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MODELLING OF QUALITY ENHANCEMENT SYSTEM FOR QUEUING SITUATION IN WIRELESS COMMUNICATION NETWORK

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ABSTRACT. In this study, we considered queuing in wireless communication network system with two different arrival points S_1 and S_2 linked with a service point S_3 which locates two different departure points S_4 and S_5. The stochastic processes of the arrival follow a Poisson distribution while the stochastic process of departure follows exponential distribution. Various probability functions derived were reduced to steady state equations. By using the generation functions techniques, we obtained the operating characteristics like the expected number of calls in the system, in queue and the expected time spent.

Keywords and phrases: Third grade fluid, PDE, Sphere.

2010 Mathematical Subject Classification: A80

1. INTRODUCTION

In recent years, as noted by Kalden et al.(2000), mobile network face exponential traffic increase and growing importance to users. In a country like ours, the number of mobile subscription has recently exceeded the number of fixed lines. The growth in awareness and usage of the mobile networks is a major issue as regards congestion in all the communication networks. Internet technology has emerged as the major driving force behind the new development in the area of telecommunication networks. The volume of packets data transfer has increased at extreme rates. In order to meet this challenging traffic platform, more and more network operators adopt their strategies and plan to migrate to IP-base backbone networks.

Furthermore, the growth of the internet and success of the mobile networks suggest that the next trend will be an increasing demand for access to internet application. It is, therefore, increasingly important that mobile networks support these applications in

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an efficient manner. Thus, mobile radio systems currently under develpment include supports for packets data sevices. The widely deployed standard for third-generation mobile radio network is the global system for mobile communication (GSM). Networks based on this standard have now been extended with general pcckets radio service (GPRS). Valberg et al.(2007) noted that wireless technology can now allow delivery of voice, text, images, etc. and relies on an extensive network base station to do this. However, when a user of a wireless phone makes a call, it is transmitted to the nearest sub-station which receives and transmits radio signals in its area thus acting as two way radio. The area covered by a base station is called a cell, which is why wireless phones are sometimes called cell or cellular phones. The largest cells are known as macrocells, smaller cells particularly in urban area are called microcells or Picocells. The number of cell varies in different area, depending on the number of users. Area with high volume of mobile phone user will have more cells. As the phone users move around, the radio signals can be switched from one cell to another, maintaining a good connection. The cell is connected with the local phone network and delivered by phone lines using a base antenna or when cell is going to another wireless phone by radio signals. Each base station caovers a limited radius. Beyond that, the power density becomes too small for the mobile to work and when this happens, the phone is automatically switched to a close base station. The schematic representation of the network's cellular structure is shown in figure 1. Fig.1 Schematic Representation of Network's Cellular Structure

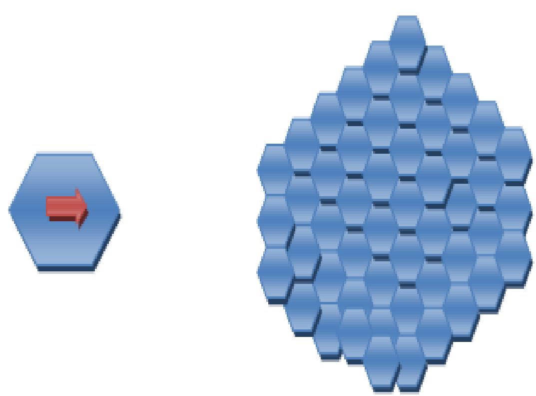

Fig. 1. Schematic Representation of Network's Cellular Structure

On the left side of the structure, is the base station which represents the main server and on the right hand side is the position of the cells. Each Hexagon structure represents the radius covered by an antenna which enables individual to make and receive calls within the radius. The antennas are linked with the base station. The major issue about these networks is their performance. Bersekas et al (1992) viewed the communication link as a pipe over which a given number of bits per seconds can be transmitted. The number is called transmission capacity of the link. It depends on both the physical channel and the interface (e.g. virtual circuits or group of virtual circuits) multiplexed on the link. The manner of allocation of capacity among these traffic streams has profound effect on packet delay.

Packets of all traffic streams are merged into a single queue and trasmitted on a first-come first-served basis. A variation of this scheme, which has roughly the same average delay per packets, maintains a separate queue for each at a time. However, if the queue of traffic stream is empty, the next traffic stream is served and no communication resource is wasted.

According to Walke et al. (2007), the properties analytically modeled using probability theory. On the lower level, the characteristics of a wireless communication link are modeled using hidden markov model, matrix probability, moment generating moment, among others. The analysis of the queueing system of packet can be described as Poisson queues. Lucey (1992) stated that queue form when the rate of arrival of items requiring services is greater than the rate of services. The imbalance may only be temporary but the temporary imbalance period, a queue will form. Generally, queue may consist of people, car component awaiting machine, telephone cells, and aeroplane indeed any discrete items. A que system can be divided into four elements namely: Arrivals, Queue, Service, and Outlet.

Arrival is the element concerned with how items (people, component, cars etc.) arrival in the system. Generally, when dealing with queueing problems, we are concerned with the rate of arrival or the time gap between arrivals, which amount to the same thing.

Queueing is the element concerned with what happens between the arrival of items requiring service and the time when the service is carried out. The manner in which the customer are being rendered service could be the first come first served (e.g. in a computer stack or a short buffer in a production line), random order, and priorities (e.g. rush orders first shortest processing time first).

Service is the element concerned with the time taken to serve a customer. The time may be constant (e.g. a machine process) or

more likely, it may vary. Service times may be reduced by better training, more personnel, more or better equipment, all of which increase cost. Typically, a queueing problem involves striking a balance between the cost of making reduction in service time and the benefit to be gained from such a reduction (Lucey 1992).
Outlet is the exit from the system. Generally this factor can be ignored but on occasions. This may influence service and/or arrival times. Amongst the wide applicability notwithstanding, the most successful and the most important application of queuing models has been and continues to be telecommunications, the arrival process consists of the generation of calls or messages. Calla and messages are stored in a buffer prior to transmission over telephone lines. The call and message handling capacity of these lines is a key element in determination of the performance of the system.
According to Bersekas et al (1992), in the context of a data network, customers represent packets assignment to a communication link for transmission. Service time corresponding to the transmission and is equal to $\frac{L}{C}$ where L is the length in bits and C is the link transmission capacity in bit/sec. In a somewhat different context, customers represent ongoing conversation between point in a network and service times correspond to the duration of the conversation. In a related context, customer represent active calls in a telephone or circuit switched network and duration of the call. As noted in Ikpotokin (2003), queuing system is classified into four main category namely;

- Single channel with single server (SCSS)
- Single channel with multiple servers (SCMS)
- Multiple channel with single server (MCSS)
- Multiple channel with multiple servers (MCMS)

A combination of the multiple channel and single channel queues is a network of queues is a network of queues. As noted in White et al (1975), it is important to consider a simple model in the analysis of queuing system. On many different varieties of queuing problems, a queue that is relatively simple model is the $(M/M/I)$ queue. Following the identifying scheme as noted by Bersekas et al (1992), the $M/M/I$ queuing system consists of a single queuing station with a single server, (in a communication context, a single transmission line). Customers arrives according to a Poisson process with rate λ and the probability distribution of the service time is exponential with mean $\frac{1}{\mu}sec.$ the name $M/M/C$ reflects standard queuing theory nomenclatures whereby: first letter represents the arrival process.

- M – Memory less (i.e Poisson process)
- G – General distribution of inter arrival time.
- D – Deterministic inter arrival time

Second letter represent the service time distribution

- M – Exponential
- G – General Distribution
- D – Deterministic inter arrival

Third letter represent number of service channels.

The $M/M/m$ queuing system is identical to the $M/M/I$ system except that there are m servers (or channels of transmission lines in data communication context). A customer at the head of the queue is routed to any server that is available. In the case of $M/M/m/m$ (The m-server loss system) where if an arrival finds all servers busy, it does not enter the system and is lost instead, the last m in the $M/M/m/m$ notation indicates the limit on the number of customers in the system, model is in wide use I telephony (also work generally, in circuit switched networks) In the context, customers in the system correspond to active telephone conversation and the m servers represent a single transmission line consisting of m circuits. The average service time $\frac{1}{\mu}$ is the average duration of a telephone conversation. The principal quantity of interest here is the steady state probability that all circuit are busy, in which case, an arriving call is refused service. Note that in an $M/M/m/m$ based model, the assumption is that blocked calls are lost (not reattempted) in data networks, the $M/M/m/m$ system can be used as model where arrivals correspond to request for virtual circuit connection between two nodes and the maximum number of virtual circuit allowed is m.

