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OUTINGS AFTER "Behind the Closed Door(s)": A REVIEW OF THE RESEARCH PUBLICATIONS OF PROFESSOR HAROON ỌLADIPỌ TẸJUMỌLA - II

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1. Preambles

In 1997, at the retirement of Professor Tẹjumọla, he gave his valedictory lecture, which he titled: *"Behind the Closed Door(s)"*. Thus, most of what he must have done up to that time must have been REQUIRED by his type of job. Then, after his retirement, his employers have now officially said, you are free, live the rest of your life with your own self-written constitution. It will be excellent then, to say that whatever publications he did *after*, were outings after *"Behind the Closed Door(s)"*, made by himself, for himself by those he may label by 'himself' (his students), and those who may be labeled with himself (those who care to read his works).

After Professor Tẹjumọla retired, he was able to open more areas of research for those who care to 'learn and see'. These were outings *after 'Behind the closed door(s)'*. He concentrated on the properties of solutions different in method and scope from the ones he did before *'Behind the Closed Door(s)'*. Precisely, he studied the following properties

(F) *Existence and Non-existence of periodic and non-periodic solutions,* and

(G) *Instability and Stability of solutions*[1]

for different orders of ordinary differential equations, with or without delay terms, including third, fourth, fifth, sixth, seventh and generalizations of these to even $2r^{th}$ and odd $(2r+1)^{th}$ order nonlinear ordinary differential equations.

As if he knew that time was running out, his works were done by combining ordinary differential equations of different orders with like properties in the same paper. This can be seen from his titles

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[1]properties (A) through (E) were given in the first review [T15]

in: [T6, T7, T11, T12, T13, T14] and the generalized equations in [T2, T3, T4, T10].

Finally, he started releasing open problems that if he had more time, he would have attempted to contribute to. Thus, he ended up putting his students, grand students and great-grand-students more challenges for their future and the future of the subject.

Now that Professor Tẹjumọla is gone,(may his soul rest in perfect peace), I will review the outings *after 'Behind the Closed Door(s)'* (that is his research papers after 1997). I will also attach what he did before *'Behind the Closed Door(s)'*, which was the first review of his research publications before retirement in 1997, as an appendix, (which was published in: *Directions in Mathematics (1999), 1-14*). This will complete his reviews, as we do not envisage a third stage.

2. Introduction

At the transition from retirement (at the *Behind the Closed Door (s)*) and outings after, (i.e. at the Door(s)), Professor Tẹjumọla with the second successful Ph.D student (Iyase, S. A.), had in [T2] studied even -ordered differential equations of the form

$$\begin{aligned} x^{(2r)} \quad + \quad & a_1 x^{(2r-1)} + \cdots + a_{2r-3}x^{(3)} + f_{2r-2}(\dot{x})\ddot{x} + \\ & f_{2r-1}(x)\dot{x} + f_{2r}(x) = p(t, x, \dot{x}, \ddot{x}, \cdots, x^{(2r-1)}) \end{aligned} \tag{2.1}$$

where $a_1, a_2, \cdots, a_{2r-3}$ are constants, $f_{2r-2}, f_{2r-1}, f_{2r}, p$ are real-valued continuous functions, and p is $\omega-$periodic in t. Under some conditions on the co-efficients, they proved a result on property (F).

Next was the joint work with his last Ph.D student Tchenagni, Bruno (from Republic of Benin), on the existence and Uniqueness of solutions of arbitrary even $2r^{th}$ - order, $2r$-point boundary value problem. This involved equations of the form

$$x^{(2r)} + f(x^{(2r-2)})x^{(2r-1)} = g(t, x, x', \cdots, x^{(2r-1)}) + e(t) \tag{2.2}$$

with $2r$-point boundary conditions,

$$x(0) = x(\eta_1) = \cdots = x(\eta_{2r-2}) = x(1) = 0$$

where $0 < \eta_1 < \eta_2 < \cdots < \eta_{2r-2} < 1$, and $g : [0,1] \times \mathbb{R}^{2r} \to \mathbb{R}$ is Carathéodory, $f \in C(\mathbb{R}, \mathbb{R})$ and $e \in L^1([0,1], \mathbb{R})$.

