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ON BOUNDEDNESS AND STABILITY OF SOLUTIONS OF CERTAIN THIRD ORDER DELAY DIFFERENTIAL EQUATION

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ABSTRACT. In this paper, sufficient criteria which guarantee the existence of uniform asymptotic stability and boundedness of solution of a scalar real third-order delay differential equation were established with the aid of a suitable Lyapunov function. With the Lyapunov function, conditions on the nonlinear terms to guarantee stability and boundedness of the solution and its derivative were given.

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1. INTRODUCTION

In this paper, we are concerned with the uniform asymptotic stability and boundedness of solutions of the equation

$$\dddot{x} + a\ddot{x} + g(\dot{x}(t-\tau)) + h(x(t-\tau)) = p(t), \tag{1}$$

where functions g, h and p are continuous and depend (at most) only on the arguments displayed explicitly, a being a constant and $\tau > 0$ a fixed delay. Here and elsewhere, all the solutions considered and all the functions which appear are supposed real. The dots indicate differentiation with respect to t. When $\tau = 0$ the above equation reduces to an ordinary nonlinear third order differential equation which have received great attention by researchers (see for instance [1],[7], [11]-[14], [18]-[19], [21], [24]-[26], [31] and the references contained therein). The Lyapunov second method was used extensively by the above researchers to discuss the qualitative properties of various form of nonlinear third order differential equations without delay. Some of these results have been summarized in [23].

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The Lyapunov second method had also been found useful and applicable to study the qualitative properties of the equation with delay (see [2]-[3], [8]-[10], [20], [22], [27]-[30]).
In [2], the second order analogue of this study was carried out and the author constructed a Lyapunov functions which was later converted to a Lyapunov functional.
Also in [21], the author gave a fundamental procedure where a nonlinear differential equation with delay could be discussed as approximation to linear differential equations.
In [20], the author adapted [2] and [21] and use a suitable complete Lyapunov function to establish criteria which guarantee existence of unique solution that is bounded together with its derivatives on the real line, globally stable and periodic under explicit conditions on the nonlinear terms of the equation considered (here $g(\dot{x}(t-\tau)) = b\dot{x}$).
In [18], the authors with the use of a complete Lyapunov function established that the nonlinear equation without delay has solutions that are bounded and stable.

In this work, we want to adopt the approach in [2] and [21] to extend the result in [18] to the equation (1) and give sufficient criteria which guarantee the existence of uniform asymptotic stability and boundedness of the solution with their derivatives on the real line.
An associated system to the equation (1) of interest to us is given as

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= z \\ \dot{z} &= -az - g(y) - h(x) + N(t) \end{aligned} \tag{2}$$

where

$$N(t) = \int_{-\tau}^{0} [g'(y(t+\theta))z(t+\theta) + h'(x(t+\theta))y(t+\theta)]d\theta + p(t)$$

At this juncture we will like to refer the reader to [4]-[6],[15]-[17] and [32] for terminologies, techniques and standard results.
The paper is organized as follow, section Two presents some basic definitions and theorem relevant to this work. The formulation of our results is presented in section Three while In section Four, preliminary results where necessary lemmas vital to the proof of the main results are given. The last section contain the proofs of our main results.

2. SOME BASIC DEFINITIONS AND THEOREMS

For completeness sake, we shall give some basic definitions as well as an important result in our our development. For $x \in \Re^n$, let $|x|$ denotes the Euclidean norm in $\Re^n$. For a given $\tau > 0$, let C denotes the space of continuous functions mapping the interval $[-\tau, 0]$ into $\Re^n$, and, for $\varphi \in C, \|\varphi\| = \sup_{-\tau\leq\theta\leq 0} \|\varphi(\theta)|$. Let also C_H denote the set of $\varphi \in C$ such that $\|\varphi\| \leq H$ where $H > 0$. If x is a continuous function of u defined on $-\tau \leq t < A,\ A > 0$, and if t is a fixed number satisfying $0 \leq t < A$, which implies that $x_t \in C$ and can be defined as $x_t(\theta) = x(t+\theta)$ for $-\tau \leq \theta \leq 0$. x_t denotes the restriction of x to the interval $[t-\tau,\ t],\ \tau > 0$.
Let $\dot{x}(t)$ denote the right-hand derivative of $x(v)$ at $v = t$ and consider the functional differential equation