There has recently been considerable interests in the primary task of a wireless communication network architecture as a provision as well as utilization of network resources efficiently to satisfy the demands imposed on it, in this regard, Kanchan et al (2009) studied Active Queue Management (AQM) for wireless networks, Unlike wired link which is assumed to have a fixed capacity, a wireless link has a capacity that is time varying due to multipath fading and mobility. Thus, the controller is required to meet performance objectives in the presence of these capacity variations. They proposed a robust controller design that maintains the queue length close to an operating point. They treated capacity variations as an

external disturbance and design a robust controller using H1 control techniques. They also consider the effect of round trip time in their model. Their method of incorporating the delay into the discredited model simplifies controller design by allowing direct use of systematic controller design methods and/or design packages.
Jain et al (2009) develop a finite queueing model having single and batch service modes for telecommunication system, where two types of traffic, i.e voice and data arrive in Poisson fashion. The server performs service singly until there are C packet in the system. After then type 2 packets are discarded an all type 1 packets are served in a batch. The arrival rates of packets depend upon the server's status. The transient state probabilities of system states are obtained by solving a set of linear equations with the help of Laplace Transform technique. Performance indices such as average queue length expected idle time, and expected busy period are determined. They also investigated the optimal value of threshold parameter N and C after which the server changes the node of service in order to minimize the expected the expected cost.
Yaroslavtev et al (2010) presented the structured queuing network as effective mathematical model of telecommunication system; this model is oriented on the analysis of delay in telecommunication networks. A set of special component were identified for providing structurization of such models. QoS parameters: the average delivery time and the utilization coefficient were calculated for MPEG video LAN implementing structured queuing model. In their work, they propose the use of structured queuing networks, which is more adequate than conventional analytical models, and expends less computational time than simulation models.
Our specific objectives of this research are to:

- Consider the case $m = 2$ and $n = 2$
- Obtain the expected number of customers in the system;
- Obtain the expected number of customers waiting;
- Obtain the expected amount of time spent in the system;
- Obtain the expected amount of time spent waiting;
- Obtain the probability of delay

In this regard, as the main contribution of this paper, we introduced a framework for analyzing telecommunication networks. An electronic solver was developed for speedy implementation of the operating characteristics of the model. For the purpose of this study, based on wireless communication network involving mobile communication systems, we present the figure below.

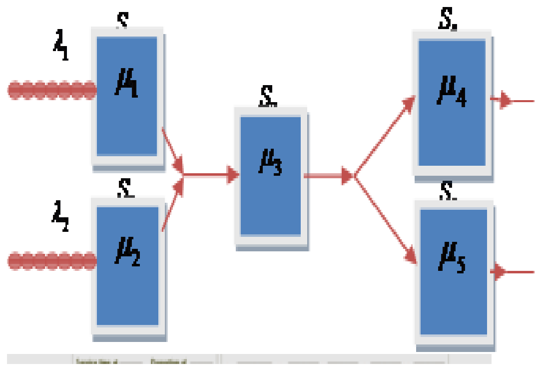

Fig. 2. A Model of a Queuing Process of Packets in Wireless Communication System

Fig. 2 represents a simple structure with two different cell units S_1 and S_2 connected to a base station S_3. Calls from different callers arrive at S_1 or S_2 depending on the one within the caller's cell radius. In either case, the call is connected to base station S_3 which in turn detects the cell unit closest to the receiver at S_4 or S_5 and the signal is so transmitted. In this setting, we assume that the stochastic process of arrival follows a Poisson distribution while stochastic process of service follows an exponential distribution. By using generating function techniques on the steady state equations derived, the measures of performance of the system are to be obatined.

1

1.1. **Research Method.** To carry out our research satisfactorily,

- We applied the generating function to the steady state probability of the wireless network and mobile communication system with multi-server queueing station linked from M to N parallel servers.
- We use L'Hopital rule to reolve the intermediate form of any equation during the solution process to obtaining the operating characteristics of the system.

1.2. **Problem Formulation and Solution.** A simple structure of the system is shown in the Figure 2. Here, S_1 represents the first substation which transmits incoming calls within its radius to the base station represented by S_3, similarly, S_2 is another substation which also transmits incoming calls within its radius to the same base station which in turn locates the receiver either at substation

S_4 or S_5. It is assumed that the caller and the receiver are both not within the radius of one of the substation, that is, the caller is confined to either S_1 or S_2 while the receiver is confined to either S_4 or S_5.

The important notations used in this work are:

- $n_i(i = 1, 2, 3, 4, 5)$, is the number of customers in the ith queue. n_1 and n_2 represent the arriving customers at cell S_1 and S_2 respectively, n_3 is the number of customer awaiting service at the base station S_3 having gone through S_1 and S_2 while n_4 and n_5 are the departing customer at S_4 and S_5 respectively.
- $\lambda_i(i = 1, 2)$, is the rate at which customers arrive at the ith queue.
- $\mu_i(i = 1, 2, 3, 4, 5)$, is the service rate for customers at the ith queue.
- $\alpha_i(i = 1, 2)$ is the proportion of customers that goes through S_4 or S_5 depending on the substation the receiver is within it's radius.
- $P(n_1, n_2, n_3, n_4, n_5; t)$, is the transient state probability that n_1, n_2, n_3, n_4, n_5 calls are in the same queue at time t.
- $P(n_1, n_2, n_3, n_4, n_5)$, is the steady state probability.

The others that are not yet known but need to be determined are

- $L_i(i = 1, 2, 3, 4)$, the expected number of customer in the ith phase.
- $L_q(q = 1, 2, 3, 4)$, the expected number of customers in the queue.
- W_s, the expected waiting time per queue.
- $D_i(i = 1, 2, 3, 4)$, the probability that an entering customer will be delayed in ith phase.

2

2.1. **Steady State Equations.** The Steady State Equations of the model are:

$$(\lambda_1+\lambda_2+\mu_1+\mu_2+\mu_3+\mu_4+\mu_5)P(n_1,n_2,n_3,n_4,n_5) = \lambda_1 P(n_1-1,n_2,n_3,n_4,n_5)$$
$$+\lambda_2 P(n_1,n_2-1,n_3,n_4,n_5)+\mu_1 P(n_1+1,n_2,n_3-1,n_4,n_5)+\mu_2 P(n_1,n_2+1,n_3-1,n_4,n_5)$$
$$+\mu_3 P(n_1,n_2,n_3+1,n_4-1,n_5)+\mu_3 P(n_1,n_2,n_3,n_4,n_5-1)+\mu_4 P(n_1,n_2,n_3,n_4+1,n_5)$$
$$+\mu_5 P(n_1,n_2,n_3,n_4,n_5+1) \tag{1}$$

$$(\lambda_1+\lambda_2+\mu_1+\mu_2+\mu_3+\mu_4+\mu_5)P(0,n_2,n_3,n_4,n_5) = \lambda_2 P(n_1,n_2-1,n_3,n_4,n_5)$$
$$+\mu_1 P(1,n_2,n_3-1,n_4,n_5)+\mu_2 P(0,n_2+1,n_3-1,n_4,n_5)+\mu_3 P(0,n_2,n_3+1,n_4-1,n_5)$$
$$+\mu_3 P(0,n_2,n_3+1,n_4,n_5-1)+\mu_4 P(0,n_2,n_3,n_4+1,n_5)+\mu_5 P(0,n_2,n_3,n_4,n_5+1) \tag{2}$$

$$(\lambda_1+\lambda_2+\mu_1+\mu_2+\mu_3+\mu_4+\mu_5)P(n_1,0,n_3,n_4,n_5) = \lambda_1 P(n_1-1,0,n_3,n_4,n_5)$$
$$+\mu_1 P(n_1+1,n_2,n_3-1,n_4,n_5)+\mu_2 P(n_1,n_2+1,n_3-1,n_4,n_5)+\mu_3 P(n_1,0,n_3+1,n_4-1,n_5)$$

$$+\mu_3 P(n_1,0,n_3+1,n_4,n_5-1)+\mu_4 P(n_1,0,n_3,n_4+1,n_5)+\mu_5 P(n_1,0,n_3,n_4,n_5+1) \quad (3)$$

$$\begin{aligned}(\lambda_1+\lambda_2+\mu_1+\mu_2+\mu_3+\mu_4+\mu_5)P(n_1,n_2,0,n_4,n_5) &= \lambda_1 P(n_1-1,n_2,0,n_4,n_5)\\ +\lambda_2 P(n_1,n_2-1,0,n_4,n_5)+\mu_3 P(n_1,n_2,1,n_4-1,n_5)&+\mu_3 P(n_1,n_2,1,n_4,n_5-1)\\ +\mu_4 P(n_1,n_2,0,n_4+1,n_5)&+\mu_5 P(n_1,n_2,0,n_4,n_5+1)\end{aligned} \quad (4)$$

$$\begin{aligned}(\lambda_1+\lambda_2+\mu_1+\mu_2+\mu_3+\mu_4+\mu_5)P(n_1,n_2,n_3,0,n_5) = \lambda_1 P(n_1-1,n_2,n_3,0,n_5)\\ +\lambda_2 P(n_1,n_2-1,n_3,0,n_5)+\mu_1 P(n_1+1,n_2,n_3-1,0,n_5)+\mu_2 P(n_1,n_2+1,n_3-1,0,n_5)\\ +\mu_3 P(n_1,n_2,n_3+1,0,n_5-1)+\mu_4 P(n_1,n_2,n_3,1,n_5)+\mu_5 P(n_1,n_2,n_3,0,n_5+1)\end{aligned} \quad (5)$$

$$\begin{aligned}(\lambda_1+\lambda_2+\mu_1+\mu_2+\mu_3+\mu_4+\mu_5)P(n_1,n_2,n_3,n_4,0) = \lambda_1 P(n_1-1,n_2,n_3,n_4,0)\\ +\lambda_2 P(n_1,n_2-1,n_3,n_4,0)+\mu_1 P(n_1+1,n_2,n_3-1,n_4,0)+\mu_2 P(n_1,n_2+1,n_3-1,n_4,0)\\ +\mu_3 P(n_1,n_2,n_3+1,n_4-1,0)+\mu_4 P(n_1,n_2,n_3,n_4+1,0)+\mu_5 P(n_1,n_2,n_3,n_4,1)\end{aligned} \quad (6)$$

