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STABILITY AND ULTIMATE BOUNDEDNESS OF SOLUTIONS OF A CERTAIN THIRD ORDER NONLINEAR VECTOR DIFFERENTIAL EQUATION

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ABSTRACT. The aim of this paper is to present some new results related to the stability and ultimate boundedness of solutions of a certain third order non-linear ordinary vector differential equation by using Lyapunov's second method. The results obtained in this paper improve and generalize the results of Tunc[23] and Omeike and Afuwape[14].

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1. INTRODUCTION

We shall consider here systems of real differential equations of the form

$$\dddot{X} + \Psi(X, \dot{X})\ddot{X} + \Phi(\dot{X}) + cX = P(t, X, \dot{X}, \ddot{X}) \tag{1}$$

which is equivalent to the system

$$\begin{aligned} \dot{X} &= Y \\ \dot{Y} &= Z \\ \dot{Z} &= -\Psi(X, Y)Z - \Phi(Y) - cX + P(t, X, Y, Z). \end{aligned} \tag{2}$$

This is obtained as usual by setting $\dot{X} = Y$, $\ddot{X} = Z$ in (1) in which $t \in \mathbb{R}^+$, $\mathbb{R}^+$ denote the real line $0 < t < \infty$ and $X \in \mathbb{R}^n$, c is a positive constant, Φ is a continuous vector function and Ψ is $n \times n$ - continuous symmetric positive definite matrix functions for the argument displayed explicitly and the dots indicate differentiation with respect to t, and $P : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ in Eq.(1). It is also assumed that the Jacobian matrix $J\Phi(Y)$ exist and is symmetric positive definite and continuous. It will be assumed that the

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conditions for the existence and the uniqueness of the solutions of Eq.(1) are satisfied. (see Picard-Lindelof in Rao[15]).
Equations of the form (1) in the special case with $\Psi(X,\dot{X})\ddot{X} = A\dot{X}$ and $\Phi(\dot{X}) = B\dot{X}$ have been studied by various authors in [2], [6-11] and [13]. They have obtained some results related to the boundedness, periodicity and stability properties of solutions. In the case $n = 1$, (1) and various other third order differential equations and their various vector analogue have received a considerable amount of attention during the past forty years. See, for example [1], [3], [4], [5], [12], [17], [18], [19], [20], [21], [22] and [24]. Many of these results are summarized in [16].
Motivation for the study of (1) comes from the work of Tunc[23] and Omeike and Afuwape[14]. They studied the particular case of the equation (1) in the form

$$\dddot{X} + \Psi(\dot{X})\ddot{X} + B\dot{X} + cX = P(t). \tag{3}$$

Tunc[23], using an incomplete Lyapunov function obtained sufficient conditions on the stability and boundedness of solutions for the cases $P \equiv 0$ and $P \neq 0$ respectively while Omeike and Afuwape[14] employed a complete Lyapunov function for the case $P \neq 0$ and proved that the system is ultimately bounded.
Our main objective in this paper is to study the stability and ultimate boundedness results of equation (1), by using suitable Lyapunov functions. These will generalize earlier results of Tunc[23] and those of Omeike and Afuwape[14].

2. PRELIMINARY

The following results will be basic to the proofs of Theorems.

Lemma 1([2]): Let A be a real symmetric positive definite $n \times n$-matrix. Then for any $X \in \mathbb{R}$

$$\delta_a\|X\|^2 \leq \langle AX, X\rangle \leq \triangle_a\|X\|^2$$

where δ_a and $\triangle_a$ are respectively, the least and greatest eigenvalues of the matrix A.

Lemma 2([19]): Subject to earlier conditions on $\Psi(X,Y)$ the following is true:

$$\frac{d}{dt}\int_0^1 \langle\sigma\Psi(X,\sigma Y)Y, Y\rangle d\sigma = \langle\Psi(X,Y)Y, Z\rangle$$