$$\dot{x}(t) = F(t, x_t), \tag{3}$$

where F is a continuous functional in $(t,\ \varphi)$ for $-\infty < t < \infty$. Moreover F takes bounded sets into bounded sets.
By solution of equation (3) we mean a function $x(t, \varphi)$, which satisfies equation (3). To specify a solution of equation (3), we require a $t_0 \in \Re$ and a function $\varphi \in C$.
Definition 1: A function $x(t_0, \varphi)$ is said to be a solution of equation (3) with initial condition $\varphi \in C_H$ at $t = t_0,\ t_0 \geq 0$, if there is a $B > 0$ such that $x(t_0, \varphi)$ is a function from $[t_0 - \tau,\ t_0 + B)$ into $\Re^n$ with the following properties

(i) $x_t(t_0,\ \varphi) \in C_H$ for $t_0 \leq t < t_0 + B$
(ii) $x_t(t_0,\ \varphi) = \varphi,$
(iii) $x_t(t_0,\ \varphi)$ satisfies equation (3) for $t_0 \leq t < t_0 + B$

We denote by $x_t(t; t_0,\ \varphi)$ the value of $x_t(t_0, \varphi)$ at t.

Definition 2: A Liapunov functional is a continuous $V(t, \varphi)$: $[0, \infty) \times C_H \longrightarrow [0, \infty)$ whose derivative along a solution of equation (3) will be denoted by $\dot{V}_{(3)}$ and is defined by

$$\dot{V}_{(3)}(t, \varphi) = \lim_{h \to 0^+} \sup \frac{1}{h} \{V(t+h, x_{t+h}(t, \varphi)) - V(t, \varphi)\} \tag{4}$$

where $x(t_0, \varphi)$ is the solution of equation (3) with $x_{t_0}(t_0, \varphi) = \varphi$

Definition 3: Let $F(t, 0) = 0$. The zero solution of the equation (3) is said to be:

(a) stable if and only if for any $t_0 \geq 0$ and $\epsilon > 0$, there is a positive $\delta = \delta(\epsilon, t_0)$ such that $\varphi \in C_\delta$ implies $|x(t; t_0, \varphi)| < \epsilon$ for $t \geq t_0$.
(b) asymptotically stable if it is stable and if for each $t_0 \geq 0$ there is a $\delta > 0$ such that $\varphi \in C_\delta$ implies $x(t; t_0, \varphi) \longrightarrow 0$ as $t \longrightarrow \infty$
(c) uniformly stable if the number δ defined in the Definition 3 is independent of t_0.
(d) uniformly asymtotically stable if it is uniformly stable and if there is a $\delta_0 > 0$ and for every $\eta > 0$ there exists a $T(\eta) > 0$ such that $\varphi \in C_{\delta_0}$ implies $|x(t; t_0, \varphi)| < \eta$ for $t \geq t_0 + T(\eta)$ and for every $t_0 \geq 0$.
(e) equi-asymptotically stable if it is stable and if η and δ in (d) depend on t_0

Definition 4: Solutions of the equation (3) are said to be:

(a) bounded if there exist a $B > 0$, such that $|x(t; t_0, \varphi)| < B$ for all $t \geq t_0$, B may depend on each solution.
(b) uniform bounded if for each $B_1 > 0$, there exists $B_2 > 0$ such that $\{t_0 \in \Re, \varphi \in C, \|\varphi\| < B_1, t \geq t_0\}$ imply that $|x(t; t_0, \varphi)| < B_2$
(c) uniform ultimately bounded for bound B if for each $B_3 > 0$ there exists $K > 0$ such that $\{t_0 \in \Re, \varphi \in C, \|\varphi\| < B_3, t \geq t_0 + K\}$ imply that $|x(t; t_0, \varphi)| < B$

We will now state a well known result due to Burton [5].

