}{1-\gamma}}(1-p^{\frac{'\gamma}{1-\gamma}})}\right]^{2n}. \tag{31}$$

Now, substituting Equation 31 into Equation 29, we have

$$h_i^* = x\left[\frac{p(1-p^{'})}{p^{'}(1-p)}\right]^{\frac{i}{1-\gamma}}\left[\frac{(1-p)^{\frac{1}{1-\gamma}}}{1-p^{'}}\right]^{2n}\left[\frac{p^{\frac{'\gamma}{1-\gamma}}}{p^{\frac{'\gamma}{1-\gamma}}(1-p)^{\frac{1}{1-\gamma}}+p^{\frac{1}{1-\gamma}}(1-p^{\frac{'\gamma}{1-\gamma}})}\right]^{2n} \tag{32}$$

Thus, an investor with intension of *power* utility return with the initial capital x of probability of increase p and decrease q and $q^{'}$ being the risk-neutral probability measure; the return on investment with an even step utility function is given by

$$\frac{U(h)}{U(x)} = \left[\frac{h_i^*}{x}\right]^{\gamma}$$

$$= \left[\frac{p(1-p^{'})}{p^{'}(1-p)}\right]^{\frac{i\gamma}{1-\gamma}}\left[\frac{(1-p)^{\frac{1}{1-\gamma}}}{1-p^{'}}\right]^{2n\gamma}\left[\frac{p^{\frac{'\gamma}{1-\gamma}}}{p^{\frac{'\gamma}{1-\gamma}}(1-p)^{\frac{1}{1-\gamma}}+p^{\frac{1}{1-\gamma}}(1-p^{\frac{'\gamma}{1-\gamma}})}\right]^{2n\gamma}. \tag{33}$$

4. CONCLUDING REMARKS

Results were obtained for various models such as negative exponential model, logarithm, power and square root models.

This study concluded that the optimization of our initial wealth is determined by the ratio $q : q^{'}$ of the probability of rising to falling as well as the ratio $p : p^{'}$ of the risk neutral probability measure of rising to the falling.

Various allocations of wealth with different Utility functions were established. The utility function considered are negative exponential, logarithm, square root and power utility functions. The N-step utility model results show the ratio of the utility functions at time point i in comparison with the initial starting time. The allocation of wealth with different utility model depends on the amount of risk an investor is willing to bear at each trading period. The ROI with different utility models shown that the Negative exponential utility model gave the best allocation of wealth.
In the subsequent paper, the model will be used to predict the performances of some selected companies in the Nigeria Capital Market.

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