Proof:

$$\frac{d}{dt}\int_0^1 \langle \sigma\Psi(\sigma X, Y)Y, Y\rangle d\sigma$$

$$= \int_0^1 \langle \sigma\Psi(\sigma X, Y)Z, Y\rangle d\sigma + \int_0^1 \langle \sigma J(\Psi(\sigma X, Y)Y|\sigma Y)Z, Y\rangle d\sigma$$
$$+ \int_0^1 \langle \sigma\Psi(\sigma X, Y)Y, Z\rangle d\sigma$$
$$= \int_0^1 \langle \sigma\Psi(\sigma X, Y)Z, Y\rangle d\sigma + \int_0^1 \sigma\langle J(\Psi(\sigma X, Y)Y|\sigma Y)Z, Y\rangle d\sigma$$
$$+ \int_0^1 \langle \sigma\Psi(\sigma X, Y)Y, Z\rangle d\sigma$$
$$= \int_0^1 \langle \sigma\Psi(\sigma X, Y)Z, Y\rangle d\sigma + \int_0^1 \sigma\frac{\partial}{\partial\sigma}\langle \sigma\Psi(\sigma X, Y)Z, Y\rangle d\sigma$$
$$= \sigma^2\langle \Psi(\sigma X, Y)Z, Y\rangle|_0^1 = \langle \Psi(X, Y)Z, Y\rangle.\square$$

Lemma 3([19]): Subject to earlier conditions on $\Phi(Y)$ the following is true:

$$\text{(i)} \qquad \frac{d}{dt}\int_0^1 \langle \Phi(\sigma Y), Y\rangle d\sigma = \langle \Phi(Y), Z\rangle$$

$$\text{(ii)} \qquad \langle \Phi(\sigma Y), Y\rangle = \int_0^1 \langle \sigma J(\Phi(\sigma Y)Y, Y)\rangle d\sigma$$

Proof:

$$\frac{d}{dt}\int_0^1 \langle \sigma\Phi(\sigma Y), Y\rangle d\sigma$$

$$= \int_0^1 \langle \sigma\Phi(\sigma Y), Z\rangle d\sigma + \int_0^1 \langle \sigma J(\Phi(\sigma Y)|\sigma Y), Z\rangle d\sigma$$
$$+ \int_0^1 \langle \sigma\Phi(\sigma Y), Z\rangle d\sigma$$
$$= \int_0^1 \langle \sigma\Phi(\sigma Y), Z\rangle d\sigma + \int_0^1 \sigma\langle J(\Phi(Y)|\sigma Y), Z\rangle d\sigma$$
$$+ \int_0^1 \langle \sigma\Phi(\sigma Y), Z\rangle d\sigma$$
$$= \int_0^1 \langle \sigma\Phi(\sigma Y), Z\rangle d\sigma + \int_0^1 \sigma\frac{\partial}{\partial\sigma}\langle \sigma\Phi(\sigma Y), Z\rangle d\sigma$$
$$= \sigma^2\langle \Phi(\sigma Y), Z\rangle|_0^1 = \langle \Phi(Y), Z\rangle.\square$$

(ii) follows from making use of the result:

$$\Phi(Y) = \int_0^1 J\Phi(\sigma Y)Y\,d\sigma$$

for arbitrary $Y \in \mathbb{R}^n$, which follows from integrating the equality

$$\frac{d}{d\sigma}\Phi(\sigma Y) = J\Phi(\sigma Y)Y.$$

with respect to σ and then using the fact that $\Phi(0) = 0$.

In the case $P \equiv 0$ in the system (1), the first result of this paper is the following theorem.

3. MAIN RESULTS

Theorem 0.1. *Let c be a positive constant, Φ a continuous vector function and Ψ an $n \times n$ - continuous symmetric positive definite matrix functions. We also assume that $J\Phi(Y)$ exist and is symmetric positive definite and continuous. We further suppose that there are positive constants a_o and b_o such that the following conditions are satisfied:*

(i) $n\times n$ continuous symmetric positive definite matrices $J\Phi(Y)$ and $\Psi(X,Y)$ commute with each other and
(ii) $\lambda_i(\Psi(X,Y)) \geq a_o$ and $\lambda_i(J\Phi(Y)) \geq b_o$, (i=1,2,...,n) with $a_o b_o - c > 0$

for all $X, Y \in \mathbb{R}^n$.