Theorem A: Let $H > 0$ and let $C_H \subset C$ with $\varphi \in C_H$, if $\|\varphi\| < H$. Suppose $V : \Re \times C_H \longrightarrow [0, \infty)$ is continuous and locally Lipschitz in φ. Let W_i be wedges:

(a) if $V(t, 0) = 0$; $W(|\varphi|) \leq V(t, \varphi)$ and $\dot{V}_{(3)}(t, X_t) \leq 0$ then the zero solution of equation (3) is stable.
(b) if $W_1(|\varphi(0)|) \leq V(t, \varphi) \leq W_2(|\varphi|)$ and $\dot{V}_{(3)}(t, X_t) \leq 0$ then the zero solution of the equation (3) is uniformly stable
(c) if $F(t, \varphi)$ is bounded for $\|\varphi\| < H$ and if $V(t, 0) = 0$ $W_1(|\varphi(0)|) \leq V(t, \varphi)$ and $\dot{V}_{(3)}(t, X_t) \leq -W_2(|X(t)|)$ then the zero solution of (3) is equi-asymptotically stable.
(d) Let $\|.\|$ be the L^2 norm on C. If $W_1(|\varphi(0)|) \leq V(t, \varphi) \leq W_2(|\varphi|)$ and $\dot{V}_{(3)}(t, X_t) \leq -W_4(|X(t)|)$ then the zero solution of (3) is uniformly asymptotically stable.

3. FORMULATION OF RESULTS

Now let the functions $g,\ h$ and p be continuous and the following conditions hold:

(i) I_0 is a subset of $\Re$ defined as $I_0 = [0, \Delta], \Delta > 0$,

(ii) $\dfrac{h(x) - h(0)}{x} = H_0 \leq \alpha \in I_0 = [0, a\alpha]\ x \neq 0$;

(iii) $\dfrac{g(y) - g(0)}{y} = G_0 \leq \beta \in I_0,\ y \neq 0$

(iv) $h(0) = g(0) = 0$.

where α, β and $a > \dfrac{H_0}{G_0}$ are all positive.

We will now state our main results.

Theorem 1: Suppose that conditions (i)-(iv) are satisfied with $p(t) \equiv 0$, then the trivial solution of the equation (1) is uniformly asymptotically stable.

Theorem 2: In addition to conditions (i)-(iv) being satisfied, suppose that the following is also satisfied

(v)

$$p(t) \leq M\ (constant)$$

for all $t \geq 0$, then there exists a constant σ, $(0 < \sigma < \infty)$ depending only on the constants $\alpha,\ \beta$ and δ such that every solution of (1) satisfies

$$x^2(t) + \dot{x}^2(t) + \ddot{x}^2(t) \leq e^{-\sigma t} \left\{ A_1 + A_2 \int_{t_0}^{t} |p(\tau)|\, e^{\frac{1}{2}\sigma\tau} d\tau \right\}^2,$$

for all t $\geq t_0$, where the constant $A_1 > 0$, depends on $\alpha,\ \beta$ and δ as well as on $t_0, x(t_0), \dot{x}(t_0), \ddot{x}(t_0)$; and the constant $A_2 > 0$ depends on $\alpha,\ \beta$ and δ.

Remark: We wish to remark here that while the Theorem 1 is on the uniform asymptotic stability of the trivial solution, Theorems 2 deals with the boundedness of the solutions.

Notations: Throughout this paper $K, K_0, K_1, \ldots K_{12}$ will denote finite positive constants whose magnitudes depend only on the functions h, g and p as well as constants $\alpha,\ \beta,\ \delta$ and Δ but are independent of solutions of the equation (1). $K_i's$ are not necessarily the same for each time they occur, but each $K_i, i = 1, 2...$ retains its identity throughout.

4. PRELIMINARY RESULTS

We shall use as a tool to prove our main results a Lyapunov function $V(x, y, z)$ defined by

$$V = V_A + V_B, \tag{5}$$

where

$$\begin{aligned} 2V_A = \frac{\delta}{\Delta} \{ &[\alpha\Delta + (1-\epsilon)\alpha^2\beta]x^2 + [a^3(1-\epsilon)^2 + a\alpha - \Delta]y^2 \\ &+ az^2 + 2a(1-\epsilon)^2\alpha xy + 2\Delta xz + 2a^2(1-\epsilon)yz \} \end{aligned} \tag{6}$$

and

$$V_B = \frac{\gamma}{2\tau}\int_{-\tau}^{0}\left\{\int_{\theta_1}^{0}[x^2(t+\theta) + y^2(t+\theta) + z^2(t+\theta)]d\theta\right\}d\theta_1, \tag{7}$$

where $a, \alpha, \beta, \epsilon, \Delta, \gamma, \delta$ and τ are all positive for all x, y, z with $0 < \epsilon < 1$, $\beta > \dfrac{(1-\epsilon)}{\alpha^2}$, $(1-\epsilon) < a < 1$ and $\Delta < a\alpha$.
The following lemmas are needed in the proofs of Theorems 1 and 2.

