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INTEGRAL CONDITIONS OF EXISTENCE AND NON-EXISTENCE OF PERIODIC SOLUTIONS OF SOME SIXTH AND FIFTH ORDER ORDINARY DIFFERENTIAL EQUATIONS

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ABSTRACT. New conditions of integral type are obtained for the existence of periodic solutions of a certain class of sixth and fifth order equations, and for non-existence of periodic solutions of their corresponding homogeneous equations.

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1. INTRODUCTION

In a recent paper [3] we examined the problem of existence of periodic solutions of fourth order nonlinear ordinary differential equations of the form

$$x^{(4)} + g_1(\dot{x}, \ddot{x})\,\dddot{x} + g_2(\dot{x})\ddot{x} + g_3(\dot{x}) + g_4(x) = p(t, x, \dot{x}, \ddot{x}, \dddot{x}), \quad (1.1)$$

$p(t+\omega, x, \dot{x}, \ddot{x}, \dddot{x}) = p(t, x, \dot{x}, \ddot{x}, \dddot{x})$ for some periodic $\omega > 0$, and the problem of nonexistence of periodic solution for the corresponding homogeneous equation (1.1) with $p \equiv 0$. Our main interest in that study was in obtaining conditions that place restrictions on the integral of g_1 and, or, g_3 rather than directly on the functions g_1 and g_3 as in previous investigations, and this resulted in relatively weaker conditions for existence and nonexistence of periodic solutions of (1.1). An additional feature of that study [3] is the full blown nonlinear terms involved in the equation. The present paper is a continuation of our study in [3] to a class of sixth and fifth order ordinary differential equations.

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To start with, consider the linear constant-coefficient sixth order ordinary differential equation

$$x^{(6)} + a_1 x^{(5)} + a_2 x^{(4)} + a_3 \dddot{x} + a_4 \ddot{x} + a_5 \dot{x} + a_6 x = p(t), \quad p(t+\omega) = p(t). \tag{1.2}$$

It can be readily verified (as in [1,4]) that if either of the conditions

(I) $a_1 \neq 0, sgn a_1 = sgn a_5, (sgn a_1) a_3 < 0$ $\quad (a_2, a_4, a_6$ arbitrary)

(II) $a_2 < 0,\ a_4 > 0,\ a_6 < 0$ $\quad (a_1, a_3, a_5$ arbitrary)

holds, then (1.2) with $p \equiv 0$, has no nontrivial periodic solutions, and the equation (1.2) with $p \neq 0$ has a unique ω-periodic solution.

In the fifth order case

$$x^{(5)} + b_1 x^{(4)} + b_2 \dddot{x} + b_3 \ddot{x} + b_4 \dot{x} + b_5 x = p(t), p(t+\omega) = p(t), \tag{1.3}$$

the corresponding conditions are

(III) $b_1 \neq 0, sgn b_1 = sgn b_5, b_3 sgn b_1 < 0$

(IV) $b_2 < 0,\ b_4 > 0$

b_1, b_2, b_3, b_4, b_5 and b_6 constants. Observe that each of the conditions (I) and (III) incorporates two conditions into one. Furthermore in the two equations (1,2), (1.3) two sets of different conditions, one involving terms with odd subscripts and the other even subscripts ((I), (III); (II), (IV)) ensure the existence or nonexistence of periodic solutions. This odd and even subscripts feature runs through the generalized criteria obtained for the nonlinear equations studied here.