Then, every solution $(X,Y,Z)\equiv(X(t),Y(t),Z(t))$ of system (2) satisfies

$$X(t) \to 0, \quad Y(t) \to 0, \quad Z(t) \to 0 \quad \text{as} \;\; t \to \infty.$$

Proof: Our main tool in the proof of the result is the Lyapunov function V=V(X,Y,Z) and its derivative $\frac{dV}{dt}$ which both imply the stability of zero solution of Eq.(2) defined by

$$\left.\begin{aligned} 2V &= c\langle X, X\rangle + 2\int_0^1 \langle \sigma\Psi(X,\sigma Y)Y, Y\rangle d\sigma \\ &\quad + 2\delta\int_0^1 \langle \Phi(\sigma Y), Y\rangle d\sigma + \delta\langle Z, Z\rangle + 2\langle Y, Z\rangle + 2\delta c\langle Y, X\rangle \end{aligned}\right\} \tag{4}$$

where

$$\frac{1}{a_o} < \delta < \frac{b_o}{c} \tag{5}$$

This function, after re-arrangements, can be re-written as

$$\begin{aligned} 2V &= \delta b_o\|Y + \frac{c}{b_o}X\|^2 + \delta\|Z + \delta^{-1}Y\|^2 \\ &+ \left\{2\int_0^1 \langle \sigma\Psi(X,\sigma Y)Y, Y\rangle d\sigma - \delta^{-1}\langle Y, Y\rangle\right\} \\ &+ c(1 - \frac{\delta c}{b_o})\langle X, X\rangle + 2\delta\int_0^1 \langle \Phi(\sigma Y), Y\rangle d\sigma - \delta b_o\langle Y, Y\rangle. \end{aligned} \tag{6}$$

We can now verify the properties of this function. First, it is clear from (6) that

$$V(0,0,0) = 0$$

Next, in view of the assumption of the Theorem 0.1 and Lemma 1 respectively, it follows that

$$\begin{aligned} &2\int_0^1 \langle \sigma\Psi(X,\sigma Y)Y, Y\rangle d\sigma - \delta^{-1}\langle Y, Y\rangle \\ &= 2\int_0^1 \langle \sigma(\Psi(X,\sigma Y) - \delta^{-1}I)Y, Y\rangle d\sigma \\ &\geq (a_o - \frac{1}{\delta})\|Y\|^2 \end{aligned}$$

and

$$\begin{aligned} &2\delta\int_0^1 \langle \Phi(\sigma Y), Y\rangle d\sigma - \delta b_o\langle Y, Y\rangle \\ &= 2\delta\int_0^1 \langle \Phi(\sigma Y), Y\rangle d\sigma - 2\delta b_o\int_0^1 \sigma\langle Y, Y\rangle d\sigma \\ &= 2\delta\int_0^1\int_0^1 \langle \sigma(J\Phi(\sigma\tau) - b_oI)Y, Y\rangle d\sigma d\tau \geq 0 \end{aligned}$$

Also, in addition

$$c(1 - \frac{\delta c}{b_o})\langle X, X\rangle = c(1 - \frac{\delta c}{b_o})\|X\|^2,$$

where by (5), $c(1 - \frac{\delta c}{b_o}) > 0$

Hence, one can get from (6) that

$$\begin{aligned} V &\geq \delta b_o\|Y + cX\|^2 + \delta\|Z + \delta^{-1}Y\|^2 \\ &+ \frac{1}{2}c(1 - \frac{\delta c}{b_o})\|X\|^2 + \frac{1}{2}(a_o - \frac{1}{\delta})\|Y\|^2. \end{aligned} \tag{7}$$

Thus it is evident from the terms contained in (7) that there exists a constant $d_1 > 0$ small enough such that

$$V \geq d_1(\|X\|^2 + \|Y\|^2 + \|Z\|^2). \tag{8}$$

Now let (X,Y,Z)=(X(t),Y(t),Z(t)) be any solution of differential system (2). Differentiating the function V=V(X(t),Y(t),Z(t)) with respect to t along system (2) and using Lemma 2, we have

$$\begin{aligned}\dot{V} &= -\langle \Phi(Y), Y\rangle + \delta c\langle Y, Y\rangle - \delta\langle \Psi(X,Y)Z, Z\rangle + \langle Z, Z\rangle \\ &= -\int_0^1 \langle (J\Phi(\sigma Y) - \delta cI)Y, Y\rangle d\sigma - \delta\langle \Psi(X,Y)Z, Z\rangle + \langle Z, Z\rangle\end{aligned}$$

it follows that

$$\dot{V} \leq -(b_o - \delta c)\|Y\|^2 - (a_o\delta - 1)\|Z\|^2 \tag{9}$$

where by (5), $(b_o - \delta c) > 0$ and $(a_o\delta - 1) > 0$,

Thus

$$\dot{V}(t) \leq 0$$

In addition, one can easily see that

$$V(X,Y,Z) \to \infty \quad as \quad \|X\|^2 + \|Y\|^2 + \|Z\|^2 \to \infty$$

Then by Theorem 0.1, the zero solution of Eq.(1) is globally asymptotic stable.□

Remark 1:
If we take $\Psi(X,\dot{X})\ddot{X} = \Psi(\dot{X})\ddot{X}$, $\Phi(\dot{X}) = B\dot{X}$ and $P(t,X,\dot{X},\ddot{X}) = 0$ in (1), Theorem 0.1 reduces to the result obtained in Tunc[23].

