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THE BOUNDEDNESS OF SOLUTIONS OF CERTAIN NONLINEAR THIRD ORDER ORDINARY DIFFERENTIAL EQUATIONS

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ABSTRACT. This paper establishes some new sufficient conditions under which all solutions of nonlinear third-order ordinary differential equation

$$x''' + \psi(x, x', x'')x'' + f(x, x') = p(t, x, x', x'')$$

are bounded. For illustration, an example is also given on the bounded solutions.

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1. INTRODUCTION

This paper is concerned with the boundedness of solutions of the third order ordinary differential equation

$$x''' + \psi(x, x', x'')x'' + f(x, x') = p(t, x, x', x'') \tag{1}$$

or its equivalent system

$$x' = y, y' = z, z' = -\psi(x, y, z)z - f(x, y) + p(t, x, y, z) \tag{2}$$

where $\psi \in C(\mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $f \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ and $p \in C([0, \infty) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$. The functions ψ, f and p depends only on the argument displayed explicitly, and the primes denote differentiation with respect to t. The derivatives ψ_x, ψ_y, ψ_z, f_x and f_y exist and are continuous. For over five decades, many authors have dealt with ordinary differential equations and obtained many interesting results, for example, see([1] - [15]) and the references cited therein. In many of these references, the authors made use of the second method of Lyapunov by considering Lyapunov functions and obtained conditions which ensure some qualitative behaviors of the

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problem. However, the construction of these Lyapunov functions remain a general problem. Many special cases of (1) exist in the literature, see[8], where authors discussed some qualitative behaviors of solutions of the equations. In particular, recently, Tunc[12] studied the differential equation

$$x''' + \psi(x, x')x'' + f(x, x') = p(t, x, x', x'') \tag{3}$$

and obtained sufficient conditions which ensure the boundedness of solutions of the equation. The motivation for the present paper has been inspired basically by the paper of Tunc[12] and the papers mentioned above. The main objective of this paper is to extend results obtained in Tunc[12] to obtain sufficient conditions for the boundedness of solutions of (1).

2. MAIN RESULTS

Our main result is the following theorem.

Theorem 1: *In addition to the basic assumptions imposed on the functions ψ, f and p appearing in (2), we assume that there exist positive constants δ_0, a, b and $c(ab > c)$ such that the following conditions hold:*

(i) $\frac{f(x,0)}{x} \geq \delta_0, (x \neq 0), f'(x,0) \leq c, \psi(x,y,z) \geq a,$
$f_y(x, \theta y) \geq b, y\psi_z(x,y,\theta z) \geq 0, 0 \leq \theta \leq 1$ *and*
$a\left[f(x,y) - f(x,0) - \int_0^y \psi_x(x,\nu,0)\nu d\nu\right] y \geq y\int_0^y f_x(x,\nu)d\nu,$

(ii) $|p(t,x,y,z)| \leq q_1(t) + q_2(t)(|y| + |z|),$
where $q_1, q_2 \in L^1(0,\infty), L^1(\infty,0)$, *is a space of integrable Lebesgue functions.*

Then, there exists a finite positive constant K such that every solution $(x(t), y(t), z(t))$ of system (2) satisfies

$$|x(t)| \leq \sqrt{K} \qquad |y(t)| \leq \sqrt{K} \qquad |z(t)| \leq \sqrt{K}$$

Remark 1: *Equation (3) is a special case of (1) if $\psi(x, x', x'') = \psi(x, x')$. Thus, if $q_2(t) = 0$, we still obtain a boundedness result obtained by Tunc[12].*

Proof: The proof of this theorem depends on a scalar differentiable Lyapunov's function $V = V(x, y, z)$. This function and its time derivative satisfy some fundamental inequalities. Let

$$V=\int_0^x f(u,0)du+\int_0^y \psi(x,\nu,0)\nu d\nu+\frac{1}{a}\int_0^y f(x,\nu)d\nu+\frac{1}{2a}z^2+yz. \tag{4}$$

This function can be rearranged as follows:

$$\begin{aligned} V &= \tfrac{1}{2a}(ay+z)^2+\tfrac{1}{2ab}(f(x,0)+by)^2+\textstyle\int_0^y\left[\psi(x,\nu,0)-a\right]\nu d\nu \\ &+ \tfrac{1}{a}\textstyle\int_0^y\left[f_\nu(x,\theta\nu)-b\right]\nu d\nu+\int_0^x\left[1-\tfrac{1}{ab}f'(u,0)\right]f(u,0)du, \end{aligned} \tag{5}$$

since $f_\nu(x,\theta\nu)=\frac{f(x,\nu)-f(x,0)}{\nu}$, $(\nu\neq 0, 0\leq\theta\leq 1)$.
Obviously, it follows from (4) that

$$V\geq\frac{1}{2a}(ay+z)^2+\frac{1}{2ab}(f(x,0)+by)^2+\frac{1}{2}\left(1-\frac{c}{ab}\right)\delta_0 x^2.$$

