Journal of the
Nigerian Mathematical Society
Vol. 31, pp. 81-90, 2012

CARATHEODORY SOLUTION OF QUANTUM STOCHASTIC DIFFERENTIAL INCLUSIONS

M. O. OGUNDIRAN[1] AND E. O. AYOOLA

ABSTRACT. This work is concerned with the existence of solution of Quantum stochastic differential inclusions in the sense of Caratheodory. The multivalued stochastic process involved which is non-convex is Scorza-Dragoni lower semicontinuous (SD-l.s.c.) hence giving rise to a directionally continuous selection. The Quantum stochastic differential inclusion is driven by annihilation, creation and gauge operators.

Keywords and phrases: Lower semicontinuous multifunctions, Scorza Dragoni's property, quantum stochastic processes.
2010 Mathematical Subject Classification: 81S25

1. INTRODUCTION

The vast applications of differential inclusions in control theory, economic model, evolution inclusions to mention a few, had made the study of differential inclusions of great interest [1], [8], [18]. Likewise, the quantum stochastic differential inclusions which is a multivalued generalization of quantum stochastic differential equation of Hudson and Parthasarathy has vast applications. This extension was first done in [9] in which the existence of solutions of Lipschitzian quantum stochastic differential inclusions was established. The study of solution set of this problem was done in [2], [3] and references cited there. The case of discontinuous quantum stochastic differential inclusions has application in the study of optimal quantum stochastic control [15]. The quantum stochastic calculus is driven by quantum stochastic processes called annihilation, creation and gauge arising from quantum field operators.
A multivalued map that is lower semicontinuous and convex-valued has continuous selection by Michael selection theorem, but if the convexity is dropped the continuous selection does not exist. But for a differential inclusion with lower semicontinuous multifunction that is not convex-valued, there is an analogue of Michael selection

Received by the editors April 20, 2012; Revised: May 31, 2012; Accepted: June 6, 2012
[1]Corresponding author

theorem called the directionally continuous selection [4] which gave rise to a class of discontinuous differential equations. A more general case of this selection for infinite dimensional space is found in [5].

The quantum stochastic differential inclusions considered in this work has its coefficients to be multivalued stochastic processes that have a special form of lower semicontinuity called Scorza-Dragoni lower semicontinuous case. It is noteworthy that the Scorza-Dragoni property is a multivalued generalization of Lusin property[14]. The directionally continuous selection of the Scorza-Dragoni of the multifunction gave rise to a class of quantum stochastic differential equations considered in [16] which have solutions in the sense of Caratheodory. Apart from the application of this work in quantum stochastic control, another motivation for the work is the application of the results in the study of non-convex quantum stochastic evolution inclusions which shall be considered in a later work.

In section 2 we give preliminaries which are essential for the work and we prove the main results in section 3.

2. PRELIMINARY

In what follows, if U is a topological space, we denote by $\mathrm{clos}(U)$, the collection of all non-empty closed subsets of U.

To each pair (D, H) consisting of a pre-Hilbert space D and its completion H, we associate the set $L_w^+(D, H)$ of all linear maps x from D into H, with the property that the domain of the operator adjoint contains D. The members of $L_w^+(D, H)$ are densely-defined linear operators on H which do not necessarily leave D invariant and $L_w^+(D, H)$ is a linear space when equipped with the usual notions of addition and scalar multiplication.

To H corresponds a Hilbert space $\Gamma(H)$ called the boson Fock space determined by H. A natural dense subset of $\Gamma(H)$ consists of linear space generated by the set of exponential vectors(Guichardet, [12]) in $\Gamma(H)$ of the form

$$e(f) = \bigoplus_{n=0}^{\infty} (n!)^{-\frac{1}{2}} \bigotimes^{n} f, \;\; f \in H$$

where $\bigotimes^0 f = 1$ and $\bigotimes^n f$ is the n-fold tensor product of f with itself for $n \geq 1$.

In what follows, $\mathbb{D}$ is some pre-Hilbert space whose completion is $\mathcal{R}$ and γ is a fixed Hilbert.