In the case $P \neq 0$, the second main result of this paper is the following theorem.

Theorem 0.2. *In addition to the conditions of Theorem 0.1, we suppose that there exist positive constants $\delta_o, \epsilon, a_o, a_1, b_o, b_1$ such that the following are satisfied:*

(i) $b_o \leq \lambda_i(J\Phi(Y)) \leq b_1$, $a_o{+}\epsilon \leq \lambda_i(\Psi(X,Y)) \leq a_1$, *(i=1,2,...,n) for all $X, Y \in \mathbb{R}^n$, and $a_ob_o - c > 0$;*

(ii) P satisfies

$$\|P(t,X,Y,Z)\| \leq \delta_o + \delta_1(\|X\| + \|Y\| + \|Z\|)$$

uniformly for all $X, Y, Z \in \mathbb{R}^n$ where $\delta_o \geq 0$, $\delta_1 \geq 0$ are constants and δ_1 is sufficiently small.

Then, there exist a constant $D > 0$ such that any solution (X(t), Y(t),Z(t)) of the system (2) ultimately satisfies

$$\|X(t)\| \leq D, \qquad \|Y(t)\| \leq D, \qquad \|Z(t)\| \leq D \qquad for \quad t \in \mathbb{R}^+$$

where the magnitude of D depends only on δ_o,δ_1,Ψ,Φ and P.

Proof: Proof of Theorem 0.2 depends on some certain fundamental properties of a continuously differentiable Lyapunov function V=V(X,Y,Z) defined by

$$V = V_1 + V_2$$

where V_1, V_2 are given by

$$2V_1 = c\langle X, X\rangle + 2\int_0^1 \langle \sigma\Psi(X, \sigma Y)Y, Y\rangle d\sigma + 2\delta\int_0^1 \langle \Phi(\sigma Y), Y\rangle d\sigma \\ + \delta\langle Z, Z\rangle + 2\langle Y, Z\rangle + 2\delta c\langle Y, X\rangle$$

where

$$\frac{1}{a_o} < \delta < \frac{b_o}{c}$$

and

$$2V_2 = a_o c\langle X, X\rangle + 2a_o\int_0^1 \langle \sigma\Psi(X, \sigma Y)Y, Y\rangle d\sigma + \alpha a_o b_o^2\langle X, X\rangle \\ + 2\int_0^1 \langle \Phi(\sigma Y), Y\rangle + \langle Z, Z\rangle + 2\alpha b_o a_o^2\langle X, Y\rangle + 2\alpha a_o b_o\langle X, Z\rangle \\ + 2a_o\langle Y, Z\rangle + 2c\langle X, Y\rangle - \alpha a_o b_o\langle Y, Y\rangle$$

where

$$0 < \alpha < min\left\{\frac{1}{a_o}, \frac{a_o}{b_o}, \frac{a_o b_o - c}{a_o b_o[a_o + c^{-1}(b_1 - b_o)^2]}, \frac{(a_o\delta - 1)c}{a_o b_o(a_1 - a_o)^2}\right\} \tag{10}$$

and $a_1 > a_o$, $b_1 \neq b_o$.

We note that the function V_1 (used here) is the same function used in the proof of Theorem 0.1 (Equation (4)),