The Leray-Schauder degree method was used to obtain property (F) for solutions, under growth conditions on g and restrictive bounds on f. This was recorded in [T3]. Also in [T10], the authours continued this work on extension to delay differential equations.

While the above were on even-ordered differential equations, Professor Tẹjumọla had earlier contributed to odd-ordered differential equations with delay in [T4]. In this, he considered equations of the form:

$$x^{(2r+1)} + a_1 x^{(2r)} + a_2 x^{(2r-1)} + \cdots + a_{2r-2} x^{(3)} + f(\dot{x})\ddot{x} + g(t, \dot{x}(t-\tau)) + h(x) = p(t) \tag{2.3}$$

where $r \geq 1$ is an arbitrary integer, $a_1, a_2, \cdots, a_{2r-2}$ are constants, $f, h : \mathbb{R} \to \mathbb{R}$ is continuous and $g : [0, 2\pi] \times \mathbb{R} \to \mathbb{R}$, $p : [0, 2\pi] \to \mathbb{R}$ are $2\pi-$periodic in $t, \tau \in [0, 2\pi)$ is a fixed delay, and g is subject to some conditions of the resonant type.

Using the Leray-Schauder degree arguments, he was able to obtain new existence and non-existence results for the linear analogues of (2.3). This gave him the opportunity for the applications of the Leray-Schauder methods. In fact this generalized his earlier works and those of Professor Ezeilo and many others.

In [T6], nonlinear differential equations of orders six and seven were studied for properties (F) and (G) of the solutions. In it, seven theorems were stated and proved. Interestingly, signum functions were employed in the proofs and many imposed conditions were used on the nonlinearities.

The sixth order nonlinear differential equation considered in [T6] were

$$x^{(6)} + a_1 x^{(5)} + a_2 x^{(4)} + g_3(\ddot{x})\dddot{x} + g_4(\dot{x})\ddot{x} + g_5(\dot{x}, \ddot{x})\dot{x} + g_6(x) = p(t, x, \dot{x}, \cdots, x^{(5)}), \tag{2.4}$$

for property (F), with suitable conditions on a_1,
$G_3(x_3) = \int_0^{x_3} g_3(s)ds,\ g_5, g_6$ and $p(t, x_1, x_2, \cdots, x_5)$;

$$x^{(6)} + a_1 x^{(5)} + a_2 x^{(4)} + a_3 \dddot{x} + \varphi_4(\dot{x}, \ddot{x}, \cdots, x^{(5)}) + g_5(x)\dot{x} + g_6(x, \dot{x}, \cdots, x^{(5)}) = 0, \tag{2.5}$$

for an unstable trivial solution $x \equiv 0$, and a non-trivial periodic solution; and

$$x^{(6)} + a_1 x^{(5)} + a_2 x^{(4)} + a_3 \dddot{x} + \bar{\varphi}_4(\dot{x}, \ddot{x})\ddot{x} + \varphi_5(x)\dot{x} + \bar{\varphi}_6(x, \dot{x}, \ddot{x}) = p(t, x, \dot{x}, \cdots, x^{(5)}); \tag{2.6}$$

for the existence of at least one $\omega-$periodic solution, with suitable conditions on $\bar{\varphi}_6, \varphi_6$ and $\omega-$periodic $p(t, x_1, x_2, \cdots, x_5)$ satisfying

$$|p(t, x_1, x_2, \cdots, x_5)| \leq C_1 + C_2(x_2^2 + x_3^2)^{1/2}$$

for sufficiently small C_2.