$L^2_\gamma(\mathbb{R}_+)$(resp. $L^2_\gamma([0,t))$, resp. $L^2_\gamma([t,\infty))$ $t \in \mathbb{R}_+$) is the space of

square integrable γ-valued maps on $\mathbb{R}_+$ (resp.$[0,t)$, resp.$[t,\infty)$). The inner product of the Hilbert space $\mathcal{R}\otimes\Gamma(L^2_\gamma(\mathbb{R}_+))$ will be denoted by $\langle .,.\rangle$ and $\|\;.\;\|$ the norm induced by $\langle .,.\rangle$.
Let $\mathbb{E},\mathbb{E}_t$ and $\mathbb{E}^t$, $t>0$ be linear spaces generated by the exponential vectors in Fock spaces $\Gamma(L^2_\gamma(\mathbb{R}_+)),\Gamma(L^2_\gamma([0,t)))$ and $\Gamma(L^2_\gamma([t,\infty)))$ respectively ;

$$\begin{aligned}\mathcal{A} &\equiv L^+_w(\mathbb{D}\underline{\otimes}\mathbb{E},\mathcal{R}\otimes\Gamma(L^2_\gamma(\mathbb{R}_+)))\\ \mathcal{A}_t &\equiv L^+_w(\mathbb{D}\underline{\otimes}\mathbb{E}_t,\mathcal{R}\otimes\Gamma(L^2_\gamma([0,t))))\otimes\mathbb{I}^t\\ \mathcal{A}^t &\equiv \mathbb{I}_t\otimes L^+_w(\mathbb{E}^t,\Gamma(L^2_\gamma([t,\infty)))),\ t>0\end{aligned}$$

where $\underline{\otimes}$ denotes algebraic tensor product and $\mathbb{I}_t$(resp.$\mathbb{I}^t$) denotes the identity map on $\mathcal{R}\otimes\Gamma(L^2_\gamma([0,t))))$(resp.$\Gamma(L^2_\gamma([t,\infty))))$, $t>0$
For every $\eta,\xi\in\mathbb{D}\underline{\otimes}\mathbb{E}$ define

$$\|\;x\;\|_{\eta,\xi}=|\;\langle\eta,x\xi\rangle\;|,\ x\in\mathcal{A}$$

then the family of seminorms

$$\{\|\;.\;\|_{\eta\xi}:\eta,\xi\in\mathbb{D}\underline{\otimes}\mathbb{E}\}$$

generates a topology τ_w , weak topology .
The completion of the locally convex spaces $(\mathcal{A},\tau_w)$, $(\mathcal{A}_t,\tau_w)$ and $(\mathcal{A}^t,\tau_w)$ are respectively denoted by $\widetilde{\mathcal{A}}$, $\widetilde{\mathcal{A}_t}$ and $\widetilde{\mathcal{A}}^t$.
We define the Hausdorff topology on clos$(\widetilde{\mathcal{A}})$ as follows:
For $x\in\widetilde{\mathcal{A}}$, $\mathcal{M},\mathcal{N}\in$ clos$(\widetilde{\mathcal{A}})$ and $\eta,\xi\in\mathbb{D}\underline{\otimes}\mathbb{E}$, define

$$\rho_{\eta\xi}(\mathcal{M},\mathcal{N})\equiv\max(\delta_{\eta\xi}(\mathcal{M},\mathcal{N}),\delta_{\eta\xi}(\mathcal{N},\mathcal{M}))$$

where

$$\begin{aligned}\delta_{\eta\xi}(\mathcal{M},\mathcal{N}) &\equiv \sup_{x\in\mathcal{M}}\mathbf{d}_{\eta\xi}(x,\mathcal{N})\text{ and}\\ \mathbf{d}_{\eta\xi}(x,\mathcal{N}) &\equiv \inf_{y\in\mathcal{N}}\|\;x-y\;\|_{\eta\xi}\ .\end{aligned}$$

The Hausdorff topology which shall be employed in what follows, denoted by, τ_H , is generated by the family of pseudometrics $\{\rho_{\eta\xi}(.):\eta,\xi\in\mathbb{D}\underline{\otimes}\mathbb{E}\}$
Moreover, if $\mathcal{M}\in clos(\widetilde{\mathcal{A}})$, then $\|\;\mathcal{M}\;\|_{\eta\xi}$ is defined by

$$\|\;\mathcal{M}\;\|_{\eta\xi}\equiv\rho_{\eta\xi}(\mathcal{M},\{0\});$$

for arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$.
For $A, B \in clos(\mathbb{C})$ and $x \in \mathbb{C}$, a complex number, define

$$\begin{aligned} d(x,B) &\equiv \inf_{y\in B} \mid x-y \mid \\ \delta(A,B) &\equiv \sup_{x\in A} d(x,B) \\ \text{and } \rho(A,B) &\equiv \max(\delta(A,B),\delta(B,A)). \end{aligned}$$