Hence, V_1 is now the expression in (7) that is,

$$V_1 \geq \delta b_o\|Y + cX\|^2 + \delta\|Z + \delta^{-1}Y\|^2 + \frac{1}{2}c(1 - \frac{\delta c}{b_o})\|X\|^2 \\ + \frac{1}{2}(a_o - \frac{1}{\delta})\|Y\|^2$$

and the term

$$2a_o\int_0^1 \langle \sigma\Psi(\sigma X, Y)Y, Y\rangle d\sigma - a_o^2\langle Y, Y\rangle$$

in the arrangement of $2V_2$ and in view of the assumption of the Theorem 0.2 and Lemma 1 respectively, it follows that

$$\begin{aligned} 2\, a_o \int_0^1 & \langle \sigma J(\Psi(X, \sigma Y)Y, Y\rangle d\sigma - 2a_o^2 \int_0^1 \sigma \langle Y, Y\rangle d\sigma \\ &= 2a_o \int_0^1 \langle \sigma J(\Psi(\sigma X, Y) - a_o I)Y, Y\rangle d\sigma \geq 0 \end{aligned}$$

combining these results, we have

$$\begin{aligned} V \geq & [c(1 - \frac{\delta c}{b_o}) + \frac{1}{2}(\alpha a_o b_o^2(1 - \alpha a_o) + \frac{1}{2}c(a_o - c\delta b_o^{-1}))]\|X\|^2 \\ & + [(a_o - \frac{1}{\delta}) + \frac{1}{2}a_o(a_o - \alpha b_o)]\|Y\|^2 + \delta b_o\|Y + cX\|^2 \\ & + \delta\|Z + \delta^{-1}Y\|^2 + \|Z + a_o Y + \alpha a_o b_o X\|^2. \end{aligned} \tag{11}$$

Thus it is evident from the terms contained in (11) that there exists sufficiently small positive constant d_3 such that

$$V \geq d_3(\|X\|^2 + \|Y\|^2 + \|Z\|^2) \tag{12}$$

where $d_3 = \frac{1}{2}min\{c(1 - \frac{\delta c}{b_o}) + (\alpha a_o b_o^2(1 - \alpha a_o) + c(a_o - cb_o^{-1})), (a_o - \frac{1}{\delta}) + a_o(a_o - \alpha b_o)\}$.

Now let (X,Y,Z)=(X(t),Y(t),Z(t)) be any solution of differential system (1.2). Differentiating the function V=V(X(t),Y(t),Z(t)) with respect to t along system (2) using lemma 2 and lemma 3 yields, for $\dot{V}_1$,

$$\begin{aligned} \dot{V}_1 = & -\langle \Phi(Y), Y\rangle + \delta c\langle Y, Y\rangle - \delta\langle \Psi(X, Y)Z, Z\rangle + \langle Z, Z\rangle \\ & + \langle Y + \delta Z, P(t, X, Y, Z)\rangle, \end{aligned}$$

and for $\dot{V}_2$,

$$\begin{aligned} \dot{V}_2 = & -\alpha a_o b_o c\langle X, X\rangle - a_o\langle \Phi(Y), Y\rangle + a_o\langle Y, Y\rangle + \alpha a_o^2 b_o\langle Y, Y\rangle \\ & - \langle(\Psi(X, Y)Z, Z\rangle - a_o\langle Z, Z\rangle - a_o b_o\langle(\Psi(X, Y)X, Z\rangle \\ & - a_o\langle X, Z\rangle - \alpha a_o b_o\langle \Phi(Y), X\rangle + \alpha a_o b_o^2\langle Y, X\rangle \\ & + \langle(\alpha a_o b_o X + a_o Y) + Z, P(t, X, Y, Z)\rangle. \end{aligned}$$

Combining $\dot{V}_1$ and $\dot{V}_2$ we obtain $\dot{V} = \dot{V}_1 + \dot{V}_2$ as

$$\begin{aligned} \dot{V} = & -\alpha a_o b_o c\langle X, X\rangle - \langle \Phi(Y), Y\rangle + \delta c\langle Y, Y\rangle - a_o\langle \Phi(Y), Y\rangle \\ & + c\langle Y, Y\rangle + \alpha a_o^2 b_o\langle Y, Y\rangle - \langle(\delta\Psi(X, Y) - I)Z, Z\rangle \\ & - \langle(\Psi(X, Y) - a_o I)Z, Z\rangle - a_o b_o\langle(\Psi(X, Y) \\ & - a_o I)X, Z\rangle - \alpha a_o b_o\langle \Phi(Y), X\rangle + \alpha a_o b_o^2\langle Y, X\rangle \\ & + \langle(\alpha a_o b_o X + (1 + a_o)Y + (1 + \delta)Z, P(t, X, Y, Z)\rangle. \end{aligned}$$