On the otherhand, Professor Tẹjumọla in [T6] simultaneously considered the seventh order nonlinear differential equations of the forms:

$$
\begin{aligned}
x^{(7)} \;\; + \;\; & a_1x^{(6)} + a_2x^{(5)} + a_3x^{(4)} + a_4 \dddot{x} + f_5(x, \dot{x}, \cdots, x^{(6)})\ddot{x} \\
& + f_6(x)\dot{x} + f_7(x, \dot{x}, \cdots, x^{(6)}) = 0; \qquad (2.7)
\end{aligned}
$$

$$
\begin{aligned}
x^{(7)} \;\; + \;\; & a_1x^{(6)} + a_2x^{(5)} + a_3x^{(4)} + a_4 \dddot{x} + \bar{f}_5(\dot{x}, \ddot{x})\ddot{x} \qquad (2.8) \\
& + f_6(x)\dot{x} + \bar{f}_7(x, \dot{x}, \ddot{x}) = p(t, x, \dot{x}, \cdots, x^{(6)});
\end{aligned}
$$

$$
\begin{aligned}
x^{(7)} \;\; + \;\; & a_1x^{(6)} + a_2x^{(5)} + a_3x^{(4)} + \psi_4(x, \dot{x}, \cdots, x^{(6)}) \dddot{x} \qquad (2.9) \\
& + \psi_5(\dot{x})\ddot{x} + \psi_6(x, \dot{x}, \cdots, x^{(6)}) + \psi_7(x) = 0;
\end{aligned}
$$

and

$$
\begin{aligned}
x^{(7)} \;\; + \;\; & a_1x^{(6)} + a_2x^{(5)} + a_3x^{(4)} + \bar{\psi}_4(\dot{x}, \ddot{x}, \dddot{x}) \dddot{x} + \psi_5(\dot{x})\ddot{x} \\
& + \bar{\psi}_6(\dot{x}, \ddot{x}, \dddot{x}) + \psi_7(x) = p(t, x, \dot{x}, \cdots, x^{(6)}); \qquad (2.10)
\end{aligned}
$$

for the existence of an unstable trivial solution $x \equiv 0$ and the existence of non trivial periodic solutions (for equations (2.7) and (2.9)) and the existence of at least one $\omega-$periodic solution, with ω being the period of $p = p(t, x, \dot{x}, \cdots, x^{(6)})$, for equations (2.8) and (2.10)).

Among the works carried out by Professor Tẹjumọla was the coordination of [T5], ordinary differential equation - the Proceedings of the National Conference in ordinary differential equation held in NMC, Abuja (July 28-29,2000), sponsored by the National Mathematical Centre, Abuja. This was the first proceedings of NMC, Abuja, and it was edited by three generations of Mathematicians, consisting of Professor J.O.C. Ezeilo (supervisor to Professor Tẹjumọla), Professor Tẹjumọla and Professor A.U. Afuwapẹ (the first successful Ph.D student of Professor Tẹjumọla). Many research findings of many Nigerian Mathematicians were presented and edited for the production of the proceedings.

In continuation of Professor Tẹjumọla's research work, (with B. Tchegnani), they discussed the stability and existence of periodic solutions of some third and fourth order nonlinear delay differential equations. The equations discussed were

$$x^{(3)} + f(t, x, \dot{x}, \ddot{x})\ddot{x} + g(t, x(t-\tau), \dot{x}(t-\tau)) + h(x(t-\tau)) = P_1; \qquad (2.11)$$

and

$$
\begin{aligned}
x^{(4)} \;\; + \;\; & \varphi(t, x, \dot{x}, \ddot{x}, x^{(3)})x^{(3)} + \psi(t, \dot{x}(t-\tau), \ddot{x}(t-\tau)) \qquad (2.12) \\
& + \chi(t, x(t-\tau), \dot{x}(t-\tau)) + h(x(t-\tau)) = P_2
\end{aligned}
$$

where $\tau > 0$ is a fixed delay, f, g, h, φ, ψ and χ are real-valued continuous functions that depend only on the arguments explicitly shown and

$$P_1 = P_1(t, x, \dot{x}, \ddot{x}, x(t-\tau), \dot{x}(t-\tau)),$$

and

$$P_2 = P_2(t, x, \dot{x}, \ddot{x}, x^{(3)}, x(t-\tau), \dot{x}(t-\tau), \ddot{x}(t-\tau)),$$

which are also real-valued continuous functions.