Then ρ is a metric on $clos(\mathbb{C})$ and induces a metric topology on the space.
Let $I \subseteq \mathbb{R}_+$. A *stochastic process* indexed by I is an $\widetilde{\mathcal{A}}$-valued measurable map on I.
A stochastic process X is called *adapted* if $X(t) \in \widetilde{\mathcal{A}}_t$ for each $t \in I$. We write $\mathrm{Ad}(\widetilde{\mathcal{A}})$ for the set of all adapted stochastic processes indexed by I.
Definition 1: A member X of $\mathrm{Ad}(\widetilde{\mathcal{A}})$ is called
(i) weakly absolutely continuous if the map $t \mapsto \langle \eta, X(t)\xi\rangle$, $t \in I$ is absolutely continuous for arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$
(ii) locally absolutely p-integrable if $\| X(.) \|_{\eta\xi}^p$ is Lebesgue - measurable and integrable on $[0,t) \subseteq I$ for each $t \in I$ and arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$.
We denote by $\mathrm{Ad}(\widetilde{\mathcal{A}})_{wac}$(resp.$L^p_{loc}(\widetilde{\mathcal{A}})$) the set of all weakly, absolutely continuous(resp. locally absolutely p-integrable) members of $\mathrm{Ad}(\widetilde{\mathcal{A}})$.
Stochastic integrators: Let $L^\infty_{\gamma,loc}(\mathbb{R}_+)$ [resp.$L^\infty_{B(\gamma),loc}(\mathbb{R}_+)$] be the linear space of all measurable , locally bounded functions from $\mathbb{R}_+$ to γ [resp. to B(γ) , the Banach space of bounded endomorphisms of γ]. If $f \in L^\infty_{\gamma,loc}(\mathbb{R}_+)$ and $\pi \in L^\infty_{B(\gamma),loc}(\mathbb{R}_+)$, then πf is the member of $L^\infty_{\gamma,loc}(\mathbb{R}_+)$ given by $(\pi f)(t) = \pi(t)f(t)$, $t \in \mathbb{R}_+$.
For $f \in L^2_\gamma(\mathbb{R})_+$ and $\pi \in L^\infty_{B(\gamma),loc}(\mathbb{R}_+)$; the annihilation , creation and gauge operators, $a(f), a^+(f)$ and $\lambda(\pi)$ in $L^+_w(\mathbb{D}, \Gamma(L^2_\gamma(\mathbb{R})_+))$ respectively, are defined as:

$$\begin{aligned} a(f)\mathbf{e}(g) &= \langle f, g\rangle_{L^2_\gamma(\mathbb{R}_+)}\mathbf{e}(g) \\ a^+(f)\mathbf{e}(g) &= \frac{d}{d\sigma}\mathbf{e}(g+\sigma f) \mid_{\sigma=0} \\ \lambda(\pi)\mathbf{e}(g) &= \frac{d}{d\sigma}\mathbf{e}(e^{\sigma\pi} f) \mid_{\sigma=0} \end{aligned}$$

$g \in L^2_\gamma(\mathbb{R})_+$
For arbitrary $f \in L^\infty_{\gamma,loc}(\mathbb{R}_+)$ and $\pi \in L^\infty_{B(\gamma),loc}(\mathbb{R}_+)$, they give rise

to the operator-valued maps A_f, A_f^+ and Λ_π defined by:

$$\begin{aligned} A_f(t) &\equiv a(f\chi_{[0,t)}) \\ A_f^+(t) &\equiv a^+(f\chi_{[0,t)}) \\ \Lambda_\pi(t) &\equiv \lambda(\pi\chi_{[0,t)}) \end{aligned}$$