Lemma 1: Subject to the assumptions of Theorem 1 there exist positive constants $K_i = K_i(\alpha, \beta, \epsilon, \Delta, \gamma, \delta, \tau), i = 1, 2$ such that

$$K_1(x^2 + y^2 + z^2) \le V(x, y, z) \le K_2(x^2 + y^2 + z^2). \tag{8}$$

Proof. Re-arranging equation (6) we have,

$$\begin{aligned} 2V_A = \frac{\delta}{\Delta} \{ &[(1-\epsilon)x + a(1-\epsilon)y + az]^2 + [\alpha\Delta + (1-\epsilon)a^2\beta \\ &-(1-\epsilon)^2]x^2 + [a^3(1-\epsilon) + a\alpha - \Delta - a^2(1-\epsilon)^2]y^2 \\ &+ a(1-a)z^2 \}. \end{aligned} \tag{9}$$

it is evident from the above that $V_A(0,0,0) \equiv 0$ and from the equation (9), we obtain

$$\begin{aligned} 2V_A \ge \frac{\delta}{\Delta} \{ &[\alpha\Delta + (1-\epsilon)a^2\beta - (1-\epsilon)^2]x^2 + [a^3(1-\epsilon) + a\alpha \\ &-\Delta - a^2(1-\epsilon)^2]y^2 + a(1-a)z^2 \} \\ &\ge K_1(x^2 + y^2 + z^2), \end{aligned} \tag{10}$$

where

$$K_1 = \frac{\delta}{2\Delta} \cdot \min\left\{\alpha\Delta + (1-\epsilon)a^2\beta - (1-\epsilon)^2, a^3(1-\epsilon)\right.$$
$$\left. + a\alpha - \Delta - a^2(1-\epsilon)^2, a(1-a)\right\}.$$

Therefore,

$$V \geq K_1(x^2+y^2+z^2)$$
$$+\frac{\gamma}{2\tau}\int_{-\tau}^{0}[x^2(t+\theta)+y^2(t+\theta)+z^2(t+\theta)]d\theta. \tag{11}$$

By using the inequality $|xy| \leq \frac{1}{2}|x^2+y^2|$ in the equation (6), we have

$$2V_A \leq \frac{\delta}{\Delta}\left\{[\alpha\Delta + (1-\epsilon)\alpha^2\beta]x^2 + [a^3(1-\epsilon)^2 + a\alpha - \Delta]y^2\right.$$
$$+az^2 + a(1-\epsilon)^2\alpha(x^2+y^2) + \Delta(x^2+z^2) \tag{12}$$
$$\left. + a^2(1-\epsilon)(y^2+z^2)\right\} .$$

Further simplification of inequality (12) gives

$$V_A \leq K_2(x^2+y^2+z^2), \tag{13}$$

where

$$K_2 = \frac{\delta}{2\Delta} \cdot \max\left\{\alpha\Delta + (1-\epsilon)\alpha^2\beta + a(1-\epsilon)^2\alpha + \Delta,\right.$$
$$\left. a^3(1-\epsilon)^2 + a\alpha - \Delta + a^2(1-\epsilon) + a(1-\epsilon)^2\alpha, a + \Delta + a^2(1-\epsilon)\right\}.$$

Therefore,

$$V \leq K_2(x^2+y^2+z^2)$$
$$+\frac{\gamma}{2\tau}\int_{-\tau}^{0}[x^2(t+\theta)+y^2(t+\theta)+z^2(t+\theta)]d\theta. \tag{14}$$