The authors achieved their aims by introducing a novel method of re-writing these equations as a perturbation of an equation with one nonlinear term and found conditions on the perturbation terms to ensure the desired properties of solutions.

Precisely, equations (2.11) and (2.12) were studied in the equivalent system forms:

$$\dot{x} = y, \dot{y} = z, \dot{z} = -az - by - h(x(t)) + N(t) \tag{2.13}$$

where

$$\begin{aligned} N(t) &= az - f(t, x, y, z)z \\ &\quad + b\int_{-\tau}^{0} z(t+\theta)d\theta - g(t, x(t-\tau, y(t-\tau)) \\ &\quad + \int_{-\tau}^{0} h'(x(t+\theta))y(t+\theta)d\theta + by(t-\tau) + P_1 \end{aligned}$$

and

$$\dot{x} = y, \dot{y} = z, \dot{z} = \omega, \dot{\omega} = -a\omega - bz - cy - h(x(t)) + M(t) \tag{2.14}$$

with

$$\begin{aligned} M(t) &= a\omega - \varphi(t, x, y, z, \omega)\omega \\ &\quad + b\int_{-\tau}^{0} \omega(t+\theta)d\theta - \psi(t, y(t-\tau, z(t-\tau)) \\ &\quad + bz(t-\tau) + c\int_{-\tau}^{0} z(t+\theta)d\theta - \chi(t, x(t-\tau, y(t-\tau)) \\ &\quad + cy(t-\tau) + \int_{-\tau}^{0} h'(x(t+\theta))y(t+\theta)d\theta + P_2, \end{aligned}$$

where a, b, c are real constants, $h(x(t))$ is the 'nonlinear term' and $N(t)$ and $M(t)$ are the 'perturbation terms'.

Appropriate Lyapunov functional were chosen to prove the results.

Next, is the joint paper [T9] with Iyase, S.A. and Tchegnani, B., Professor Tẹjumọla studied the existence and uniqueness results for some fourth order four-point and three-point boundary value problems. They considered the equations

$$\left.\begin{array}{l} u^{(iv)}+\phi(u'')u'''=g(t,u,u',u'',u''')+e(t) \\ u''(0)=u''(1)=u'(\eta_1)=u(\eta_2)=0,\ 0\le\eta_1,\eta_2\le 1; \end{array}\right\}$$

$$\left.\begin{array}{l} u^{(iv)}+\psi(u')=g(t,u,u',u'',u''')+e(t) \\ u''(0)=u''(1)=u'(0)=u(\eta)=0,\ 0\le\eta\le 1; \end{array}\right\}$$

$$\left.\begin{array}{l} u^{(iv)}+\chi(u')u''=g(t,u,u',u'',u''')+e(t) \\ u'(0)=u'(1)=u(\eta_1)=u(\eta_2)=0,\ \eta_1\ne\eta_2, \\ 0\le\eta_1<1,\ 0<\eta_2\le 1; \end{array}\right\}$$

and

$$u^{(iv)}=g(t,u,u',u'',u''')+e(t)$$

with

$$\begin{aligned} u(0) &= u(1)=u'(0)=u(\eta)=0,\ 0<\eta<1; \\ u''(0) &= u'''(1)=u'(\eta_1)=u(\eta_2)=0,\ 0\le\eta\le 1; \\ u'(0) &= u''(1)=u(\eta_1)=u(\eta_2)=0, \eta_1\ne\eta_2,\ 0\le\eta_1<1, 0<\eta_2\le 1; \\ u'(0) &= u(1)=u''(\eta_1)=u''(\eta_2)=0,\ 0\le\eta_1,\eta_2\le 1; \end{aligned}$$

where $\phi,\psi,\chi:\mathbb{R}\to\mathbb{R}$ are continuous functions, $g:[0,1]\times\mathbb{R}^4\to\mathbb{R}$ satisfies the Carathéodory conditions and $e\in L^1[0,1]$.