$$\begin{aligned}(\lambda_1+\lambda_2+\mu_1+\mu_2+\mu_3+\mu_4+\mu_5)P(0,0,n_3,n_4,n_5) = \mu_1 P(1,0,n_3-1,n_4,n_5)\\ +\mu_2 P(0,1,n_3-1,n_4,n_5)+\mu_3 P(0,0,n_3+1,n_4-1,n_5)+\mu_3 P(0,0,n_3+1,n_4,n_5-1)\\ +\mu_4 P(0,0,n_3,n_4+1,n_5)+\mu_5 P(0,0,n_3,n_4,n_5+1)\end{aligned} \quad (7)$$

$$\begin{aligned}(\lambda_1+\lambda_2+\mu_1+\mu_2+\mu_3+\mu_4+\mu_5)P(0,n_2,0,n_4,n_5) = \lambda_2 P(0,n_2-1,0,n_4,n_5)\\ +\mu_3 P(0,n_2,1,n_4-1,n_5)+\mu_3 P(0,n_2,1,n_4,n_5-1)\\ +\mu_4 P(0,n_2,0,n_4+1,n_5)+\mu_5 P(0,n_2,0,n_4,n_5+1)\end{aligned} \quad (8)$$

$$\begin{aligned}(\lambda_1+\lambda_2+\mu_1+\mu_2+\mu_3+\mu_4+\mu_5)P(0,n_2,n_3,0,n_5) = \lambda_2 P(0,n_2-1,n_3,0,n_5)\\ +\mu_1 P(1+1,n_2,n_3-1,0,n_5)+\mu_2 P(0,n_2+1,n_3-1,0,n_5)\\ +\mu_3 P(0,n_2,n_3+1,0,n_5-1)+\mu_4 P(0,n_2,n_3,1,n_5)+\mu_5 P(0,n_2,n_3,0,n_5+1)\end{aligned} \quad (9)$$

$$\begin{aligned}(\lambda_1+\lambda_2+\mu_1+\mu_2+\mu_3+\mu_4+\mu_5)P(0,n_2,n_3,n_4,0) = \lambda_2 P(0,n_2-1,n_3,n_4,0)\\ +\mu_1 P(1,n_2,n_3-1,n_4,0)+\mu_2 P(0,n_2+1,n_3-1,n_4,0)\\ +\mu_3 P(0,n_2,n_3+1,n_4-1,0)+\mu_4 P(0,n_2,n_3,n_4+1,0)+\mu_5 P(0,n_2,n_3,n_4,1)\end{aligned} \quad (10)$$

$$\begin{aligned}(\lambda_1+\lambda_2+\mu_1+\mu_2+\mu_3+\mu_4+\mu_5)P(n_1,0,0,n_4,n_5) = \lambda_1 P(n_1-1,0,0,n_4,n_5)\\ +\mu_3 P(n_1,0,1,n_4-1,n_5)+\mu_3 P(n_1,0,1,n_4,n_5-1)\\ +\mu_4 P(n_1,0,0,n_4+1,n_5)+\mu_5 P(n_1,0,0,n_4,n_5+1)\end{aligned} \quad (11)$$

$$\begin{aligned}(\lambda_1+\lambda_2+\mu_1+\mu_2+\mu_3+\mu_4+\mu_5)P(n_1,0,n_3,0,n_5) = \lambda_1 P(n_1-1,0,n_3,0,n_5)\\ +\mu_1 P(n_1+1,0,n_3-1,0,n_5)+\mu_2 P(n_1,1,n_3-1,0,n_5)\\ +\mu_3 P(n_1,0,n_3+1,0,n_5-1)+\mu_4 P(n_1,0,n_3,1,n_5)+\mu_5 P(n_1,0,n_3,0,n_5+1)\end{aligned} \quad (12)$$

$$\begin{aligned}(\lambda_1+\lambda_2+\mu_1+\mu_2+\mu_3+\mu_4+\mu_5)P(n_1,0,n_3,n_4,0) = \lambda_1 P(n_1-1,0,n_3,n_4,0)\\ +\mu_1 P(n_1+1,0,n_3-1,n_4,0)+\mu_2 P(n_1,1,n_3-1,n_4,0)\\ +\mu_3 P(n_1,0,n_3+1,n_4-1,0)+\mu_4 P(n_1,0,n_3,n_4+1,0)+\mu_5 P(n_1,0,n_3,n_4,1)\end{aligned} \quad (13)$$

$$\begin{aligned}(\lambda_1+\lambda_2+\mu_1+\mu_2+\mu_3+\mu_4+\mu_5)P(n_1,n_2,0,0,n_5) &= \lambda_1 P(n_1-1,n_2,0,0,n_5)\\ &+\lambda_2 P(n_1,n_2-1,0,0,n_5)+\mu_3 P(n_1,n_2,1,0,n_5-1)\\ &+\mu_4 P(n_1,n_2,0,1,n_5)+\mu_5 P(n_1,n_2,0,0,n_5+1)\end{aligned} \tag{14}$$

$$\begin{aligned}(\lambda_1+\lambda_2+\mu_1+\mu_2+\mu_3+\mu_4+\mu_5)P(n_1,n_2,0,n_4,0) &= \lambda_1 P(n_1-1,0,n_3,n_4,0)\\ &+\lambda_2 P(n_1,n_2-1,0,n_4,0)+\mu_3 P(n_1,n_2,1,n_4-1,0)\\ &+\mu_4 P(n_1,n_2,0,n_4+1,0)+\mu_5 P(n_1,n_2,0,n_4,1)\end{aligned} \tag{15}$$

$$\begin{aligned}(\lambda_1+\lambda_2+\mu_1+\mu_2+\mu_3+\mu_4+\mu_5)P(n_1,n_2,n_3,0,0) &= \lambda_1 P(n_1-1,n_2,n_3,0,0)\\ &+\lambda_2 P(n_1,n_2-1,n_3,0,0)+\mu_1 P(n_1+1,n_2,n_3-1,0,0)\\ &+\mu_2 P(n_1,n_2+1,n_3-1,0,0)+\mu_4 P(n_1,n_2,n_3,1,0)+\mu_5 P(n_1,n_2,n_3,0,1)\end{aligned} \tag{16}$$

$$\begin{aligned}(\lambda_1+\lambda_2+\mu_1+\mu_2+\mu_3+\mu_4+\mu_5)P(n_1,n_2,0,0,0) &= \lambda_1 P(n_1-1,n_2,0,0,0)\\ &+\lambda_2 P(n_1,n_2-1,0,0,0)+\mu_4 P(n_1,n_2,0,1,0)+\mu_5 P(n_1,n_2,0,0,1)\end{aligned} \tag{17}$$

$$\begin{aligned}(\lambda_1+\lambda_2+\mu_1+\mu_2+\mu_3+\mu_4+\mu_5)P(n_1,0,n_3,0,0) &= \lambda_1 P(n_1-1,0,n_3,0,0)\\ &+\mu_1 P(n_1+1,0,n_3-1,0,0)+\mu_2 P(n_1,1,n_3-1,0,0)\\ &+\mu_4 P(n_1,0,n_3,1,0)+\mu_5 P(n_1,0,n_3,0,1)\end{aligned} \tag{18}$$

$$\begin{aligned}(\lambda_1+\lambda_2+\mu_1+\mu_2+\mu_3+\mu_4+\mu_5)P(n_1,0,0,n_4,0) &= \lambda_1 P(n_1-1,0,0,n_4,0)\\ &+\mu_3 P(n_1,0,1,n_4-1,0)+\mu_4 P(n_1,0,0,n_4+1,0)+\mu_5 P(n_1,0,0,n_4,1)\end{aligned} \tag{19}$$

$$\begin{aligned}(\lambda_1+\lambda_2+\mu_1+\mu_2+\mu_3+\mu_4+\mu_5)P(n_1,0,0,0,n_5) &= \lambda_1 P(n_1-1,0,0,0,n_5)\\ &+\mu_3 P(n_1,0,1,0,n_5-1)+\mu_4 P(n_1,0,0,1,n_5)+\mu_5 P(n_1,0,0,0,n_5+1)\end{aligned} \tag{20}$$

$$\begin{aligned}(\lambda_1+\lambda_2+\mu_1+\mu_2+\mu_3+\mu_4+\mu_5)P(0,n_2,n_3,0,0) &= \lambda_2 P(0,n_2-1,n_3,0,0)\\ &+\mu_1 P(1,n_2,n_3-1,0,0)+\mu_2 P(0,n_2+1,n_3-1,0,0)\\ &+\mu_4 P(0,n_2,n_3,1,0)+\mu_5 P(0,n_2,n_3,0,1)\end{aligned} \tag{21}$$

$$\begin{aligned}(\lambda_1+\lambda_2+\mu_1+\mu_2+\mu_3+\mu_4+\mu_5)P(0,n_2,0,n_4,0) &= \lambda_2 P(0,n_2-1,0,n_4,0)\\ &+\mu_3 P(0,n_2,1,n_4-1,0)+\mu_4 P(0,n_2,0,n_4+1,0)+\mu_5 P(0,n_2,0,n_4,1)\end{aligned} \tag{22}$$