2. STATEMENT OF RESULTS - SIXTH ORDER EQUATIONS

We shall be concerned with sixth order equations of the forms

$$\begin{aligned} x^{(6)} + f_1(x^{(4)}) x^{(5)} + f_2(\dddot{x}) x^{(4)} + f_3(\ddot{x}) \dddot{x} + f_4(\ddot{x}) + f_5(\dot{x}) \\ + a_6 x = p_1(t, x, \dot{x}, \ddot{x}, \dddot{x}, x^{(4)}, x^{(5)}) \end{aligned} \tag{2.1}$$

$$\begin{aligned} x^{(6)} + a_1 x^{(5)} + g_2(\dddot{x}) x^{(4)} + g_3(\ddot{x}) \dddot{x} + g_4(\dot{x}) \ddot{x} + g_5(\dot{x}) \\ + g_6(x) = p_2(t, x, \dot{x}, \ddot{x}, \dddot{x}, x^{(4)}, x^{(5)}) \end{aligned} \tag{2.2}$$

in which a_1, a_6 are constants, $f_i,\ i = 1, 2, \dots, 5$, $g_i, i = 2, \dots, 6$ and $p_i, i = 1, 2$, are real-valued continuous functions of their respective arguments, and $p_i(t+\omega, x, \dots, x^{(5)}) = p_i(t, x, \dots, x^{(5)}), i = 1, 2$ for some constant $\omega > 0$. Corresponding respectively to the conditions (I) and (II) above we obtain the following results for the equations (2.1), (2.2). Assume that $f_5'(y), g_6'(x)$ exist and are continuous for all y and x.

Theorem 1. Let $a \neq 0$ be an arbitrary constant and suppose that

$$(sgna)\frac{F_1(v)}{v} > 0, (sgna)f_5'(y) > 0, (sgna)f_3(u) < 0, \qquad F_1(v) \equiv \int_0^v f_1(s)ds, \tag{2.3}$$

for all y, u and $v \neq 0$. Then the equation (2.1) with $p_1 \equiv 0$ has no nontrivial periodic solution of whatever period.

In the non-autonomous case $p_1 \not\equiv 0$, we have the following result.

Theorem 2. Let $a_1 \neq 0, a_3, a_5$ be constants satisfying (I) such that

$$\left.\begin{aligned} &(sgna_1)\frac{F_1(v)}{v} \geq |a_1| \text{ for } |y| \geq 1, \\ &(sgna_1)f_3(u) \leq (sgna_1)a_3 \text{ for } |u| \geq 1, \\ &(sgna_1)f_5'(y) \geq |a_5| \text{ for } |y| \geq 1. \end{aligned}\right\} \tag{2.4}$$

Suppose further that there are constants $A_1 > 0$, $A_2 \geq 0$, with A_2 sufficiently small, such that

$$|p_1(t, x, y, z, u, v, w)| \leq A_1 + A_2(|z| + |u| + |v|) \tag{2.5}$$

for all t, x, y, z, u, v. Then the equation (2.1) has at least one periodic solution of period ω.

Note that the conditions (2.3) and (2.4) are generalizations of the criteria (I); they also involve terms with odd subscripts in equation (2.1). Note also the absence of any conditions on the constant a_6 and the terms with even subscripts f_2, f_4 in (2.1). Our results in the other direction, that is involving terms with even subscripts, concern the equation (2.2), and are as follows.

Theorem 3. Suppose that

$$\frac{G_2(u)}{u} < 0, \ g_4(y) > 0, \ g_6'(x) < 0, \ G_2(u) \equiv \int_0^u g_2(s)ds, \tag{2.6}$$

for all x, y and $u \neq 0$. Then the equation (2.2), with $p_2 \equiv 0$, has no nontrivial periodic solution of whatever period.

Theorem 4. Let a_2, a_4, a_6 be constants satisfying (II) such that

$$\left.\begin{aligned} &\frac{G_2(u)}{u} \leq a_2 \text{ for } |u| \geq 1, \ g_4(y) > a_4 \text{ for } |y| \geq 1 \\ &g_6'(x) < a_6 \text{ for } |x| \geq 1, \ G_2(u) \equiv \int_0^u g_2(s)ds, \end{aligned}\right\} \tag{2.7}$$

and let

$$g_6(x)sgnx \to +\infty(-\infty) \text{ as } |x| \to \infty.$$

Suppose that there exist constants $A_1^* > 0,\ A_2^* > 0$, with A_2^* sufficiently small, such that

$$|p_2(t,x,y,z,u,v,w)| \leq A_1^* + A_2^*(|y| + |z| + |u|) \tag{2.8}$$

for all t, x, y, z, u, v, w. Then the equation (2.2) has at least one periodic solution of period ω.