The Mawhin's version of the Leray-Schauder continuation theorem was used in the proofs.

With further re - arrangements, uniqueness of solutions results were proved for these equations.

In the next paper, [T8], which turned out to be the last published joint work with Prof. J. O. C. Ezeilo, after many of such together since 1966, (35 years of joint work - waoo, the longest close relationship), was like opening up the secrets of the subject. A method for discussing the non-resonant oscillations for some fourth order differential equations with strong arguments on the co-efficients of the linear constant co-efficient fourth order b.v.p.

$$\left.\begin{array}{l} x^{(4)}+a_1x^{(3)}+a_2\ddot{x}+a_3\dot{x}+a_4x=p(t) \\ x^{(r)}(2\pi)=x^{(r)}(0),\ (r=0,1,2,3). \end{array}\right\} \tag{2.15}$$

Using the earlier results that (2.15) have a solution for arbitrary a_1 and a_3 if a_2 and a_4 satisfy

$$\chi(m)=m^4-a_2m^2+a_4\ne 0,\ (m=0,1,2,\cdots);$$

they were able to establish the existence of solutions of two general b.v.p of form (2.15) in which a_1,a_2,a_3 and a_4 are not all necessarily constants and $p:[0,1]\times\mathbb{R}^4\to\mathbb{R}$.

Precisely, they considered the equations

$$x^{(4)} + \psi(\ddot{x})x^{(3)} + \gamma(x, \dot{x}, \ddot{x}, x^{(3)})\ddot{x} + \theta(\dot{x}) + \varphi(x) = p(t, x, \dot{x}, \ddot{x}, x^{(3)})$$

and

$$x^{(4)} + \psi(\ddot{x})x^{(3)} + a_2\ddot{x} + \theta(\dot{x}) + \varphi(x) = p(t, x, \dot{x}, \ddot{x}, x^{(3)})$$

with boundary conditions $x^{(r)}(2\pi) = x^{(r)}(0),\ (r = 0, 1, 2, 3)$.

Again, in the next paper, [T11], Professor Tẹjumọla used the Leray-Schauder continuation technique, with a novel feature of using some new special functions on equations of the fifth, fourth and third order ordinary differential equations.

Specifically, he discussed equations of the forms:

$$x^{(5)} + f_1(x^{(4)}) + f_2(\dddot{x}) + f_3(\ddot{x}) + a_4\dot{x} + a_5x = p_1(t, x, \dot{x}, \ddot{x}, \dddot{x}, x^{(4)}) \tag{2.16}$$

$$x^{(4)} + g_1(\dddot{x}) + g_2(\ddot{x}) + g_3(\dot{x}) + b_4x = p_2(t, x, \dot{x}, \ddot{x}, \dddot{x}) \tag{2.17}$$

and

$$\dddot{x} + h_1(\ddot{x}) + h_2(\dot{x}) + c_3x = p_3(t, x, \dot{x}, \ddot{x}) \tag{2.18}$$

with periodic boundary conditions

$$D^r x(0) = D^r x(2\pi),\ D^r \equiv \frac{d^r}{dt^r},$$

where $r = 0, 1, 2, 3, 4; r = 0, 1, 2, 3$ and $r = 0, 1, 2$ respectively. He proved results on the existence and non-existence of periodic solutions with period equal to the period of p_1, p_2 or p_3. Signum functions were used in the proofs.

In continuation, in [T12], Professor Tẹjumọla discussed property (F) for some fourth and third order nonlinear ordinary differential equations of the forms:

$$x^{(4)} + g_1(\dot{x}, \ddot{x})\dddot{x} + g_2(\dot{x})\ddot{x} + g_3(\dot{x}) + g_4(x) = p_1(t, x, \dot{x}, \ddot{x}, \dddot{x}) \tag{2.19}$$

and

$$\dddot{x} + h_1(x, \ddot{x})\ddot{x} + h_2(x)\dot{x} + h_3(x) = p_2(t, x, \dot{x}, \ddot{x}) \tag{2.20}$$

where $g_i,\ i = 1, 2, 3, 4,\ h_i,\ i = 1, 2, 3$ and $p_1,\ p_2$ are periodic real-valued continuous functions in their respective arguments, and with p_1, p_2 periodic in t with period 2π.