$$\begin{aligned}(\lambda_1+\lambda_2+\mu_1+\mu_2+\mu_3+\mu_4+\mu_5)P(0,n_2,0,0,n_5) &= \lambda_2 P(0,n_2-1,0,0,n_5)\\ &+\mu_3 P(0,n_2,1,0,n_5-1)+\mu_4 P(0,n_2,0,1,n_5)+\mu_5 P(0,n_2,0,0,n_5+1)\end{aligned} \tag{23}$$

$$\begin{aligned}(\lambda_1+\lambda_2+\mu_1+\mu_2+\mu_3+\mu_4+\mu_5)P(0,0,n_3,n_4,n_5) &= \mu_1 P(1,0,n_3-1,n_4,0)\\ &+\mu_2 P(0,n1,n_3-1,n_4,0)+\mu_3 P(0,0,n_3+1,n_4-1,0)\\ &+\mu_4 P(0,0,n_3,n_4+1,0)+\mu_5 P(0,0,n_3,n_4,1)\end{aligned} \tag{24}$$

$$\begin{aligned}(\lambda_1+\lambda_2+\mu_1+\mu_2+\mu_3+\mu_4+\mu_5)P(0,0,n_3,0,n_5) &= \mu_1 P(1,0,n_3-1,0,n_5)\\ &+\mu_2 P(0,1,n_3-1,0,n_5)+\mu_3 P(0,0,n_3,0,n_5-1)\end{aligned}$$

$$+\mu_4 P(0,0,n_3,1,n_5)+\mu_5 P(0,0,n_3,0,n_5+1) \tag{25}$$

$$(\lambda_1+\lambda_2+\mu_1+\mu_2+\mu_3+\mu_4+\mu_5)P(0,0,0,n_4,n_5)=\mu_3 P(0,0,1,n_4-1,n_5)$$
$$+\mu_3 P(0,0,1,n_4,n_5-1)+\mu_4 P(0,0,0,n_4+1,n_5)+\mu_5 P(0,0,0,n_4,n_5+1) \tag{26}$$

$$(\lambda_1+\lambda_2+\mu_1+\mu_2+\mu_3+\mu_4+\mu_5)P(n_1,0,0,0,0)=$$
$$\lambda_1 P(n_1-1,0,0,0,0)+\mu_4 P(n_1,0,0,1,0)+\mu_5 P(n_1,0,0,0,1) \tag{27}$$

$$(\lambda_1+\lambda_2+\mu_1+\mu_2+\mu_3+\mu_4+\mu_5)P(0,n_2,0,0,0)=$$
$$\lambda_2 P(0,n_2-1,0,0,0)+\mu_4 P(0,n_2,0,1,0)+\mu_5 P(0,n_2,0,0,1) \tag{28}$$

$$(\lambda_1+\lambda_2+\mu_1+\mu_2+\mu_3+\mu_4+\mu_5)P(0,0,n_3,0,0)=\mu_1 P(1,0,n_3-1,0,0)$$
$$+\mu_2 P(0,1,n_3-1,0,0)+\mu_4 P(0,0,n_3,1,0)+\mu_5 P(0,0,n_3,0,1) \tag{29}$$

$$(\lambda_1+\lambda_2+\mu_1+\mu_2+\mu_3+\mu_4+\mu_5)P(0,0,0,n_4,0)=\mu_3 P(0,0,1,n_4-1,0)$$
$$+\mu_4 P(0,0,0,n_4+1,0)+\mu_5 P(0,0,0,n_4,1) \tag{30}$$

$$(\lambda_1+\lambda_2+\mu_1+\mu_2+\mu_3+\mu_4+\mu_5)P(0,0,0,0,n_5)=$$
$$\mu_3 P(0,0,0,0,n_5-1)+\mu_4 P(0,0,0,1,n_5)+\mu_5 P(0,0,0,0,n_5+1) \tag{31}$$

$$(\lambda_1+\lambda_2+\mu_1+\mu_2+\mu_3+\mu_4+\mu_5)P(0,0,0,0,0)=$$
$$\mu_4 P(0,0,0,1,0)+\mu_5 P(0,0,0,0,1) \tag{32}$$

2.2. **Generating Functions.**

$$A(n_2,n_3,n_4,n_5;x)=\sum_{n_1=0}^{\infty} P(n_1,n_2,n_3,n_4,n_5)x^{n_1}, \tag{33}$$

$$\begin{aligned} B(n_3,n_4,n_5;x,y) &= \sum_{n_1=0}^{\infty} P(n_2,n_3,n_4,n_5;x)y^{n_2} \\ C(n_1,n_3,n_4,n_5;y) &= \sum_{n_2=0}^{\infty} P(n_1,n_2,n_3,n_4,n_5)y^{n_2} \end{aligned} \tag{34}$$

$$D(n_4,n_5;x,y,z)=\sum_{n_3=0}^{\infty} P(n_3,n_4,n_5)z^{n_3}$$

$$\begin{aligned} E(n_1,n_4,n_5;y,z) &= \sum_{n_3=0}^{\infty} C(n_3,n_4,n_5;y)z^{n_3} \\ F(n_2,n_4,n_5;x,z) &= \sum_{n_3=0}^{\infty} A(n_2,n_3,n_4,n_5;x)z^{n_3} \end{aligned} \tag{35}$$

$$G(n_1,n_2,n_4,n_5;y,z)=\sum_{n_3=0}^{\infty} P(n_1,n_2,n_3,n_4,n_5)z^{n_3}$$

$$H(n_5;x,y,z,s) = \sum_{n_4=0}^{\infty} D(n_4,n_5;x,y,z)s^{n_4}$$

$$I(n_1,n_5;y,z,s) = \sum_{n_4=0}^{\infty} E(n_1,n_4,n_5;y,z)s^{n_4}$$

$$J(n_2,n_5;x,z,s) = \sum_{n_4=0}^{\infty} F(n_2,n_4,n_5;x,z)s^{n_4} \tag{36}$$

$$K(n_1,n_2,n_5;z,s) = \sum_{n_4=0}^{\infty} G(n_1,n_2,n_4,n_5;z)s^{n_4}$$

$$L(n_3,n_5;x,y,s) = \sum_{n_4=0}^{\infty} B(n_3,n_4,n_5;x,y)s^{n_4}$$

$$M(x,y,z,s,w) = \sum_{n_5=0}^{\infty} D(n_5;x,y,z,s)w^{n_5}$$

$$N(n_1;y,z,s,w) = \sum_{n_5=0}^{\infty} I(n_1,n_5;y,z,s)w^{n_5}$$

$$O(n_2;x,z,s,w) = \sum_{n_5=0}^{\infty} J(n_2,n_5;x,z,s)w^{n_5}$$

$$Q(n_1,n_2;z,s,w) = \sum_{n_5=0}^{\infty} K(n_1,n_2,n_5;z,s)w^{n_5} \tag{37}$$

$$R(n_3;x,y,s,w) = \sum_{n_5=0}^{\infty} K(n_3,n_5;x,y,s)w^{n_5}$$

$$T(n_4;x,y,z,w) = \sum_{n_5=0}^{\infty} D(n_4,n_5;x,y,z)w^{n_5}$$

2.3. **Solution Procedure.** Multiply (1) by x^{n_1} and summing over n_1 from 1 to ∞ using (2) and (33), we have:

$$(\lambda_1+\lambda_2+\mu_1+\mu_2+\mu_3+\mu_4+\mu_5)A(n_2,n_3,n_4,n_5;x) - P(0,n_2,n_3,n_4,n_5;x) =$$
$$x\lambda_1 A(n_2,n_3,n_4,n_5;x) + \lambda_2 A(n_2-1,n_3,n_4,n_5;x) + \frac{\mu_1}{x}A(n_2,n_3-1,n_4,n_5;x)$$
$$-\frac{\mu_1}{x}A(0,n_2,n_3-1,n_4,n_5;x) + \mu_2 A(n_2+1,n_3-1,n_4,n_5;x) + \mu_3 A(n_2,n_3+1,n_4-1,n_5;x)$$
$$+\mu_3 A(n_2,n_3+1,n_4,n_5-1;x) + \mu_4 A(n_2,n_3,n_4+1,n_5;x) + \mu_5 A(n_2,n_3,n_4,n_5+1;x) \tag{38}$$

Multiply (3) by x^{n_1} and summing over n_1 from 1 to ∞ using (7) and (33), we have:

$$(\lambda_1+\lambda_2+\mu_1+\mu_2+\mu_3+\mu_4+\mu_5)A(0,n_3,n_4,n_5;x) - P(0,0,n_3,n_4,n_5;x) =$$

$$x\lambda_1 A(0,n_3,n_4,n_5;x)+\frac{\mu_1}{x}A(0,n_3-1,n_4,n_5;x)-\frac{\mu_1}{x}A(0,0,n_3-1,n_4,n_5;x)$$
$$+\mu_2 A(1,n_3-1,n_4,n_5;x)+\mu_3 A(0,n_3+1,n_4-1,n_5;x)+\mu_3 A(0,n_3+1,n_4,n_5-1;x)$$
$$+\mu_4 A(0,n_3,n_4+1,n_5;x)+\mu_5 A(0,n_3,n_4,n_5+1;x) \tag{39}$$

Multiply (4) by x^{n_1} and summing over n_1 from 1 to ∞ using (8) and (33), we have:

$$(\lambda_1+\lambda_2+\mu_1+\mu_2+\mu_3+\mu_4+\mu_5)A(n_2,0,n_4,n_5;x)-P(0,n_2,0,n_4,n_5;x)=$$
$$x\lambda_1 A(n_2,0,n_4,n_5;x)+\lambda_2 A(n_2-1,0,n_4,n_5;x)+\mu_3 A(n_2,1,n_4-1,n_5;x)$$
$$+\mu_3 A(n_2,1,n_4,n_5-1;x)+\mu_4 A(n_2,0,n_4+1,n_5;x)+\mu_5 A(n_2,0,n_4,n_5+1;x) \tag{40}$$