Observe the absence of any restrictions on the constant a_1 and the functions g_3, g_5 in (2.2). The conditions on F_1 in (2.3) and (2.4), and on G_2 in (2.6) and (2.7) are the integral conditions; they place restrictions on the integrals of f_1 and g_2 rather than directly on the functions f_1 and g_2.

3. STATEMENT OF RESULTS - FIFTH ORDER EQUATIONS

We now state parallel results for fifth order equations. We shall consider equations of the form

$$\begin{aligned} x^{(5)} + \varphi_1(\dddot{x})x^{(4)} + \varphi_2(\ddot{x})\dddot{x} + \varphi_3(\dot{x})\ddot{x} + \varphi_4(\dot{x}) + \varphi_5(x) \\ = q_1(t, x, \dot{x}, \ddot{x}, \dddot{x}, x^{(4)}) \end{aligned} \tag{3.1}$$

$$\begin{aligned} x^{(5)} + bx^{(4)} + \psi_2(\ddot{x})\dddot{x} + \psi_3(\dot{x})\ddot{x} + \psi_4(\dot{x}) + \psi_5(x) \\ = q_2(t, x, \dot{x}, \ddot{x}, \dddot{x}, x^{(4)}) \end{aligned} \tag{3.2}$$

where b is an arbitrary constant, $\varphi_i,\ i = 1, 2, \ldots, 5,\ \psi_i,\ i = 2, \ldots, 5,$ q_1, q_2 are real-valued continuous functions, $\varphi_5'(x)$ exists and is continuous for all x and q_1, q_2 are periodic in t, of period ω. Our first set of results concern (3.1), and are as follows.

Theorem 5. Let

$$\begin{gathered} (sgna)\frac{\Phi_1(u)}{u} > 0,\ (sgna)\varphi_3(y) < 0,\ (sgna)\varphi_5'(x) > 0, \\ \Phi_1(u) \equiv \int_0^u \varphi_1(s)ds \end{gathered} \tag{3.3}$$

for all x, y and $u \neq 0$, where a is an arbitrary constant. Then the equation (3.1), with $q_1 \equiv 0$, has no nontrivial periodic solutions of any period.

Theorem 6. Let $b_1 \neq 0, b_3, b_5$ be constants satisfying (III) such that

$$\begin{gathered} (sgnb_1)\frac{\Phi_1(u)}{u} \geq |b_1|,\ (sgnb_1)\varphi_3(y) \leq (sgnb_1)b_3, \\ (sgnb_5)\varphi_5'(x) \geq |b_5|,\ \Phi_1(u) \equiv \int_0^u \varphi_1(s)ds \end{gathered} \tag{3.4}$$

for all $|x| \geq 1$, $|y| \geq 1$ and $|u| \geq 1$. Suppose that

$$(sgnx)\varphi_5(x) \to +\infty(-\infty) \text{ as } |x| \to \infty. \tag{3.5}$$

Suppose further that there exist constants $B_1 > 0, B_2 > 0$, with B_2 sufficiently small, such that

$$|q_1(t,x,y,z,u,v)| \leq B_1 + B_2(|y| + |z| + |u|) \tag{3.6}$$

for all t,x,y,z,u,v. Then the equation (3.1) has at least one periodic solution of period ω.

Observe the absence of any restrictions on terms with even subscripts in (3.1); our results involving terms with even subscripts, and in line with (IV), concerns the equation (3.2) and are as follows.

Theorem 7. Let

$$\frac{\Psi_2(z)}{z} < 0, \ \frac{\psi_4(y)}{y} > 0, \ \Psi_2(z) \equiv \int_0^z \psi_2(s)ds \tag{3.7}$$

for all $y \neq 0$ and $z \neq 0$. Then the equation (3.2), with $q_2 \equiv 0$, and b an arbitrary constant, has no nontrivial periodic solution of any period.