The R.H.S. of the inequalities (11) and (14) are always positive, hence by the definition of K_1 and K_2, V is positive definite and so we have

$$K_1(x^2+y^2+z^2) \leq V(x,y,z) \leq K_2(x^2+y^2+z^2), \tag{15}$$

which proves Lemma 4.1. $\square$

Lemma 2: In addition to assumptions of Theorem 1, let the condition (v) of the Theorem 2 be satisfied also. Then there are positive constants $K_j = K_j(\alpha, \beta, \epsilon, \Delta, \gamma, \delta, \tau)(j = 3, 4)$ such that for any solution (x, y, z) of the system (2),

$$\begin{aligned} \dot{V}|_{(2)} &\equiv \frac{d}{dt} V|_{(2)}(x, y, z) \\ &\leq -K_3(x^2 + y^2 + z^2) + K_4 \left(x^2 + y^2 + z^2\right)^{\frac{1}{2}} N(t). \end{aligned} \tag{16}$$

Proof. By the definition of V we have that $\dot{V} = \dot{V}_A + \dot{V}_B$;
From equations (1) and (2) we have,

$$\begin{aligned} \dot{V}_2 &= \frac{\partial V}{\partial x}\dot{x} + \frac{\partial V}{\partial y}\dot{y} + \frac{\partial V}{\partial z}\dot{z} \\ &= \frac{\partial V}{\partial x} y + \frac{\partial V}{\partial y} z + \frac{\partial V}{\partial z}(-az - g(y) - h(x) + N(t).) \end{aligned}$$

with

$$\begin{aligned} \dot{V}_A = \frac{\delta}{\Delta} \big\{ &[\alpha\Delta + (1-\epsilon)\alpha^2\beta]xy + [a^3(1-\epsilon)^2 + a\alpha - \Delta]yz \\ &+ az(-az - g(y) - h(x) + N(t))a(1-\epsilon)^2\alpha[y^2 + xz] \\ &+ \Delta[x(-az - g(y) - h(x) + N(t)) + yz] \\ &+ a^2(1-\epsilon)[z^2 + y(-az - g(y) - h(x) + N(t)] \big\} \end{aligned} \tag{17}$$

Using the conditions on $h(x)$ and $g(y)$,

$$\dot{V}_A = -\frac{\delta}{\Delta}\left\{x^2 + y^2 + z^2 - (\Delta x + a^2(1-\epsilon)y + az)N(t)\right\} \tag{18}$$

$$\leq -\frac{\delta}{\Delta}\left\{x^2 + y^2 + z^2 - K_*(|x| + |y| + |z|)N(t)\right\}, \tag{19}$$

where

$$K_* = \max\left\{\Delta, a^2(1-\epsilon), a\right\}.$$

Therefore

$$\dot{V}_A \leq -K_3(x^2 + y^2 + z^2) + K_4(|x| + |y| + |z|)N(t), \tag{20}$$

where

$$K_3 = \frac{\delta}{\Delta} \text{ and } K_4 = \frac{K_*\delta}{\Delta}.$$

Also from the definition of V_b it follows that

$$\dot{V}_B \leq \frac{\gamma}{2\tau} \int_{-\tau}^{0} \{[x^2(t) - x^2(t+\theta) + y^2(t) - y^2(t+\theta)$$

$$+z^2(t) - z^2(t+\theta)]\}d\theta$$

$$= \frac{\gamma}{2\tau}\int_{-\tau}^{0}(x^2(t) + y^2(t) + z^2(t))d\theta$$
$$-\frac{\gamma}{2\tau}\int_{-\tau}^{0}(x^2(t+\theta)) + y^2(t+\theta) + z^2(t+\theta)d\theta.$$

Therefore,

$$\begin{aligned}\dot{V}_B &\le \gamma(x^2(t) + y^2(t) + z^2(t))\\ &\quad -\frac{\gamma}{2\tau}\int_{-\tau}^{0}(x^2(t+\theta)) + y^2(t+\theta) + z^2(t+\theta)d\theta.\end{aligned} \tag{21}$$