Multiply (5) by x^{n_1} and summing over n_1 from 1 to ∞ using (9) and (33), we have:

$$(\lambda_1+\lambda_2+\mu_1+\mu_2+\mu_3+\mu_4+\mu_5)A(n_2,n_3,0,n_5;x)-P(0,n_2,n_3,0,n_5;x)=$$
$$x\lambda_1 A(n_2,n_3,0,n_5;x)+\lambda_2 A(n_2-1,n_3,0,n_5;x)+\frac{\mu_1}{x}A(n_2,n_3-1,0,n_5;x)$$
$$-\frac{\mu_1}{x}A(0,n_2,n_3-1,0,n_5;x)+\mu_2 A(n_2+1,n_3-1,0,n_5;x)+\mu_3 A(n_2,n_3+1,0,n_5-1;x)$$
$$+\mu_4 A(n_2,n_3,1,n_5;x)+\mu_5 A(n_2,n_3,0,n_5+1;x) \tag{41}$$

Multiply (6) by x^{n_1} and summing over n_1 from 1 to ∞ using (10) and (33), we have:

$$(\lambda_1+\lambda_2+\mu_1+\mu_2+\mu_3+\mu_4+\mu_5)A(n_2,n_3,n_4,0;x)-P(0,n_2,n_3,n_4,0;x)=$$
$$x\lambda_1 A(n_2,n_3,n_4,0;x)+\lambda_2 A(n_2-1,n_3,n_4,0;x)+\frac{\mu_1}{x}A(n_2,n_3-1,n_4,0;x)$$
$$-\frac{\mu_1}{x}A(0,n_2,n_3-1,n_4,0;x)+\mu_2 A(n_2+1,n_3-1,n_4,0;x)+\mu_3 A(n_2,n_3+1,n_4-1,0;x)$$
$$+\mu_4 A(n_2,n_3,n_4+1,0;x)+\mu_5 A(n_2,n_3,n_4,1;x) \tag{42}$$

Multiply (11) by x^{n_1} and summing over n_1 from 1 to ∞ using (26) and (33), we have:

$$(\lambda_1+\lambda_2+\mu_1+\mu_2+\mu_3+\mu_4+\mu_5)A(0,0,n_4,n_5;x)-P(0,0,0,n_4,n_5;x)=$$
$$x\lambda_1 A(0,0,n_4,n_5;x)+\mu_3 A(0,1,n_4-1,n_5;x)+\mu_3 A(0,1,n_4,n_5-1;x)$$
$$+\mu_4 A(0,0,n_4+1,n_5;x)+\mu_5 A(0,0,n_4,n_5+1;x) \tag{43}$$

Multiply (12) by x^{n_1} and summing over n_1 from 1 to ∞ using (25) and (33), we have:

$$(\lambda_1+\lambda_2+\mu_1+\mu_2+\mu_3+\mu_4+\mu_5)A(0,n_3,0,n_5;x)-P(0,0,n_3,0,n_5;x)=$$
$$x\lambda_1 A(0,n_3,0,n_5;x)+\frac{\mu_1}{x}A(0,n_3-1,0,n_5;x)-\frac{\mu_1}{x}A(0,0,n_3-1,0,n_5;x)$$
$$+\mu_2 A(1,n_3-1,0,n_5;x)+\mu_3 A(0,n_3+1,0,n_5-1;x)+\mu_4 A(0,n_3,1,n_5;x)$$
$$+\mu_5 A(0,n_3,0,n_5+1;x) \tag{44}$$

Multiply (13) by x^{n_1} and summing over n_1 from 1 to ∞ using (24) and (33), we have:

$$(\lambda_1+\lambda_2+\mu_1+\mu_2+\mu_3+\mu_4+\mu_5)A(0,n_3,n_4,0;x)-P(0,0,n_3,n_4,0;x)=$$
$$x\lambda_1 A(0,n_3,n_4,0;x)+\frac{\mu_1}{x}A(0,n_3-1,n_4,0;x)-\frac{\mu_1}{x}A(0,0,n_3-1,n_4,0;x)$$
$$+\mu_2 A(1,n_3-1,n_4,0;x)+\mu_3 A(0,n_3+1,n_4-1,0;x)+\mu_4 A(0,n_3,n_4+1,0;x)$$
$$+\mu_5 A(0,n_3,n_4,1;x) \tag{45}$$

Multiply (14) by x^{n_1} and summing over n_1 from 1 to ∞ using (23) and (33), we have:

$$(\lambda_1+\lambda_2+\mu_1+\mu_2+\mu_3+\mu_4+\mu_5)A(n_2,0,0,n_5;x)-P(0,n_2,0,0,n_5;x)=$$
$$x\lambda_1 A(n_2,0,0,n_5;x)+\lambda_2 A(n_2-1,0,0,n_5;x)+\mu_3 A(n_2,1,0,n_5-1;x)$$
$$+\mu_4 A(n_2,0,1,n_5;x)+\mu_5 A(n_2,0,0,n_5+1;x) \tag{46}$$

Multiply (15) by x^{n_1} and summing over n_1 from 1 to ∞ using (22) and (33), we have:

$$(\lambda_1+\lambda_2+\mu_1+\mu_2+\mu_3+\mu_4+\mu_5)A(n_2,0,n_4,0;x)-P(0,n_2,0,n_4,0;x)=$$
$$x\lambda_1 A(n_2,0,n_4,0;x)+\lambda_2 A(n_2-1,0,n_4,0;x)+\mu_3 A(n_2,1,n_4-1,0;x)$$
$$+\mu_4 A(n_2,0,n_4+1,0;x)+\mu_5 A(n_2,0,n_4,1;x) \tag{47}$$

Multiply (16) by x^{n_1} and summing over n_1 from 1 to ∞ using (21) and (33), we have:

$$(\lambda_1+\lambda_2+\mu_1+\mu_2+\mu_3+\mu_4+\mu_5)A(n_2,n_3,0,0;x)-P(0,n_2,n_3,0,0;x)=$$
$$x\lambda_1 A(n_2,n_3,0,0;x)+\lambda_2 A(n_2-1,n_3,0,0;x)+\frac{\mu_1}{x}A(n_2,n_3-1,0,0;x)$$
$$-\frac{\mu_1}{x}A(0,n_2,n_3-1,0,0;x)+\mu_2 A(n_2+1,n_3-1,0,0;x)+\mu_4 A(n_2,n_3,1,0;x)$$
$$+\mu_5 A(n_2,n_3,0,1;x) \tag{48}$$

Multiply (17) by x^{n_1} and summing over n_1 from 1 to ∞ using (28) and (33), we have:

$$(\lambda_1+\lambda_2+\mu_1+\mu_2+\mu_3+\mu_4+\mu_5)A(n_2,0,0,0;x)-P(0,n_2,0,0,0;x)=$$
$$x\lambda_1 A(n_2,0,0,0;x)+\lambda_2 A(n_2-1,0,0,0;x)+\mu_4 A(n_2,0,1,0;x)$$
$$+\mu_5 A(n_2,0,0,1;x) \tag{49}$$

Multiply (18) by x^{n_1} and summing over n_1 from 1 to ∞ using (30) and (33), we have:

$$(\lambda_1+\lambda_2+\mu_1+\mu_2+\mu_3+\mu_4+\mu_5)A(0,n_3,0,0;x)-P(0,0,n_3,0,0;x)=$$
$$x\lambda_1 A(0,n_3,0,0;x)+\frac{\mu_1}{x}A(0,n_3-1,0,0;x)-\frac{\mu_1}{x}A(0,0,n_3-1,0,0;x)$$
$$+\mu_2 A(1,n_3-1,0,0;x)+\mu_4 A(0,n_3,1,0;x)+\mu_5 A(0,n_3,0,1;x) \tag{50}$$

Multiply (19) by x^{n_1} and summing over n_1 from 1 to ∞ using (30) and (33), we have:

$$(\lambda_1+\lambda_2+\mu_1+\mu_2+\mu_3+\mu_4+\mu_5)A(0,0,n_4,0;x)-P(0,0,0,n_4,0;x)=$$
$$x\lambda_1 A(0,0,n_4,0;x)+\mu_3 A(0,1,n_4-1,0;x)$$
$$+\mu_4 A(0,0,n_4+1,0;x)+\mu_5 A(0,0,n_4,1;x) \tag{51}$$

Multiply (20) by x^{n_1} and summing over n_1 from 1 to ∞ using (31) and (33), we have:

$$(\lambda_1+\lambda_2+\mu_1+\mu_2+\mu_3+\mu_4+\mu_5)A(0,0,0,n_5;x)-P(0,0,0,0,n_5;x)=$$
$$x\lambda_1 A(0,0,0,n_5;x)+\mu_3 A(0,1,0,n_5-1;x)$$
$$+\mu_4 A(0,0,1,n_5;x)+\mu_5 A(0,0,0,n_5+1;x) \tag{52}$$