Theorem 8. Let b_2, b_4 be constants satisfying (IV) such that

$$\begin{gathered}\frac{\Psi_2(z)}{z} \leq b_2 \ (|z| \geq 1), \ \frac{\psi_4(y)}{y} \geq b_4 \ (|y| \geq 1),\\ \Psi_2(z) \equiv \int_0^z \psi_2(s)ds.\end{gathered} \tag{3.8}$$

Suppose that

$$\psi_5(x)sgnx \to +\infty(-\infty) \text{ as } |x| \to \infty. \tag{3.9}$$

Suppose further that there are constants $B_1^* > 0, B_2^* \geq 0$ with B_2^* sufficiently small, such that

$$|q_2(t,x,y,z,u,v)| \leq B_1^* + B_2^*(|y| + |z|) \tag{3.10}$$

for all t,x,y,z,u,v. Then the equation (3.2) has at least one periodic solution of period ω.

The procedure for the proof of the theorems is as in [1,2] and [3,4]: for each nonexistence result we need to exhibit a real-valued function with appropriate properties, while for existence the desired a-priori bound will be obtained for a suitably defined parameter-dependent equation.

In what follows $D_i, i = 1, 2, \ldots$ will denote finite positive constants whose magnitude depend on the constants and functions in an equation but are independent of solutions and of parameter μ in the equations.

4. OUTLINE OF PROOF OF THEOREMS 1 - 4

We start with Theorems 1 and 2. Consider first the equation (2.1) with $p_1 \equiv 0$ in the system form

$$\left.\begin{aligned} &\dot{x}_i = x_{i+1}, \quad i = 1, 2, \ldots, 5, \quad x \equiv x_1 \\ &\dot{x}_6 = -f_1(x_5)x_6 - f_2(x_4)x_5 - f_3(x_3)x_4 - f_4(x_3) \\ &\qquad -f_5(x_2) - a_6 x_1, \end{aligned}\right\} \tag{4.1}$$

and define the function $V = V(x_1, x_2, \ldots, x_6)$ by

$$V = U sgn a_1, \tag{4.2}$$

where

$$\left.\begin{aligned} U = x_4x_6 - \frac{1}{2}x_5^2 + a_6x_1x_3 - \frac{1}{2}a_6x_2^2 + x_4F_1(x_5) \\ +F_2(x_4) + F_4(x_3) + x_3f_5(x_2) \\ F_1(x_5) \equiv \int_0^{x_5} f_1(s)ds, \ F_2(x_4) \equiv \int_0^{x_2} sf_2(s)ds, \\ F_4(x_3) \equiv \textstyle\int_0^{x_3} f_4(s)ds \end{aligned}\right\} \tag{4.3}$$

Let $(x_1, x_2, \ldots, x_6) = (x_1(t), x_2(t), \ldots, x_6(t))$ be an arbitrary periodic solution of (4.1), then from (4.2) and (4.3) it is clear on differentiation and using (4.1) that

$$\begin{aligned} \dot{V} = (sgna_1)x_5F_1(x_5) + (sgna_1)f_5'(x_2)x_3^2 \\ - (sgna_1)f_3(x_3)x_4^2 \geq \ 0 \end{aligned} \tag{4.4}$$

by (2.3). The conclusion of Theorem 1 now follows in view of the arguments in [1,4].

To prove Theorem 2, consider, instead of (2.1), the parameter μ-dependent equation

$$\begin{aligned} x^{(6)} + f_1^{\mu}(x^{(4)})x^{(5)} + \mu f_2(\ddot{x})x^{(4)} + f_3^{\mu}(\ddot{x})\,\dddot{x} + \mu f_4(\ddot{x}) \\ +f_5^{\mu}(\dot{x}) + a_6x = \mu\rho_1, \quad 0 \leq \mu \leq 1, \end{aligned} \tag{4.5}$$