Combining inequalities (20) and (21) for $\gamma > 0$, we have that

$$\begin{aligned}\dot{V} &\le -(K_3 - \gamma)(x^2 + y^2 + z^2)\\ &\quad -\frac{\gamma}{2\tau}\int_{-\tau}^{0}\{(x^2(t+\theta)) + y^2(t+\theta)\\ &\quad +z^2(t+\theta)\}d\theta + K_4(|x| + |y| + |z|)N(t)\end{aligned} \tag{22}$$

for $K_3 > \gamma$ we have

$$\begin{aligned}\dot{V} \le -K_5(x^2 + y^2 + z^2) - \frac{\gamma}{2\tau}\int_{-\tau}^{0}\{(x^2(t+\theta) + y^2(t+\theta)\\ +z^2(t+\theta)\}d\theta + K_4(|x| + |y| + |z|)N(t)\end{aligned} \tag{23}$$

where

$$K_5 = K_3 - \gamma > 0.$$

Consequently

$$\dot{V} \le -K_5(x^2 + y^2 + z^2) + K_4(|x| + |y| + |z|)N(t). \tag{24}$$

Since (see [23],[19])

$$(|x| + |y| + |z|) \le \sqrt{3}\left(x^2 + y^2 + z^2\right)^{\frac{1}{2}},$$

then the inequality (24) becomes

$$\dot{V} \le -K_5(x^2 + y^2 + z^2) + K_6\left(x^2 + y^2 + z^2\right)^{\frac{1}{2}} N(t), \tag{25}$$

where

$$K_6 = \sqrt{3}K_4.$$

This completes the proof of Lemma 3.2. □

5. PROOF OF MAIN RESULTS

We shall now give the proofs of the main results.

Proof of Theorem 1. The proof of Theorem 1 follows from Lemmas 1 and 2 where it has been established that the trivial solution of the equation (1) is uniformly asymptotically stable. i.e every solution $(x(t), \dot{x}(t), \ddot{x}(t))$ of the system (2) satisfies $x^2(t) + \dot{x}^2(t) + \ddot{x}^2(t) \longrightarrow 0$ as $t \longrightarrow \infty$. $\square$

Proof of Theorem 2. Clearly from equations (11) and (14),we have that

$$V \geq K_1(x^2 + y^2 + z^2) \tag{26}$$

Combining inequalities (26) and (14) we have

$$\begin{aligned} K_1(x^2 + y^2 + z^2) &\leq V(t, x, y, z, x_t, y_t, z_t) \\ &\leq K_1(x^2 + y^2 + z^2) \qquad (27) \\ &+\frac{\gamma}{2\tau}\int_{-\tau}^{0}(x^2(t+\theta)) + y^2(t+\theta) + z^2(t+\theta)d\theta \end{aligned}$$

Indeed from the inequality (25),

$$\frac{dV}{dt} \leq -K_5 V + K_6(x^2 + y^2 + z^2)^{\frac{1}{2}} \left|N(t)\right|.$$

Also from inequality (15),

$$K_1(x^2 + y^2 + z^2) \leq V,$$

which implies that

$$(x^2 + y^2 + z^2)^{\frac{1}{2}} \leq \left(\frac{V}{K_1}\right)^{\frac{1}{2}}$$

Thus inequality (25) becomes

$$\frac{dV}{dt} \leq -K_5 V + K_7 V^{\frac{1}{2}} \left|N(t)\right|, \tag{28}$$

where

$$K_7 = \frac{K_6}{\sqrt{K_1}}.$$

This can be written as

$$\dot{V} \leq -2K_8 V + K_7 V^{\frac{1}{2}} \left|N(t)\right|, \tag{29}$$

with

$$K_8 = \frac{1}{2}K_5.$$

Therefore

$$\dot{V} + K_8 V \leq -K_8 V + K_7 V^{\frac{1}{2}} \left|N(t)\right| \tag{30}$$

$$\leq K_7 V^{\frac{1}{2}} \left\{\left|N(t)\right| - K_9 V^{\frac{1}{2}}\right\}, \tag{31}$$

where

$$K_9 = \frac{K_8}{K_7}.$$

Thus inequality (31) becomes,

$$\dot{V} + K_8 V \le K_7 V^{\frac{1}{2}} V^*, \tag{32}$$

where

$$V^* = \left\{ |N(t)| - K_9 V^{\frac{1}{2}} \right\} \tag{33}$$
$$\le |N(t)|.$$

Let us note that when $|N(t)| \le -K_9 V^{\frac{1}{2}}$

$$V^* \le 0. \tag{34}$$

and when $|N(t)| \ge K_9 V^{\frac{1}{2}}$,

$$V^* \le |N(t)| \cdot \frac{1}{K_9}, \tag{35}$$

Substituting (35) into (32) we have

$$\dot{V} + K_8 V \le K_{10} V^{\frac{1}{2}} |N(t)|, \tag{36}$$

where $K_{10} = \dfrac{K_7}{K_9}$.