Multiply (21) by x^{n_1} and summing over n_1 from 1 to ∞ using (31) and (33), we have:

$$(\lambda_1+\lambda_2+\mu_1+\mu_2+\mu_3+\mu_4+\mu_5)A(0,0,0,0;x)-P(0,0,0,0,0;x)=$$
$$x\lambda_1 A(0,0,0,0;x)+\mu_4 A(0,0,1,0;x)+\mu_5 A(0,0,0,1;x) \tag{53}$$

Multiply (38) by y^{n_2} and summing over n_2 from 1 to ∞ using (39) and (34), we have:

$$\begin{aligned}(\lambda_1+\lambda_2+\mu_1+\mu_2+\mu_3+\mu_4+\mu_5)B(n_3,n_4,n_5;x,y)-\mu_2 C(n_1,n_3,n_4,n_5;y)\\ -\mu_1 A(n_2,n_3,n_4,n_5;x)=x\lambda_1 B(n_3,n_4,n_5;x,y)+y\lambda_2 A(n_3,n_4,n_5;x,y)\\ +\frac{\mu_1}{x}B(n_3-1,n_4,n_5;x,y)-\frac{\mu_1}{x}C(0,n_3-1,n_4,n_5;y)+\frac{\mu_1}{x}P(0,0,n_3-1,n_4,n_5)\\ +\frac{\mu_2}{y}B(n_3-1,n_4,n_5;x,y)+\frac{\mu_2}{y}A(0,n_3-1,n_4,n_5;x)+\mu_3 B(n_3+1,n_4-1,n_5;x,y)\\ +\mu_3 B(n_3+1,n_4,n_5-1;x,y)+\mu_4 B(n_3,n_4+1,n_5;x,y)+\mu_5 B(n_3,n_4,n_5+1;x,y)\end{aligned}\tag{54}$$

Multiply (40) by y^{n_2} and summing over n_2 from 1 to ∞ using (43) and (34), we have:

$$\begin{aligned}(\lambda_1+\lambda_2+\mu_1+\mu_2+\mu_3+\mu_4+\mu_5)B(0,n_4,n_5;x,y)-\mu_2 C(0,0,n_4,n_5;y)\\ -\mu_1 A(0,0,n_4,n_5;x)=x\lambda_1 B(0,n_4,n_5;x,y)+y\lambda_2 A(0,n_4,n_5;x,y)\\ +\mu_3 B(1,n_4-1,n_5;x,y)+\mu_3 B(1,n_4,n_5-1;x,y)\\ +\mu_4 B(0,n_4+1,n_5;x,y)+\mu_5 B(0,n_4,n_5+1;x,y)\end{aligned}\tag{55}$$

Multiply (41) by y^{n_2} and summing over n_2 from 1 to ∞ using (44) and (34), we have:

$$\begin{aligned}(\lambda_1+\lambda_2+\mu_1+\mu_2+\mu_3+\mu_4+\mu_5)B(n_3,0,n_5;x,y)-\mu_2 C(0,n_3,0,n_5;y)\\ -\mu_1 A(0,n_3,0,n_5;x)=x\lambda_1 B(n_3,0,n_5;x,y)+y\lambda_2 A(n_3,0,n_5;x,y)\\ +\frac{\mu_1}{x}B(n_3-1,0,n_5;x,y)-\frac{\mu_1}{x}C(0,n_3-1,0,n_5;y)+\frac{\mu_1}{x}P(0,0,n_3-1,0,n_5)\\ +\frac{\mu_2}{y}B(n_3-1,0,n_5;x,y)+\frac{\mu_2}{y}A(0,n_3-1,0,n_5;x)+\mu_3 B(n_3+1,0,n_5-1;x,y)\\ +\mu_4 B(n_3,1,n_5;x,y)+\mu_5 B(n_3,0,n_5+1;x,y)\end{aligned}\tag{56}$$

Multiply (42) by y^{n_2} and summing over n_2 from 1 to ∞ using (45) and (34), we have:

$$\begin{aligned}(\lambda_1+\lambda_2+\mu_1+\mu_2+\mu_3+\mu_4+\mu_5)B(n_3,n_4,0;x,y)-\mu_2 C(0,n_3,n_4,0;y)\\ -\mu_1 A(0,n_3,n_4,0;x)=x\lambda_1 B(n_3,n_4,0;x,y)+y\lambda_2 A(n_3,n_4,0;x,y)\\ +\frac{\mu_1}{x}B(n_3-1,n_4,0;x,y)-\frac{\mu_1}{x}C(0,n_3-1,n_4,0;y)+\frac{\mu_1}{x}P(0,0,n_3-1,n_4,0)\\ +\frac{\mu_2}{y}B(n_3-1,n_4,0;x,y)+\frac{\mu_2}{y}A(0,n_3-1,n_4,0;x)+\mu_3 B(n_3+1,n_4-1,0;x,y)\\ +\mu_4 B(n_3,n_4+1,0;x,y)+\mu_5 B(n_3,n_4,1;x,y)\end{aligned}\tag{57}$$

Multiply (46) by y^{n_2} and summing over n_2 from 1 to ∞ using (53) and (34), we have:

$$\begin{aligned}(\lambda_1+\lambda_2+\mu_1+\mu_2+\mu_3+\mu_4+\mu_5)B(0,0,n_5;x,y)-\mu_2 C(0,0,0,n_5;y)\\ -\mu_1 A(0,0,0,n_5;x)=x\lambda_1 B(0,0,n_5;x,y)+y\lambda_2 A(0,0,n_5;x,y)\\ +\mu_3 B(1,0,n_5-1;x,y)+\mu_4 B(0,1,n_5;x,y)+\mu_5 B(0,0,n_5+1;x,y)\end{aligned}\tag{58}$$

Multiply (47) by y^{n_2} and summing over n_2 from 1 to ∞ using (52) and (34), we have:

$$\begin{aligned}(\lambda_1+\lambda_2+\mu_1+\mu_2+\mu_3+\mu_4+\mu_5)B(0,n_4,0;x,y)-\mu_2 C(0,0,n_4,0;y)\\ -\mu_1 A(0,0,n_4,0;x)=x\lambda_1 B(0,n_4,0;x,y)+y\lambda_2 A(0,n_4,0;x,y)\\ +\mu_3 B(1,n_4-1,0;x,y)+\mu_4 B(0,n_4+1,0;x,y)+\mu_5 B(0,n_4,1;x,y)\end{aligned}\tag{59}$$

Multiply (49) by y^{n_2} and summing over n_2 from 1 to ∞ using (51) and (34), we have:

$$(\lambda_1+\lambda_2+\mu_1+\mu_2+\mu_3+\mu_4+\mu_5)B(n_3,0,0;x,y)-\mu_2 C(0,n_3,0,0;y)$$

$$-\mu_1 A(0,n_3,0,0;x) = x\lambda_1 B(n_3,0,0;x,y) + y\lambda_2 A(n_3,0,0;x,y) + \frac{\mu_1}{x}B(n_3-1,0,0;x,y) - \frac{\mu_1}{x}C(0,n_3-1,0,0;y) + \frac{\mu_1}{x}P(0,0,n_3-1,0,0) + \frac{\mu_2}{y}B(n_3-1,0,0;x,y) + \frac{\mu_2}{y}A(0,n_3-1,0,0;x) + \mu_4 B(n_3,1,0;x,y) + \mu_5 B(n_3,0,1;x,y) \tag{60}$$

Multiply (50) by y^{n_2} and summing over n_2 from 1 to ∞ using (54) and (34), we have:

$$(\lambda_1+\lambda_2+\mu_1+\mu_2+\mu_3+\mu_4+\mu_5)B(0,0,0;x,y) - \mu_2 C(0,0,0,0;y) - \mu_1 A(0,0,0,0;x) = x\lambda_1 B(0,0,0;x,y) + y\lambda_2 A(0,0,0;x,y) + \mu_4 B(0,0,1,0;x,y) + \mu_5 B(0,0,1;x,y) \tag{61}$$

Multiply (55) by z^{n_3} and summing over n_3 from 1 to ∞ using (56) and (35), we have:

$$(\lambda_1+\lambda_2+\mu_1+\mu_2+\mu_3+\mu_4+\mu_5)D(n_4,n_5;x,y,z) - \mu_2 E(0,n_4,n_5;y,z) - \mu_1 F(1,0,n_5;x,z) = x\lambda_1 D(n_4,n_5;x,y,z) + y\lambda_2 D(n_4,n_5;x,y,z) + \frac{z\mu_1}{x}D(n_4,n_5;x,y,z) - \frac{z\mu_1}{x}E(0,n_4,n_5;y,z) + \frac{z\mu_1}{x}G(0,0,0,n_4,n_5;z) + \frac{z\mu_2}{y}D(n_4,n_5;x,y,z) - \frac{\mu_2}{y}F(1,n_4,n_5;x,z) + \frac{\mu_3}{z}D(n_4-1,n_5;x,y,z) - \frac{\mu_3}{z}B(0,n_4-1,n_5;x,y) + \frac{\mu_3}{z}D(n_4,n_5-1;x,y,z) - \frac{\mu_3}{z}D(0,n_4,n_5-1;x,y) + \mu_4 D(n_4+1,n_5;x,y,z) + \mu_5 D(n_4,n_5+1;x,y,z) \tag{62}$$