or, as is more convenient, the equivalent system

$$\left.\begin{aligned} &\dot{x}_i = x_{i+1},\ i = 1, 2, \dots, 5,\ \ x_1 \equiv x \\ &\dot{x}_6 = -f_1^{\mu}(x_5)x_6 - \mu f_2(x_4)x_5 - f_3^{\mu}(x_3)x_4 - \mu f_4(x_3) \\ &\qquad\qquad -f_5^{\mu}(x_2) - a_6 x + \mu p_1,\ \ 0 \le \mu \le 1, \end{aligned}\right\} \tag{4.6}$$

where

$$\begin{aligned} f_1^{\mu}(x_5) &= (1-\mu)a_1 + \mu f_1(x_5), \\ f_3^{\mu}(x_3) &= (1-\mu)a_3 + \mu f_3(x_3), \\ f_5^{\mu}(x_2) &= (1-\mu)a_5 x_2 + \mu f_5(x_2). \end{aligned} \tag{4.7}$$

Now let $V^{\mu} = V^{\mu}(x_1, x_2, \dots, x_6)$ be defined by

$$V^{\mu} = U^{\mu}(sgna_1) \tag{4.8}$$

where U^{μ} is obtained from U in (4.3) by replacing $F_1(x_5)$, $F_2(x_4)$, $F_4(x_3)$, $f_5(x_2)$ respectively with $F_1^{\mu}(x_5) = (1-\mu)a_1x_5 + \mu F_1(x_5)$, $\mu F_2(x_4)$, $\mu F_4(x_3)$, $f_5^{\mu}(x_2 = (1-\mu)a_5x_2 + \mu f_5(x_2)$. By the continuity of F_1^{μ}, f_3^{μ} and $f_5^{'\mu}$, it can be verified from (2.4) that for some constants $D_i > 0,\ \ i = 1, 2, 3$ and for all x_2, x_3, x_5,

$$\left.\begin{aligned} &(sgna_1)x_5F_1^{\mu}(x_5) \ge |a_1|x_5^2 - D_1, \\ &(sgna_5)f_5^{'\mu}(x_2)x_3^2 \ge |a_5|x_3^2 - D_2, \\ &((sgna_1)f_3^{\mu}(x_3))x_4^2 \le (sgna_1)a_3x_4^2 + D_3. \end{aligned}\right\} \tag{4.9}$$

Let $(x, x_1, \dots, x_6)$ be an ω-periodic solution of (4.6). Then on differentiating (4.8) and using (4.2), (4.3), it will follow from (4.9) that

$$\begin{aligned} \dot{V}^{\mu} &= (sgna_1)x_5F_1^{\mu}(x_5) + x_3^2(sgna_1)f_5^{'\mu}(x_2) - x_4^2(sgna_1)f_3^{\mu}(x_3) - \mu x_4 p_1 \\ &\ge |a_1|x_5^2 + |a_5|x_3^2 + \gamma x_4^2 - A_1|x_4| - 2A_2(x_3^2 + x_4^2 + x_5^2) - D_4, \\ &\qquad \gamma = -(sgna_1)a_3 > 0, \end{aligned}$$

so that if

$$A_2 < \frac{1}{4}\min(|a_1|, |a_5|, \gamma) \equiv D_5$$

then

$$\dot{V}^{\mu} \ge D_5(x_3^2 + x_4^2 + x_5^2) - D_6,\ \ D_6 = D_4 + \gamma^{-1}A_1^2. \tag{4.10}$$

By the ω-periodicity of V^{μ}, (4.10) implies that

$$\int_0^{\omega} (x_3^2(t) + x_4^2(t) + x_5^2(t))dt \le D_7,$$

and hence

$$|x_2(t)| \leq D_8,\ |x_3(t)| \leq D_8,\ |x_4(t)| \leq D_8$$

for some constants D_7, D_8. The rest of the arguments follow as in [1§5].