Using lemma 3(ii) and re-arranging, we write $\dot{V}$, thus,

$$\begin{aligned}\dot{V} = &-\frac{1}{2}\alpha a_o b_o c\langle X, X\rangle - \langle(a_o J\Phi - cI - \alpha a_o^2 b_o I)Y, Y\rangle + a_o\langle Z, Z\rangle \\ &- (b_o - \delta c)\langle Y, Y\rangle - \langle(\Psi(X,Y) - a_o I)Z, Z\rangle - \frac{1}{4}\alpha a_o b_o\left\{\langle cX, X\rangle\right. \\ &\left.+4\langle(\Psi(X,Y) - a_o I)X, Z\rangle\right\} - (a_o\delta - 1)\langle Z, Z\rangle \\ &- \frac{1}{4}\alpha a_o b_o\{\langle cX, X\rangle + 4\langle(J\Phi - b_o I)X, Y\rangle\} \\ &+ \langle\alpha a_o b_o X + (1 + a_o)Y + (1 + \delta)Z, P(t, X, Y, Z)\rangle.\end{aligned}$$

We note that

$$\begin{aligned}&\langle cX, X\rangle + 4\langle(\Psi(X,Y) - a_o I)X, Z\rangle \\ &= c\{\langle X, X\rangle + 4c^{-1}\langle(\Psi(X,Y) - a_o I)X, Z\rangle\} \\ &= c\|X + 2c^{-1}(\Psi(X,Y) - a_o I)Z\|^2 - \|2c^{-\frac{1}{2}}(\Psi(X,Y) - a_o I)Z\|^2\end{aligned}$$

and that

$$\begin{aligned}&\langle cX, X\rangle + 4\langle(J\Phi - b_o I)X, Y\rangle \\ &= c\|X + 2c^{-1}(J\Phi - b_o I)Y\|^2 - \|2c^{-\frac{1}{2}}(J\Phi - b_o I)Y\|^2\end{aligned}$$

it follows that

$$\begin{aligned}\dot{V} \le &-\frac{1}{2}\alpha a_o b_o c\langle X, X\rangle - \langle(a_o J\Phi - cI - \alpha a_o^2 b_o I)Y, Y\rangle \\ &- (b_o - \delta c)\langle Y, Y\rangle - \langle(\delta\Psi(X,Y) - I)Z, Z\rangle \\ &- \langle(\Psi(X,Y) - a_o I)Z, Z\rangle + \frac{1}{4}\alpha a_o b_o\|2c^{-\frac{1}{2}}(\Psi(X,Y) - a_o I)Z\|^2 \\ &+ \frac{1}{4}\alpha a_o b_o\|2c^{-\frac{1}{2}}(J\Phi - b_o I)Y\|^2 \\ &+ \langle\alpha a_o b_o X + (1 + a_o)Y + (1 + \delta)Z, P(t, X, Y, Z)\rangle.\end{aligned}$$

Since

$$\|2c^{-\frac{1}{2}}(J\Phi - b_o I)Y\|^2 = 4\langle c^{-\frac{1}{2}}(J\Phi - b_o I)Y, c^{-\frac{1}{2}}(J\Phi - b_o I)Y\rangle$$

and

$$\begin{aligned}&\|2c^{-\frac{1}{2}}(\Psi(X,Y) - a_o I)Z\|^2 \\ &= 4\langle c^{-\frac{1}{2}}(\Psi(X,Y) - a_o I)Z, c^{-\frac{1}{2}}(\Psi(X,Y) - a_o I)Z\rangle\end{aligned}$$

we have that

$$\begin{aligned}\dot{V} \leq & -\frac{1}{2}\alpha a_o b_o c\langle X, X\rangle - \langle (a_o J\Phi - cI - \alpha a b_o[a_o I \\ & + c^{-1}(b_1 - b_o)^2]Y, Y\rangle - (b_o - \delta c)\langle Y, Y\rangle \\ & - \langle a_o\delta - 1 - \alpha c^{-1}(\Psi(X,Y) - a_o I)^2 Z, Z\rangle \\ & - \langle (\delta\Psi(X,Y) - I)Z, Z\rangle + \langle \alpha a_o b_o X + (1 + a_o)Y \\ & + (1+\delta)Z, P(t, X, Y, Z)\rangle.\end{aligned}$$