$t \in \mathbb{R}_+$, where χ_I denotes the indicator function of the Borel set $I \subseteq \mathbb{R}_+$. The maps A_f, A_f^+ and Λ_π are stochastic processes , called annihilation, creation and gauge processes , respectively, when their values are identified with their amplifications on $\mathcal{R}\otimes\Gamma(L^2_\gamma(\mathbb{R}_+))$. These are the stochastic integrators in Hudson and Parthasarathy[13] formulation of boson quantum stochastic integration.
For processes $p, q, u, v \in L^2_{loc}(\widetilde{\mathcal{A}})$, the quantum stochastic integral:

$$\int_{t_0}^{t} \big(p(s)d\Lambda_\pi(s) + q(s)dA_f(s) + u(s)dA_g^+(s) + v(s)ds\big),\ t_0, t \in \mathbb{R}_+$$

is interpreted in the sense of Hudson-Parthasarathy[13] The definition of Quantum stochastic differential Inclusions follows as in [9].
A relation of the form

$$\begin{aligned} dX(t) &\in E(t, X(t))d\Lambda_\pi(t) + F(t, X(t))dA_f(t) \\ &\quad + G(t, X(t))dA_g^+(t) + H(t, X(t))dt \text{ almost all } t \in I \qquad (1) \\ X(t_0) &= x_0 \end{aligned}$$

is called Quantum stochastic differential inclusions(QSDI) with coefficients E, F, G, H and initial data (t_0, x_0).
Equation(1) is understood in the integral form:

$$\begin{aligned} X(t) \in x_0 + \int_{t_0}^{t} &\big(E(s, X(s))d\Lambda_\pi(s) + F(s, X(s))dA_f(s) \\ &+G(s, X(s))dA_g^+(s) + H(s, X(s))ds\big),\ t \in I \end{aligned}$$

called a stochastic integral inclusion with coefficients E, F, G, H and initial data (t_0, x_0)
An equivalent form of (1) has been established in [9], Theorem 6.2

as :

$$\begin{aligned}
(\mu E)(t,x)(\eta,\xi) &= \{\langle \eta, \mu_{\alpha\beta}(t)p(t,x)\xi\rangle : p(t,x) \in E(t,x)\} \\
(\nu F)(t,x)(\eta,\xi) &= \{\langle \eta, \nu_{\beta}(t)q(t,x)\xi\rangle : q(t,x) \in F(t,x)\} \\
(\sigma G)(t,x)(\eta,\xi) &= \{\langle \eta, \sigma_{\alpha}(t)u(t,x)\xi\rangle : u(t,x) \in G(t,x)\} \\
\mathbb{P}(t,x)(\eta,\xi) &= (\mu E)(t,x)(\eta,\xi) + (\nu F)(t,x)(\eta,\xi) \\
&\quad + (\sigma G)(t,x)(\eta,\xi) + H(t,x)(\eta,\xi) \\
H(t,x)(\eta,\xi) &= \{v(t,x)(\eta,\xi) : v(.,X(.)) \\
&\qquad \text{is a selection of } H(.,X(.)) \forall\ X \in L^2_{loc}(\widetilde{\mathcal{A}})\}
\end{aligned} \tag{2}$$

Then Problem (1) is equivalent to

$$\begin{aligned}
\frac{d}{dt}\langle \eta, X(t)\xi\rangle &\in \mathbb{P}(t,X(t))(\eta,\xi) \\
X(t_0) &= x_0
\end{aligned} \tag{3}$$

for arbitrary $\eta,\xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, almost all $t \in I$. Hence the existence of solution of (1) implies the existence of solution of (3) and vice-versa.

As explained in [9], for the map $\mathbb{P}$,

$$\mathbb{P}(t,x)(\eta,\xi) \neq \widetilde{\mathbb{P}}(t,\langle \eta, x\xi\rangle)$$

for some complex-valued multifunction $\widetilde{\mathbb{P}}$ defined on $I \times \mathbb{C}$ for $t \in I$, $x \in \widetilde{\mathcal{A}}$, $\eta,\xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$.

Definition 2: For an arbitrary $\eta,\xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, let $M > 0$, we define a set $\Gamma^M_{\eta\xi}$, as

$$\Gamma^M_{\eta\xi} = \{(t,x) \in I \times \widetilde{\mathcal{A}} : \mid \langle \eta, x\xi\rangle \mid \leq Mt\}$$

Let $(t_0,x_0) \in I \times \widetilde{\mathcal{A}}$ and $\epsilon > 0$. For an arbitrary $\eta,\xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, $(t_0,x_0) \in I \times \widetilde{\mathcal{A}}$ and $\delta > 0$, the family of conical neighbourhoods;

$$\begin{aligned}
\Gamma^M_{\eta\xi}((t_0,x_0),\delta) = \{(t,x) \in I \times \widetilde{\mathcal{A}} : \parallel x - x_0 \parallel_{\eta\xi} \leq M(t-t_0), \\
t_0 \leq t < t_0 + \delta\}
\end{aligned}$$

generates a topology, τ^+, which satisfies the following property:

(P) For every pair of sets $A \subset B$, with A closed and B open(in the original topology), there exists a set C, closed-open with respect to τ^+, such that $A \subset C \subset B$.