Turning now to Theorems 3 and 4, consider first the equation (2.2), with $p_2 \equiv 0$, in the system form

$$\left.\begin{aligned} &\dot{x}_i = x_{i+1},\ i = 1, 2, \ldots, 5, \quad x \equiv x_1 \\ &\dot{x}_6 = -a_1 x_6 - g_2(x_4)x_5 - g_3(x_3)x_4 - g_4(x_2)x_3 \\ &\qquad\quad -a_5 x_5 - g_6(x_1) \end{aligned}\right\} \tag{4.11}$$

Let $V = V(x_1, x_2, \ldots, x_6)$ be defined by

$$V = -U \tag{4.12}$$

where

$$\left.\begin{aligned} &U = x_3 G_2(x_4) + G_3(x_3) + G_5(x_2) + x_2 g_6(x_1) + x_3 x_6 \\ &\qquad + a_1 x_3 x_5 - \tfrac{1}{2} a_1 x_4^2 \\ &G_2(x_4) \equiv \int_0^{x_4} g_2(s)ds,\ G_3(x_3) \equiv \int_0^{x_3} s g_3(s)ds, \\ &\qquad G_5(x_2) \equiv \int_0^{x_2} g_5(s)ds \end{aligned}\right\} \tag{4.13}$$

Let $(x_1, x_2, \ldots, x_6) = (x_1(t), x_2(t), \ldots, x_6(t))$ be an arbitrary solution of (4.11) of period α say. Then, on differentiating (4.12) and using (4.13) and (4.11), it will follow, after some calculation, that

$$\dot{V} = -x_4 G_2(x_4) + x_3^2 g_4(x_2) - x_2^2 g_6'(x_1) \geq 0$$

by (2.6). The rest of the argument is as in [1,4].

For Theorem 4, consider the parameter μ-dependent system

$$\left.\begin{aligned} &\dot{x}_i = x_{i+1}, i = 1, 2, \ldots, 5 \quad x \equiv x_1 \\ &\dot{x}_6 = -a_1 x_6 - g_2^\mu(x_4)x_5 - \mu g_3(x_3)x_4 - g_4^\mu(x_2)x_3 \\ &\qquad\quad -\mu g_5(x_2) - g_6^\mu(x_1) + \mu p_2(t, x_1, \ldots, x_6) \end{aligned}\right\} \tag{4.14}$$

where

$$\begin{aligned} g_2^\mu(x_4) &= (1-\mu)a_2 + \mu g_2(x_4), \\ g_4^\mu(x_2) &= (1-\mu)a_4 + g_4(x_2), \\ g_6^\mu(x_1) &= (1-\mu)a_6 x_1 + \mu g_6(x_1), \end{aligned}$$

a_1 is an arbitrary constant and a_2, a_4, a_6 are constants satisfying (II). Let $V^\mu = V^\mu(x_1, x_2, \ldots, x_6)$ be defined by $V^\mu = -U^\mu$, where

$$\begin{aligned} U^\mu &= x_3 G_2^\mu(x_4) + \mu G_3(x_3) + \mu G_5(x_2) + x_2 g_6^\mu(x_1) \\ &\quad + x_3 x_6 + a_1 x_3 x_5 - \frac{1}{2} a_1 x_4^2 \end{aligned}$$

and $G_2^\mu(x_4) = (1-\mu)a_2 x_4 + \mu G_2(x_4)$. By the continuity of g_2, g_6' and g_4, and by (2.7), it will be clear that for some constants D_9 and D_{10},

$$\begin{aligned} x_4 G_2^\mu(x_4) &\geq |a_2| x_4^2 - D_9, \\ -x_2^2 g_6'^\mu(x_1) &\geq |a_6| x_2^2 - D_9, \\ g_4^\mu(x_2) x_3^2 &\geq -(sgn a_2) a_4 x_3^2 - D_{10} \end{aligned} \tag{4.16}$$

for all x_1, x_2, x_3 and x_4. Thus, for any arbitrary ω-periodic solution $(x_1, x_2, \ldots, x_6) = (x_1(t), x_2(t), \ldots, x_6(t)$ of (4.14), it can shown that

$$\begin{aligned} \dot{V}^\mu &= -x_4 G_2^\mu(x_4) - x_2^2 g_6'^\mu(x_1) + x_3^2 g_4^\mu(x_2) - \mu x_3 p_2 \\ &\geq |a_2| x_4^2 + \beta x_3^2 + |a_6| x_2^2 - A_1^* |x_3| - 2A_2^*(x_2^2 + x_3^2 + x_4^2) - D_{11} \\ &\geq D_{12}(x_2^2 + x_3^2 + x_4^2) - D_{11}, \end{aligned}$$