Multiply (57) by z^{n_3} and summing over n_3 from 1 to ∞ using (59) and (35), we have:

$$(\lambda_1+\lambda_2+\mu_1+\mu_2+\mu_3+\mu_4+\mu_5)D(0,n_5;x,y,z) - \mu_2 E(0,0,n_5;y,z) - \mu_1 F(1,0,n_5;x,z) = x\lambda_1 D(0,n_5;x,y,z) + y\lambda_2 D(0,n_5;x,y,z) + \frac{z\mu_1}{x}D(0,n_5;x,y,z) - \frac{z\mu_1}{x}E(0,0,n_5;y,z) + \frac{z\mu_1}{x}G(0,0,0,0,n_5;z) + \frac{z\mu_2}{y}D(0,n_5;x,y,z) - \frac{\mu_2}{y}F(1,0,n_5;x,z) + \frac{\mu_3}{z}D(0,n_5-1;x,y,z) - \frac{\mu_3}{z}D(0,0,n_5-1;x,y) + \mu_4 D(1,n_5;x,y,z) + \mu_5 D(0,n_5+1;x,y,z) \tag{63}$$

Multiply (58) by z^{n_3} and summing over n_3 from 1 to ∞ using (60) and (35), we have:

$$(\lambda_1+\lambda_2+\mu_1+\mu_2+\mu_3+\mu_4+\mu_5)D(n_4,0;x,y,z) - \mu_2 E(0,n_4,0;y,z) - \mu_1 F(1,0,0;x,z) = x\lambda_1 D(n_4,0;x,y,z) + y\lambda_2 D(n_4,0;x,y,z) + \frac{z\mu_1}{x}D(n_4,0;x,y,z) - \frac{z\mu_1}{x}E(0,n_4,0;y,z) + \frac{z\mu_1}{x}G(0,0,0,n_4,0;z) + \frac{z\mu_2}{y}D(n_4,0;x,y,z) - \frac{\mu_2}{y}F(1,n_4,0;x,z) + \frac{\mu_3}{z}D(n_4-1,0;x,y,z) - \frac{\mu_3}{z}B(0,n_4-1,0;x,y) + \mu_4 D(n_4+1,0;x,y,z) + \mu_5 D(n_4,1;x,y,z) \tag{64}$$

Multiply (61) by z^{n_3} and summing over n_3 from 1 to ∞ using (62) and (35), we have:

$$(\lambda_1+\lambda_2+\mu_1+\mu_2+\mu_3+\mu_4+\mu_5)D(0,0;x,y,z) - \mu_2 E(0,0,0;y,z) - \mu_1 F(1,0,0;x,z) = x\lambda_1 D(0,0;x,y,z) + y\lambda_2 D(0,0;x,y,z) + \frac{z\mu_1}{x}D(0,0;x,y,z) - \frac{z\mu_1}{x}E(0,0,0;y,z) + \frac{z\mu_1}{x}G(0,0,0,0,0;z) + \frac{z\mu_2}{y}D(0,0;x,y,z) - \frac{\mu_2}{y}F(1,0,0;x,z)$$

$$+\mu_4 D(1,0;x,y,z)+\mu_5 D(0,1;x,y,z) \tag{65}$$

Multiply (63) by s^{n_4} and summing over n_4 from 1 to ∞ using (64) and (36), we have:

$$\begin{aligned}&(\lambda_1+\lambda_2+\mu_1+\mu_2+\mu_3+\mu_4+\mu_5)H(n_5;x,y,z,s)-\mu_2 I(0,n_5;y,z,s)\\&-\mu_1 J(1,n_5;x,z,s)=x\lambda_1 H(n_5;x,y,z,s)+y\lambda_2 H(n_5;x,y,z,s)\\&+\frac{z\mu_1}{x}H(n_5;x,y,z,s)-\frac{z\mu_1}{x}I(0,n_5;y,z,s)+\frac{z\mu_1}{x}K(0,0,n_5;z,s)\\&+\frac{z\mu_2}{y}D(n_5;x,y,z,s)-\frac{\mu_2}{y}J(1,n_5;x,z,s)+\frac{s\mu_3}{z}H(n_5;x,y,z,s)\\&-\frac{s\mu_3}{z}zL(0,n_5;x,y,s)+\frac{\mu_3}{z}H(n_5-1;x,y,z,s)-\frac{\mu_3}{z}L(0,n_5-1;x,y,s)\\&+\frac{\mu_4}{s}H(n_5;x,y,z,s)-\frac{\mu_4}{s}D(0,n_5+1;x,y,z)+\mu_5 H(n_5+1;x,y,z,s)\end{aligned} \tag{66}$$

Multiply (65) by s^{n_4} and summing over n_4 from 1 to ∞ using (66) and (36), we have:

$$\begin{aligned}&(\lambda_1+\lambda_2+\mu_1+\mu_2+\mu_3+\mu_4+\mu_5)H(0;x,y,z,s)-\mu_2 I(0,0;y,z,s)\\&-\mu_1 J(1,0;x,z,s)=x\lambda_1 H(0;x,y,z,s)+y\lambda_2 H(0;x,y,z,s)\\&+\frac{z\mu_1}{x}H(0;x,y,z,s)-\frac{z\mu_1}{x}I(0,0;y,z,s)+\frac{z\mu_1}{x}K(0,0,0;z,s)\\&+\frac{z\mu_2}{y}D(0;x,y,z,s)-\frac{\mu_2}{y}J(1,0;x,z,s)+\frac{s\mu_3}{z}H(0;x,y,z,s)\\&-\frac{s\mu_3}{z}zL(0,0;x,y,s)+\frac{\mu_4}{s}H(0;x,y,z,s)\\&-\frac{\mu_4}{s}D(0,1;x,y,z)+\mu_5 H(1;x,y,z,s)\end{aligned} \tag{67}$$

Multiply (66) by s^{n_4} and summing over n_4 from 1 to ∞ using (67) and (36), we have:

$$\begin{aligned}&(\lambda_1+\lambda_2+\mu_1+\mu_2+\mu_3+\mu_4+\mu_5)M(x,y,z,s,w)-\mu_2 N(0;y,z,s,w)\\&-\mu_1 J(1;x,z,s,w)=x\lambda_1 M(x,y,z,s,w)+y\lambda_2 M(x,y,z,s,w)\\&+\frac{z\mu_1}{x}M(x,y,z,s,w)-\frac{z\mu_1}{x}N(0;y,z,s,w)+\frac{z\mu_1}{x}Q(0,0;z,s,w)\\&+\frac{z\mu_2}{y}M(x,y,z,s,w)-\frac{\mu_2}{y}J(1;x,z,s,w)+\frac{s\mu_3}{z}M(x,y,z,s,w)\\&-\frac{s\mu_3}{z}L(0;x,y,s,w)+\frac{w\mu_3}{z}M(x,y,z,s,w)-\frac{w\mu_3}{z}R(0;x,y,s,w)\\&+\frac{\mu_4}{s}M(x,y,z,s,w)-\frac{\mu_4}{s}T(0;x,y,z,w)\\&+\frac{\mu_5}{w}M(x,y,z,s,w)-\frac{\mu_5}{w}H(0,x,y,z,s)\end{aligned} \tag{68}$$

$$\begin{aligned}&\Big(\lambda_1-\lambda_1 x+\lambda_2-\lambda_2 y+\lambda_{21}+\mu_1-\frac{z\mu_1}{x}+\mu_2-\frac{\mu_2}{y}+\mu_3-\frac{s\mu_3}{z}-\frac{w\mu_3}{z}+\mu_4-\frac{\mu_4}{s}\\&+\mu_5-\frac{\mu_5}{w}\Big)M(x,y,z,s,w)=\mu_1 J(1;x,z,s,w)-\frac{z\mu_1}{x}N(0;y,z,s,w)+\mu_2 N(0;y,z,s,w)\\&-\frac{z\mu_2}{y}O(1;x,z,s,w)-\frac{s\mu_3}{z}R(0;x,y,s,w)-\frac{w\mu_3}{z}R(0;x,y,s,w)-\frac{\mu_4}{s}T(0;x,y,z,w)\\&-\frac{\mu_5}{w}H(0;x,y,z,s,),\end{aligned}$$

$$\begin{aligned}
M(x,y,z,s,w) &= \Big(\mu_1 J(1;x,z,s,w) - \frac{z\mu_1}{x}N(0;y,z,s,w) + \mu_2 N(0;y,z,s,w) \\
&- \frac{\mu_2}{y}O(1;x,z,s,w) - \frac{s\mu_3}{z}R(0;x,y,s,w) - \frac{w\mu_3}{z}R(0;x,y,s,w) \\
&- \frac{\mu_4}{s}T(0;x,y,z,w) - \frac{\mu_5}{w}M(x,y,z,s,w)\Big) / (\lambda_1 - \lambda_1 x + \lambda_2 \qquad (69) \\
&- \lambda_2 y + \lambda_{21} + \mu_1 - \frac{z\mu_1}{x} + \mu_2 - \frac{\mu_2}{y}\mu_3 - \frac{s\mu_3}{z} - \frac{w\mu_3}{z} + \mu_4 \\
&- \frac{\mu_4}{s} + \mu_5 - \frac{\mu_5}{w}
\end{aligned}$$

$$M(x,y,z,s,w) = \frac{U(x,y,z,s,w)}{V(x,y,z,s,w)} \tag{70}$$

where

$$\begin{aligned}
&U(x,y,z,s,w) = \mu j(1;x,z,s,w) - \frac{z\mu_1}{x}N(0;y,z,s,w) + \mu_2 N(0;y,z,s,w) \\
&-\frac{z\mu_2}{y}O(1;x,z,s,w) - \frac{s\mu_3}{z}R(0;x,y,s,w) - \frac{w\mu_3}{z}R(0;x,y,s,w) \\
&-\frac{\mu_4}{s}T(0;x,y,z,w) - \frac{\mu_5}{w}M(x,y,z,s,w)
\end{aligned}$$

and

$$\begin{aligned}
&V(x,y,z,s,w) = \lambda_1 - \lambda_1 x + \lambda_2 - \lambda_2 y + \lambda_{21} + \mu_1 \\
&-\frac{z\mu_1}{x} + \mu_2 - \frac{\mu_2}{y}\mu_3 - \frac{s\mu_3}{z} - \frac{w\mu_3}{z} \\
&+\mu_4 - \frac{\mu_4}{s} + \mu_5 - \frac{\mu_5}{w}
\end{aligned}$$