Thus, there exist a positive constant K_1 such that

$$V\geq K_1\left(x^2+y^2+z^2\right).$$

Now, let $(x,y,z)=(x(t),y(t),z(t))$ be any solution of the system (2). Differentiating the function V given by (4) along the system (2) with respect to t, we obtain

$$\begin{aligned} \tfrac{d}{dt}V(x,y,z) &= -\tfrac{1}{a}(\psi(x,y,z)-a)z^2-\psi_z(x,y,\theta z)yz^2 \\ &- \left[f(x,y)-f(x,0)-\textstyle\int_0^y\psi_x(x,\nu,0)\nu d\nu\right] \\ &+\tfrac{1}{a}y\textstyle\int_0^y f_x(x,\nu)d\nu+\tfrac{1}{a}(ay+z)p(t,x,y,z). \end{aligned}$$

Making use of assumption (i) of Theorem 1, we have that

$$\frac{d}{dt}V(x,y,z)\leq\frac{1}{a}(ay+z)p(t,x,y,z).$$

On using assumption (ii) of Theorem 1, the inequality $2|u\nu|\leq u^2+\nu^2$ and the fact that

$$y^2+z^2\leq x^2+y^2+z^2\leq K_1^{-1}V(x,y,z), \tag{6}$$

we easily obtain

$$\begin{aligned} \tfrac{d}{dt}V(.) &\leq \left(|y|+\tfrac{1}{a}|z|\right)(q_1(t)+q_2(t))(|y|+|z|) \\ &\leq K_2(|y|+|z|)(q_1(t)+q_2(t))(|y|+|z|) \\ &\leq K_2q_1(t)\left(2+y^2+z^2\right)+2K_2q_2(t)\left(y^2+z^2\right) \\ &\leq K_2\left(2+K_1^{-1}V(x,y,z)\right)q_1(t)+2K_2^{-1}V(x,y,z)q_2(t) \\ &= 2K_2q_1(t)+K_2K_1^{-1}V(x,y,z)(q_1(t)+2q_2(t)) \end{aligned} \tag{7}$$

where $K_2 = \min\left\{1, \frac{1}{a}\right\}$. Integrating(7) from 0 to t, using the assumption $q_1, q_2 \in L^1(0,\infty)$ and Gronwall-Reid-Bellman inequality, we have

$$V(x,y,z) \leq (V(0,0,0) + 2K_2A_1)\exp\left(K_2K_1^{-1}(A_1 + 2A_2)\right) = K_3 < \infty \tag{8}$$

where $K_3 > 0$ is a constant, $A_1 = \int_0^\infty q_1(s)ds$ and $A_2 = \int_0^\infty q_2(s)ds$. In view of the inequalities (6) and (8), we get

$$x^2(t) + y^2(t) + z^2(t) \leq K_1^{-1}V(x,y,z) \leq K$$

where $K = K_3K_1^{-1}$. Aforementioned inequality implies that

$$|x(t)| \leq \sqrt{K}, \qquad |y(t)| \leq \sqrt{K}, \qquad |z(t)| \leq \sqrt{K}$$

for all $t \geq t_0 \geq 0$. Hence

$$|x(t)| \leq \sqrt{K}, \qquad |x'(t)| \leq \sqrt{K}, \qquad |x''(t)| \leq \sqrt{K}$$

for all $t \geq t_0 \geq 0$. Thus, the proof of the theorem is now complete. □

Example 1: Consider equation (2) with

$$\psi(x,y,z) = \ln\left(1+x^2\right) + e^{yz} + 2, f(x,y) = x + \frac{x}{1+x^2}\left(1+y^2\right) + y + \frac{1}{3}y^3$$

and

$$p(t,x,y,z) = \frac{1}{1+t^2+x^2+y^2+z^2}.$$

It is easy to check that the hypotheses in Theorem 1 are satisfied. Since $\frac{f(x,0)}{x} = 1 + \frac{1}{1+x^2} > 1 = \delta_0, (x \neq 0), f'(x,0) = 1 + \frac{1-x^2}{(1+x^2)^2} \leq 2 = c, \psi(x,y,z) > 2 = a$ and

$$\begin{array}{rcl} 2[f(x,y) - f(x,0) & - & \int_0^y \psi_x(x,\nu,0)\nu d\nu]y \\ & = & 2\left[\frac{x}{1+x^2}y^2 + y + \frac{1}{3}y^3 - \frac{x}{1+x^2}y^2\right]y \\ & = & 2\left(y^2 + \frac{1}{3}y^4\right) \\ & \geq & y^2 + \frac{1-x^2}{(1+x^2)^2}\left(y^2 + \frac{1}{3}y^4\right) = y\int_0^y f_x(x,\nu)d\nu. \end{array}$$

Finally, we have

$$|p(t,x,y,z)| \leq \frac{1}{1+t^2} + \frac{2}{1+t^2}\left(|y(t)| + |z(t)|\right),$$

$$\int_0^\infty q_1(s)ds = \int_0^\infty \frac{1}{1+s^2}ds = \frac{\pi}{2} < \infty$$

and

$$\int_0^\infty q_2(s)ds = \int_0^\infty \frac{2}{1+s^2}ds = \pi < \infty$$

that is, $q_1, q_2 \in L^1(0,\infty)$.

Hence all the hypotheses in Theorem 1 are satisfied, and so for every solution $x(t)$ of equation(1) there is a constant $K > 0$ such that

$$|x(t)| \le \sqrt{K}, \qquad |x'(t)| \le \sqrt{K}, \qquad |x''(t)| \le \sqrt{K} \;\; for \;\; t \ge 0.$$

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