$\beta \equiv -(sgn a_2) a_4 > 0$, by (4.16) and (2.8), for some constants D_{11}, D_{12}. The rest of the proof follows as in [1,4].

5. INDICATION OF PROOF OF THEOREMS 5 - 8

Since the arguments are essentially the same as those in §4, we shall merely indicate the appropriate equivalent system and the scalar function V required for nonexistence and, for existence, the parameter μ-dependent system and its corresponding scalar function V^μ.

We start with Theorem 5. The equivalent system (to (3.1) with $q_1 \equiv 0$) is

$$\left.\begin{aligned} &\dot{x}_i = x_{i+1}, \quad i = 1, 2, \ldots, 4 \qquad x_1 \equiv x \\ &\dot{x}_5 = \varphi_1(x_4) x_5 - \varphi_2(x_3) x_4 - \varphi_3(x_2) x_3 - \varphi_4(x_2) - \varphi_5(x_1), \end{aligned}\right\} \tag{5.1}$$

and the scalar function $V = V(x_1, x_2, \ldots, x_5)$ is defined by

$$V = (sgn b_1) U, \tag{5.2}$$

$$U = x_3 \Phi_1(x_4) + \Phi_2(x_3) + \Phi_4(x_2) + x_2 \varphi_5(x_1) + x_3 x_5 - \frac{1}{2} x_4^2, \tag{5.3}$$

$$\left.\begin{aligned}\Phi_1(x_4) &\equiv \int_0^{x_4} \varphi_1(s)ds,\\ \Phi_2(x_3) &\equiv \int_0^{x_3} s\varphi_2(s)ds,\\ \Phi_4(x_2) &\equiv \int_0^{x_2} \varphi_4(s)ds.\end{aligned}\right\} \tag{5.4}$$

For Theorem 6 the appropriate equivalent system to consider is

$$\left.\begin{aligned}&\dot{x}_i = x_{i+1}, \quad i = 12, 3, 4 \qquad x \equiv x_1\\ &\dot{x}_5 = -\varphi_1^\mu(x_4)x_5 - \mu\varphi_2(x_3)x_4 - \varphi_3^\mu(x_2)x_3\\ &\qquad -\mu\varphi_4(x_2) - \varphi_5^\mu(x_1) + \mu q_1, \quad 0 \leq \mu \leq 1,\end{aligned}\right\} \tag{5.5}$$

$$\left.\begin{aligned}\varphi_1^\mu(x_4) &= (1-\mu)b_1 + \mu\varphi_1(x_4),\\ \varphi_3^\mu(x_2) &= (1-\mu)b_3 + \mu\varphi_3(x_2),\\ \varphi_5^\mu(x_1) &= (1-\mu)b_5 + \mu\varphi_5(x_1)\end{aligned}\right\} \tag{5.6}$$

and the scalar function $V^\mu = V^\mu(x_1, x_2, x_3, x_4, x_5)$ is defined by

$$V^\mu = (sgnb_1)U^\mu,$$

with U^μ obtained from (5.3) by replacing Φ_1 with Φ_1^μ, Φ_2 with $\mu\Phi_2$, Φ_4 with Φ_4^μ and φ_5 with φ_5^μ, noting the appropriate definitions in (5.4) and (5.6). It can be readily verified that

$$\begin{aligned}\dot{V}^\mu &= (sgnb_1)x_4\Phi_1^\mu(x_4) - (sgnb_1)x_3^2\varphi_3^\mu(x_2) + (sgnb_1)x_2^2\varphi_5^{'\mu}(x_1)\\ &\quad -\mu x_3 q_1 \geq D_{13}(x_2^2 + x_3^2 + x_4^2) - D_{14}\end{aligned}$$

for some constants D_{13}, D_{14}, if B_2 is sufficiently small.