Using L'Hopital Rule for indeterminate form, $\frac{0}{0}$, we find for (70), the following for $J(0,1,1,1,1)$, $R(0,1,1,1,1)$, $T(0,1,1,1,1)$ and $H(0,1,1,1,1)$ using the fact that $M(0,1,1,1,1) = 1$ and bearing in mind that the proportion which goes through S_4 is α_1 and S_5 is α_2, we have therefore,

$$N(0,1,1,1,1) = \lambda_1/\mu_1 \tag{71}$$

$$J(0,1,1,1,1) = \lambda_2/\mu_2 \tag{72}$$

$$R(0,1,1,1,1) = (\lambda_1 + \lambda_2/)\mu_3 \tag{73}$$

$$T(0,1,1,1,1) = [\alpha_1(\lambda_1 + \lambda_2)]/\mu_4 \tag{74}$$

$$H(0,1,1,1,1) = [\alpha_2(\lambda_1 + \lambda_2)]/\mu_5 \tag{75}$$

Therefore, the mean queue length L is given by

$$\begin{aligned}
L &= l_1 + l_2 + l_3 + l_4 + l_5 \\
&= \frac{\partial M(x,y,z,s,w)}{\partial x} + \frac{\partial M(x,y,z,s,w)}{\partial y} + \frac{\partial M(x,y,z,s,w)}{\partial z} \\
&+ \frac{\partial M(x,y,z,s,w)}{\partial s} + \frac{\partial M(x,y,z,s,w)}{\partial w}
\end{aligned}$$

Using (33),(34),(35),(36),(37),(70),(71),(72),(73),(74),(75), we get,

$$\left\{\begin{aligned} l_1 &= \frac{\lambda_1}{\mu_1 - \lambda_1}, \\ l_2 &= \frac{\lambda_2}{\mu_1 - \lambda_1}, \\ l_3 &= \frac{\lambda_1 + \lambda_2}{\mu_3 - \lambda_1 - \lambda_2}, \\ l_4 &= \frac{\lambda_1 + \lambda_2}{\mu_4 - \lambda_1 - \lambda_2}, \\ l_5 &= \frac{\lambda_1 + \lambda_2}{\mu_5 - \lambda_1 - \lambda_2}. \end{aligned}\right. \tag{76}$$

Since the queue on the long run behaved like a Poisson queue, we have at steady state, the following:

- The expected number of calls at first substation S_1 expressed as:

$$\begin{aligned} n_i &= \sum_{n_i=0}^{\infty} n_i P n_i, \\ &= \sum_{n_i=0+}^{\infty} n_i \left(1 - \frac{\lambda_1}{\mu_1}\right)\left(\frac{\lambda_1}{\mu_1}\right)^{n_i}, \\ &= 1 - \frac{\lambda_1}{\mu_1} \sum_{n_i=0}^{\infty} n_i \left(\frac{\lambda_1}{\mu_1}\right)^{n_i}, \\ &= \frac{\lambda_1}{\mu_1 - \lambda_1}. \end{aligned}$$

- Expected Number of calls at sub-station SI can be expressed as the average number of calls in system less the average of calls being serviced, i.e.

$$L_{q_i} = \frac{\lambda_1}{\mu_1 - \lambda_1} - \frac{\lambda_1}{\mu_1} = \frac{\lambda_1^2}{\mu_1\left(\mu_1 - \lambda_1\right)}$$

- The expected time an arriving call spends waiting before departing the first sub-station SI is composed of the average waiting time and average time required for service, i.e.

$$W_{s_i} = \frac{\lambda_1}{\mu_1\left(\mu_1 - \lambda_1\right)} + \frac{1}{\mu_1} = \frac{1}{\mu_1 - \lambda_1}$$

- Using the Little's formula, the average time spent waiting before service in S_1 is given by

$$W_{q_i} = \frac{W_{s_i}}{\lambda_1 \mu_1} = \frac{\lambda_1}{\mu_1 (\mu_1 - \lambda_1)}$$

- The probability that the queue at S_1 is empty is $\frac{\lambda_1}{\mu_1}$.
- The probability of delay is $1 - \frac{\lambda_1}{\mu_1}$

Using similar argument, the parameters at S_2, S_3, S_4 and S_5, were obtained as depicted in table 1.

Table 1: Summary of the Operating Characteristics for the Model

	S_1	S_2	S_3	S_4	S_5
L	$\frac{\lambda_1}{\mu_1-\lambda_1}$	$\frac{\lambda_2}{\mu_2-\lambda_2}$	$\frac{\lambda_1+\lambda_2}{\mu_3-(\lambda_1+\lambda_2)}$	$\frac{\alpha_1(\lambda_1+\lambda_2)}{\mu_4-\alpha_1(\lambda_1+\lambda_2)}$	$\frac{\alpha_2(\lambda_1+\lambda_2)}{\mu_5-\alpha_2(\lambda_1+\lambda_2)}$
Lq	$\frac{\lambda_1^2}{\mu_1(\mu_1-\lambda_1)}$	$\frac{\lambda_1^2}{\mu_2(\mu_2-\lambda_2)}$	$\frac{(\lambda_1+\lambda_2)^2}{\mu_3(\mu_3-\lambda_1-\lambda_2)}$	$\frac{\alpha_1^2(\lambda_1+\lambda_2)^2}{\mu_4-\alpha_1^2(\lambda_1+\lambda_2)^2}$	$\frac{\alpha_2^2(\lambda_1+\lambda_2)^2}{\mu_5-\alpha_2^2(\lambda_1+\lambda_2)^2}$
W_q	$\frac{\lambda_1}{\mu_1(\mu_1+\lambda_1)}$	$\frac{\lambda_2}{\mu_2(\mu_2+\lambda_2)}$	$\frac{(\lambda_1+\lambda_2)}{\mu_3(\mu_3-\lambda_1-\lambda_2)}$	$\frac{\alpha_1(\lambda_1+\lambda_2)}{\mu_4(\mu_4-\alpha_1(\lambda_1+\lambda_2))}$	$\frac{\alpha_2(\lambda_1+\lambda_2)}{\mu_4(\mu_5-\alpha_2(\lambda_1+\lambda_2))}$
W	$\frac{1}{\mu_1-\lambda_1}$	$\frac{1}{\mu_2-\lambda_2}$	$\frac{1}{\mu_3-\lambda_1-\lambda_2}$	$\frac{1}{\mu_4-\alpha_1(\lambda_1+\lambda_2)}$	$\frac{1}{\mu_5-\alpha_2(\lambda_1+\lambda_2)}$
Po	$1-\frac{\lambda_1}{\mu_1}$	$1-\frac{\lambda_2}{\mu_2}$	$1-\frac{\lambda_1+\lambda_2}{\mu_3}$	$1-\frac{\alpha_1(\lambda_1+\lambda_2)}{\mu_4}$	$1-\frac{\alpha_2(\lambda_1+\lambda_2)}{\mu_5}$
D	$\frac{\lambda_1}{\mu_1}$	$\frac{\lambda_2}{\mu_2}$	$\frac{\lambda_1+\lambda_2}{\mu_3}$	$\frac{\alpha_1(\lambda_1+\lambda_2)}{\mu_4}$	$\frac{\alpha_2(\lambda_1+\lambda_2)}{\mu_5}$

3

3.1. **Implementation of our Model.** We developed software for numerical computation for our model using Microsoft Visual Basic 6.0 to automate the computation of the performance measures, such as expected number of customers in the system, expected number of customers waiting, the expected amount of time spent in the system. A sample of the interface is presented in Figure 3.

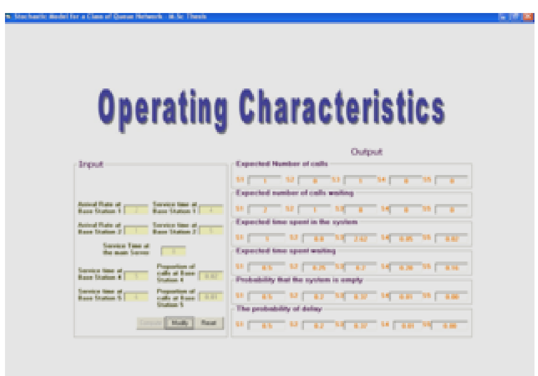

Fig. 3. Interface For the Queue Network Model

3.2. **Conclusion.** In this work, we modeled a queueing system for GSM communication network. The measures of performance were obtained by using the generating function techniques. Further, we developed software for the computation of operating characteristics or performance measure using Microsoft Visual Basic 6.0. The software will request for inputs parameters for the different service channels S_1, S_2,...., S_5. However, the issue of pre-empted calls may be considered by other researchers.

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