In view of lemma 1, we have

$$\begin{aligned}\dot{V} \leq & -\frac{1}{2}\alpha a_o b_o c\|X\|^2 - (b_o - \delta c)\|Y\|^2 \\ & + \{(a_o b_o - c) - \alpha a_o b_o[a_o + c^{-1}(b_1 - b_o)^2\}\|Y\|^2 \\ & - \epsilon\|Z\|^2 - \{(a_o\delta - 1) - \alpha a_o b_o c^{-1}(a_1 - a_o)^2\}\|Z\|^2 \\ & + (\alpha a_o b_o\|X\| + (1 + a_o)\|Y\| + (1+\delta)\|Z\|)\|P(t, X.Y, Z)\|.\end{aligned}$$

Next, in view of the assumptions of Theorem 0.2, it follows that

$$\begin{aligned}\dot{V} \leq & -\frac{1}{2}\alpha a_o b_o c\|X\|^2 - (b_o - \delta c)\|Y\|^2 - \epsilon\|Z\|^2 \\ & + (\alpha a_o b_o\|X\| + (1 + a_o)\|Y\| + (1+\delta)\|Z\|)\|P(t, X.Y, Z)\|.\end{aligned}$$

where by (5)and (10) there exist a positive constant d_4 such that

$$\begin{aligned}\dot{V} \leq & -2d_4(\|X\|^2 + \|Y\|^2 + \|Z\|^2) + d_5(\|X\| + \|Y\| + \|Z\|) \\ & \times [\delta_o + \delta_1(\|X\| + \|Y\| + \|Z\|)]\end{aligned}$$

$$\begin{aligned}\dot{V} \leq & -2d_4(\|X\|^2 + \|Y\|^2 + \|Z\|^2) + \delta_1 d_5(\|X\|^2 + \|Y\|^2 + \|Z\|^2) \\ & + \delta_o d_5(\|X\|^2 + \|Y\|^2 + \|Z\|^2)^2\end{aligned}$$

$$\begin{aligned}\dot{V} \leq & -2d_4(\|X\|^2 + \|Y\|^2 + \|Z\|^2) + \delta_1 d_5(\|X\|^2 + \|Y\|^2 + \|Z\|^2) \\ & + 3\delta_o d_5(\|X\|^2 + \|Y\|^2 + \|Z\|^2)\end{aligned}$$

$$\begin{aligned}\dot{V} \leq & -2d_4(\|X\|^2 + \|Y\|^2 + \|Z\|^2) + 3\delta_1 d_5(\|X\|^2 + \|Y\|^2 + \|Z\|^2) \\ & + 3^{\frac{1}{2}}\delta_o d_5(\|X\|^2 + \|Y\|^2 + \|Z\|^2)^{\frac{1}{2}}\end{aligned}$$

where

$$d_4 = \frac{1}{2}min\{\alpha a_o b_o c; b_o - \delta c; 2\epsilon\} > 0$$

and $d_5 = max\{\alpha a_o b_o, (1 + a_o), (1 + \delta)\}$.

$$\dot{V} \leq -2d_6(\|X\|^2 + \|Y\|^2 + \|Z\|^2) + d_7(\|X\|^2 + \|Y\|^2 + \|Z\|^2)^{\frac{1}{2}} \quad (13)$$

where

$$d_6 = \frac{1}{2}(d_4 - 3\delta_1 d_5), \quad d_1 < 3^{-1}d_5 d_4, \quad d_7 = 3^{\frac{1}{2}}\delta_o d_5$$

If we choose

$$(\|X\|^2 + \|Y\|^2 + \|Z\|^2)^{\frac{1}{2}} \geq d_8 = d_7 d_6^{-1}$$

the inequality (13) implies that

$$\dot{V} \leq -d_6(\|X\|^2 + \|Y\|^2 + \|Z\|^2) \tag{14}$$

Then, there exists d_9 such that

$$\dot{V} \leq -1 \quad if \quad \|X\|^2 + \|Y\|^2 + \|Z\|^2 \geq d_9^2$$

The remainder of the proof of the Theorem 0.2 may now be obtained by the use of the estimates (12) and (14) and an obvious adaptation of the Yoshizawa type reasoning in [13].□

4. REMARK 2

If we take $\Psi(X, \dot{X})\ddot{X} = \Psi(\dot{X})\ddot{X}$, $\Phi(\dot{X}) = B\dot{X}$ and $P(t, X, \dot{X}, \ddot{X}) = P(t)$ in (1), Theorem 0.2 reduces to the result obtained in Omeike and Afuwape[14].

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