We turn lastly to Theorems 7 and 8. The equivalent system (to (3.2) with $q_2 \equiv 0$) is

$$\left.\begin{aligned}&\dot{x}_i = x_{i+1}, \; i = 1, 2, 3, 4 \qquad x_1 \equiv x\\ &\dot{x}_5 = -bx_5 - \psi_2(x_3)x_4 - \psi_3(x_2)x_3 - \psi_4(x_2) - \psi_5(x_1),\end{aligned}\right\} \tag{5.7}$$

and the appropriate scalar function V is given by

$$\begin{aligned}V = &-x_2\Psi_2(x_3) - x_2x_5 - \Psi_3(x_2) - \Psi_5(x_1)\\ &- b_1x_2x_4 + \frac{1}{2}b_1x_3^2 + x_3x_4,\end{aligned} \tag{5.8}$$

$$\Psi_2(x_3) = \int_0^{x_3} \psi_2(s)ds, \; \Psi_3(x_2) = \int_0^{x_2} s\psi_3(s)ds,$$

$$\Psi_5(x_1) = \int_0^{x_1} \psi_5(s)ds.$$

For Theorem 8, the equivalent parameter μ-dependent system to consider is

$$\left.\begin{aligned} &\dot{x}_i = x_{i+1}, \quad i = 1, 2, 3, 4, \qquad x_1 \equiv x \\ &\dot{x}_5 = -bx_5 - \psi_2^\mu(x_3)x_4 - \mu\psi_3(x_2)x_3 - \psi_4^\mu(x_2) \\ &\qquad\qquad -\mu\psi_5(x_1) + \mu q_2, \quad 0 \leq \mu \leq 1, \end{aligned}\right\} \tag{5.9}$$

$$\psi_2^\mu(x_3) = (1-\mu)b_2 + \mu\psi_2(x_3), \; \psi_4^\mu(x_2) = (1-\mu)b_4 + \mu\psi_4(x_2). \tag{5.10}$$

The corresponding function $V^\mu = V^\mu(x_1, x_2, x_3, x_4, x_5)$ is given by

$$\begin{aligned} V^\mu = &-x_2\Psi_2^\mu(x_3) - \mu\Psi_3(x_2) - \mu\Psi_5(x_1) \\ &- b_1x_2x_4 + \frac{1}{2}b_1x_3^2 + x_3x_4, \end{aligned} \tag{5.11}$$

with $\Psi_2^\mu(x_3) \equiv \int_0^{x_3} \psi_2^\mu(s)ds$, and ψ_2^μ defined by (5.10).

From (5.11), (5.9), (5.10) and (3.9) it will be clear that

$$\begin{aligned} \dot{V}^\mu &= x_4^2 - x_3\Psi_2^\mu(x_3) + x_2^2\Psi_4^\mu(x_1) - \mu x_2q_2 \\ &\geq x_4^2 - b_2x_3^2 + b_4x_2^2 - B_1^*|x_2| - 2B_2^*(x_2^2 + x_3^2 + x_4^2), \end{aligned}$$

so that if B_2^* is sufficiently small, then for some constants D_{15}, D_{16},

$$\dot{V}^\mu \geq D_{15}(x_2^2 + x_3^2 + x_4^2) - D_{16}.$$

Because of some technical difficulties we have not been able to extend the results in §3 to full blown nonlinear sixth order equations of the form

$$\begin{aligned} x^{(6)} + f_1(x^{(4)})x^{(5)} + f_2(\dddot{x})x^{(4)} + f_3(\ddot{x})\,\dddot{x} + f_4(\ddot{x}) + f_5(\dot{x}) \\ + f_6(x) = p(t, x, \dot{x}, \ldots, x^{(5)}), \end{aligned}$$

efforts are still continuing in this direction. This remark also holds for equation (3.2).

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