The main distinguishing feature of this study is the restrictions on the integrals of g_1 and h_1 as against the well known restrictions on the functions g_1 and h_1 in earlier studies.

In [T13], Professor Tẹjumọla discussed what he called *Nearly Periodic* bvp for some fourth and third order ordinary differential equations.

In fact he studied a mixed three-point bvp with periodic boundary conditions in the derivatives of the forms

$$\left.\begin{array}{l} u^{(iv)} + f(\ddot{x})\,\dddot{u} + g(\dot{u})\ddot{u} = p(t, u, \dot{u}, \ddot{u}, \dddot{u}) \\ u(0) = u(\eta),\ u^{(i)}(0) = u^{(i)}(1),\ i = 1, 2, 3,\ 0 < \eta < 1; \end{array}\right\} \tag{2.21}$$

and

$$\left.\begin{array}{l} \dddot{u} + \varphi(\dot{u})\ddot{u} = \theta(t, u, \dot{u}, \ddot{u}) \\ u(0) = u(\eta),\ u^{(i)}(0) = u^{(i)}(1),\ i = 1, 2,\ 0 < \eta < 1; \end{array}\right\} \tag{2.22}$$

with $f, g, \varphi \in C(\mathbb{R}, \mathbb{R}),\ p : I \times \mathbb{R}^4 \to \mathbb{R},\ \theta : I \times \mathbb{R}^3 \to \mathbb{R}$, are continuous functions and $I = [0, 1]$.

Two theorems were stated and proved using Mawhin's continuation theorem. Unique decompositions of p and θ were used in the proofs.

Finally, Professor Tẹjumọla continued his investigations of ordinary differential equations in his last paper [T14],(posthumously), of some sixth and fifth order ordinary differential equations of the forms:

$$\begin{aligned} x^{(6)} \quad + \quad & f_1(x^{(4)})x^{(5)} + f_2(\dddot{x})x^{(4)} + f_3(\ddot{x})\,\dddot{x} + f_4(\ddot{x}) \\ & + f_5(\dot{x}) + a_6 x = p_1(t, x, \dot{x}, \ddot{x}, \dddot{x}, x^{(4)}, x^{(5)}); \end{aligned} \tag{2.23}$$

and

$$\begin{aligned} x^{(6)} \quad + \quad & a_1 x^{(5)} + g_2(\dddot{x})x^{(4)} + g_3(\ddot{x})\,\dddot{x} + g_4(\dot{x})\ddot{x} \\ & + g_5(\dot{x}) + g_6(x) = p_2(t, x, \dot{x}, \ddot{x}, \dddot{x}, x^{(4)}, x^{(5)}); \end{aligned} \tag{2.24}$$

He used as bases for his arguments the equivalent conditions

(I) $\quad a_1 \neq 0,\ sgn a_1 = sgn a_5,\ (sgn a_1)a_3 < 0,\ (a_2, a_4, a_6$ arbitrary)

(II) $\quad a_2 < 0,\ a_4 > 0,\ a_6 < 0,\ (a_1, a_3, a_5$ arbitrary)

on the linear constant co-efficient sixth order ordinary differential equation

$$x^{(6)} + a_1 x^{(5)} + a_2 x^{(4)} + a_3\,\dddot{x} + a_4\ddot{x} + a_5\dot{x} + a_6 x = p(t),\ p(t+\omega) = p(t).$$

He proved that the equations (2.23) and (2.24) have

(1) no non-trivial periodic solutions for $p \equiv 0$ and
(2) a unique $\omega-$periodic solution with $p \not\equiv 0$.