This implies that (36) can be written as

$$V^{-\frac{1}{2}} \dot{V} + K_8 V^{\frac{1}{2}} \le K_{10} |N(t)|. \tag{37}$$

Multiplying both sides of the inequality (37) by $e^{\frac{1}{2}K_8 t}$, gives

$$e^{\frac{1}{2}K_8 t} \left\{ V^{-\frac{1}{2}} \dot{V} + K_8 V^{\frac{1}{2}} \right\} \le e^{\frac{1}{2}K_8 t} K_{10} \left| N(t) \right|, \tag{38}$$

i.e

$$2 \frac{d}{dt} \left\{ V^{\frac{1}{2}} e^{\frac{1}{2}K_8 t} \right\} \le e^{\frac{1}{2}K_8 t} K_{10} \left| N(t) \right|. \tag{39}$$

Integrating both sides of (39) from t_0 to t, gives

$$\left\{ V^{\frac{1}{2}} e^{\frac{1}{2}K_8 \omega} \right\}_{t0}^{t} \le \int_{t0}^{t} \frac{1}{2} e^{\frac{1}{2}K_8 \tau} K_{10} \left| N(\tau) d\tau \right|, \tag{40}$$

which implies that

$$\left\{ V^{\frac{1}{2}}(t) \right\} e^{\frac{1}{2}K_8 t} \le V^{\frac{1}{2}}(t_0) e^{\frac{1}{2}K_8 t_0} + \frac{1}{2} K_{10} \int_{t0}^{t} \left| N(\tau) \right| e^{\frac{1}{2}K_8 \tau} d\tau,$$

or

$$V^{\frac{1}{2}}(t) \le e^{-\frac{1}{2}K_8 t} \left\{ V^{\frac{1}{2}}(t_0) e^{\frac{1}{2}K_8 t_0} + \frac{1}{2} K_{10} \int_{t0}^{t} \left| N(\tau) \right| e^{\frac{1}{2}K_8 \tau} d\tau \right\}.$$

On utilizing inequalities (11) and (13), we have

$$K_1(x^2(t)+\dot{x}^2(t)+\ddot{x}^2(t)) \le e^{-K_8 t}\left\{ K_2(x^2(t_0)+\dot{x}^2(t_0)+\ddot{x}^2(t_0))^{\frac{1}{2}} \right.$$

$$\left. e^{\frac{1}{2}K_8 t_0} + \frac{1}{2}K_{10}\int_{t0}^{t} |N(\tau)|\, e^{\frac{1}{2}K_8\tau} d\tau \right\}^2, \qquad (41)$$

for all $t \ge t_0$.
Thus,

$$x^2(t)+\dot{x}^2(t)+\ddot{x}^2(t) \le \frac{1}{K_1}\left\{ e^{-K_8 t}\left\{ K_2(x^2(t_0)+\dot{x}^2(t_0)+\ddot{x}^2(t_0))^{\frac{1}{2}} \right.\right.$$

$$\left. e^{\frac{1}{2}K_8 t_0} + \tfrac{1}{2}K_{10}\int_{t0}^{t} |N(\tau)|\, e^{\frac{1}{2}K_8\tau} d\tau \right\}^2$$

$$\le e^{-K_8 t}\left\{ A_1 + A_2\int_{t0}^{t} |N(\tau)|\, e^{\frac{1}{2}K_8\tau} d\tau \right\}^2, \qquad (42)$$

where A_1 and A_2 are constants depending on $\{K_1, K_2, \ldots K_{10}$ and $(x^2(t_0)+\dot{x}^2(t_0)+\ddot{x}^2(t_0))$.
By substituting $K_8 = \sigma$ in the inequality (42), we have

$$x^2(t)+\dot{x}^2(t)+\ddot{x}^2(t) \le e^{-\sigma t}\left\{ A_1 + A_2\int_{t0}^{t} |N(\tau)|\, e^{\frac{1}{2}\sigma\tau} d\tau \right\}^2.$$

$$\le K \qquad (43)$$

for sufficiently large t where K is a constant This completes the proof. □

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