He also in like manner gave results on the fifth order nonlinear equations

$$\begin{aligned} x^{(5)} \quad + \quad & \varphi_1(\dddot{x})x^{(4)} + \varphi_2(\ddot{x})\,\dddot{x} + \varphi_3(\dot{x})\ddot{x} + \varphi_4(\dot{x}) \\ & + \varphi_5(x) = q_1(t, x, \dot{x}, \ddot{x}, \dddot{x}, x^{(4)}) \end{aligned} \tag{2.25}$$

and

$$x^{(5)} + b_1 x^{(4)} + \psi_2(\ddot{x})\dddot{x} + \psi_3(\dot{x})\ddot{x} + \psi_4(\dot{x}) + \psi_5(x) = q_2(t, x, \dot{x}, \ddot{x}, \dddot{x}, x^{(4)}). \tag{2.26}$$

He again used as bases for his proves the equivalent conditions

(III) $b_1 \neq 0,\ sgn b_1 = sgn b_5,\ (sgn b_1) b_3 < 0,\ (b_2, b_4$ arbitrary)

(IV) $b_2 < 0,\ b_4 > 0,\ (b_1, ba_3, b_5$ arbitrary)

on the linear constant co-efficient fifth order ordinary differential equation

$$x^{(5)} + b_1 x^{(4)} + b_2 \dddot{x} + b_3 \ddot{x} + b_4 \dot{x} + b_5 x = q(t),\ q(t+\omega) = q(t).$$

Integral equivalent conditions to (I), (II), (III) and (IV) were introduced for the respective nonlinear equations.

3. Open Problems released by Professor H.O. Tẹjumọla

In this section, we list the problems envisaged by Professor Tẹjumọla for contribution to, but for which destiny did not allow him to carry out. Some of these may be good challenges to the generations that follow.

1) In [T11], Professor Tẹjumọla wrote that because of technical difficulties, it has not been possible to extend the results on equations

$$x^{(5)} + f_1(x^{(4)}) + f_2(\dddot{x}) + f_3(\ddot{x}) + a_4\dot{x} + a_5 x = p_1(t, x, \dot{x}, \ddot{x}, \dddot{x}, x^{(4)}) \tag{3.1}$$

and

$$x^{(4)} + g_1(\dddot{x}) + g_2(\ddot{x}) + g_3(\dot{x}) + b_4 x = p_2(t, x, \dot{x}, \ddot{x}, \dddot{x}) \tag{3.2}$$

to equations in which $b_4 x$ is replaced by nonlinear term $g_4(x)$, and $a_4\dot{x},\ a_5 x$ replaced by nonlinear terms $f_4(\dot{x})$ and $f_5(x)$.

2) Other problems on (3.1) and (3.2) are
   (a) Stability of solutions when $p_1 \equiv 0 \equiv p_2$ and
   (b) Ultimate boundedness of solutions when $p_1 \not\equiv 0, p_2 \not\equiv 0$.
3) In [T4], while considering equations of the form

$$x^{(2r+1)} + a_1 x^{(2r)} + a_2 x^{(2r-1)} + \cdots + a_{2r-2} x^{(3)} + f(\dot{x})\ddot{x} + g(t, \dot{x}(t-\tau)) + h(x) = p(t), \tag{3.3}$$

and the constant co-efficient analogue in the autonomous case

$$x^{(2r+1)} + a_1 x^{(2r)} + a_2 x^{(2r-1)} + \cdots + a_{2r-1}\ddot{x} + a_{2r}\dot{x}(t-\tau) + a_{2r+1} x = 0$$

with the auxiliary equation

$$\psi(\beta) = \beta^{2r+1} + a_1\beta^{2r}a_2\beta^{2r-1} + \cdots + a_{2r-1}\beta^2 + a_{2r}\beta e^{-\beta\tau} + a_{2r+1} = 0.$$

A consideration of the real part of $\psi(in)$ led to a set of conditions on the constants $a_1, a_2, a_3, \cdots$, but we have so far not been able to extend these conditions to any equation (3.3).

4) In [T5], while discussing the instability and existence of non trivial periodic solutions for sixth and seventh order nonlinear differential equations, Professor Tẹjumọla wrote: *Although these results can be extended to equations of orders eight and nine, it is certainly more interesting to obtain such results for equations of arbitrary even (or odd) order. Our result in this direction will be presented elsewhere.*

   This did not see the light of day before he died.

4. Conclusion

Having lived a fulfilled life, Professor Tẹjumọla has given his contributions to the qualitative properties of nonlinear differential equations of higher orders. Precisely, his contributions on equations of orders two through eight, even and odd ordered equations with or without delays, remain a great landmark to research. His contributions and open problems will remain as challenges to future generations. I also hope that those coming behind will follow the footsteps Professor Tẹjumọla left behind.

I wish him a perfect rest in PEACE.

5. Publications Of Professor H. O. Tẹjumọla, Outings after Closing the Door(s) in 1997

[T1] Behind the Closed Door(s). *Directions in Mathematics (1999), 338-356* (edited by G. O. S. Ekhaguere, O. O. Ugbebor and D. O. A. Ajayi).

[T2] Periodic solutions of a class of even order differential equations. *Ann. Differential Equations 13 (1997), 3, 209-221* (with S. A. Iyase)

[T3] Existence and Uniqueness of solutions to an arbitrary even 2rth order , 2r-point boundary value problem. *Directions in Mathematics (1999), 37-48* (edited by G. O. S. Ekhaguere, O. O. Ugbebor and D. O. A. Ajayi). (with B. Tchenagni)

[T4] Existence of periodic solutions of some odd-ordered differential equations with delay. *J. Nigerian Math. Soc. 18 (1999) 21-30.*

[T5] *Ordinary Differential Equations. Proceedings of the National Mathematical Centre, Abuja, Nigeria volume 1, number 1 (Abuja 2000),* (edited with J. O. C. Ezeilo, and A. U. Afuwapẹ)

[T6] Instability and periodic solutions of certain nonlinear differential equations of orders six and seven. Proceedings of National Mathematical Centre, Abuja, vol. 1, no. 1,(2000), 56-65.

[T7] Stability, Boundedness and existence of periodic solutions of some third and fourth order nonlinear delay differential equations . *J. Nigerian Math. Soc. 19 (2000) 9 - 19.* (with B. Tchegnani)

[T8] Existence and Uniqueness results for some fourth order four-point and three-point boundary value problems , *J. Nigerian Math. Soc. 20 (2001) 29 - 44.* (with B. Tchegnani)

[T9] Non resonant Oscillations for some fourth order differential equations II, (with J. O. C. Ezeilo) *J. Nigerian Math. Soc. 20 (2001) 45 - 58.*

[T10] Non resonant Oscillations for some arbitrary even $2r^{th}$ order differential equations with delay , *J. Nigerian Math. Soc. 21 (2002) 1 - 12.* (with B. Tchegnani)

[T11] Periodic boundary value problems for some fifth, fourth and third order ordinary differential equations *J. Nigerian Math. Soc. 25 (2006) 37 - 46.*

[T12] Existence results for some fourth and third order differential equations. *J. Nigerian Math. Soc. 27 (2008) 19 - 31.*

[T13] Nearly periodic boundary value problems for some fourth and third order differential equations. *Advances in mathematics. vol.1. Proceedings of a Memorial Conference in Honour of Late Professor C. O. A. Sowunmi, (2009), 38-45.* (ed. by E. O. Ayoola, V. F. Payne and D. O. A. Ajayi).

[T14] Integral conditions for Existence and non-existence of periodic solutions of some sixth and fifth order ordinary differential equations. *J. Nigerian Math. Soc. (2012) THIS ISSUE. (posthumously)*

[T15] A.U. Afuwapẹ, A Review of Research Publications of Professor Haroon Ọladipọ Tẹjumọla. *Directions in Mathematics (1999), 1-14*, (edited by G. O. S. Ekhaguere, O. O. Ugbebor and D. O. A. Ajayi).

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