This topology follows from [5] and the references cited there.

Definition 3: (i) For an arbitrary pair $\eta,\xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$ a map $\Phi : I \times \widetilde{\mathcal{A}} \to \widetilde{\mathcal{A}}$ will be said to be $\Gamma^M_{\eta\xi}$-continuous(directionally continuous or τ^{+}-continuous) at a point $(t_0, x_0) \in I \times \widetilde{\mathcal{A}}$, if for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$\| \Phi(t,x) - \Phi(t_0,x_0) \|_{\eta\xi} \leq \epsilon \text{ if } t_0 \leq t \leq t_0 + \delta \text{ and } \| x - x_0 \|_{\eta\xi} \leq M(t - t_0)$$

(ii) For an arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, $S \subset \widetilde{\mathcal{A}}$, a sesquilinear-form valued map $\Psi : S \to 2^{sesq(\mathbb{D}\underline{\otimes}\mathbb{E})^2}$ will be said to be lower semicontinuous on S if for every closed subset C of $\mathbb{C}$ the set $\{s \in S : \Psi(s)(\eta,\xi) \subset C\}$ is closed in S.
We remark that if E, F, G, H are lower semicontinuous on S, then the sesquilinear-form valued $\mathbb{P}$ is lower semicontinuous on S.
A multivalued generalization of Lusin property which is called Scorza - Dragoni property [14] employed in [6] is used to define the form of lower semicontinuity in this work. The well-known Lusin property is the following.
Definition 4:(Lusin's property)Let X and Y be two separable metric spaces and let $f : I \times X \to Y$ be function such that
(i)$t \to f(t,u)$ is measurable for every $u \in X$
(ii) $u \to f(t,u)$ is continuous for almost every $t \in I$, $I \subseteq \mathbb{R}_+$.
Then, for each $\epsilon > 0$, there exists a closed set $A \subseteq I$ such that $\lambda(I \setminus A) < \epsilon$, (λ is the Lebesgue measure on $\mathbb{R}$) and the restriction of f to $A \times X$ is continuous.
Definition 5: A sequilinear- form valued map $\Psi : [0,T] \times \widetilde{\mathcal{A}} \to 2^{sesq(\mathbb{D}\underline{\otimes}\mathbb{E})^2}$ is Scorza-Dragoni lower semicontinuous (SD-l.s.c.) on $[0,T] \times \widetilde{\mathcal{A}}$ if there exists a sequence of disjoint compact sets $J_n \subset [0,T]$, with meas$([0,T] \setminus \bigcup_{n\in\mathbb{N}J_n}) = 0$ such that Ψ is lower semicontinuous on each set $J_n \times \widetilde{\mathcal{A}}$.
If Ψ is lower semicontinuous and convex-valued then by Michael selection theorems, there exists continuous selection of Ψ. But if the convexity is removed and Ψ is not decomposable valued multifunction then the existence of continuous selection is not guaranteed. However, a non-convex analogue of Michael selection is Directional continuous selection result in [4] and for infinite dimensional space in [5]. We established in this work that such selection exists for SD-lsc multivalued stochastic process.
For an arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, if $\Psi \in \mu E, \nu F, \sigma G, H$ appearing in (1) are SD-lsc then the map $(t,x) \to \mathbb{P}(t,x)(\eta,\xi)$ is SD-lsc.
A quantum stochastic differential inclusion will be said to be SD-lower semicontinuous if the coefficients are SD-lsc.

3. MAIN RESULTS

Theorem 1: For almost all $t \in I$, $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$. Suppose the following holds:
(i) The maps $X \to \Psi(t,X)(\eta,\xi)$, $\Psi \in \{\mu E, \nu F, \sigma G, H\}$ are non-empty lower semicontinuous multivalued stochastic processes
(ii)The maps $t \to \Psi(t,X)(\eta,\xi)$ are closed
(iii) τ^+ is a topology on $I \times \widetilde{\mathcal{A}}$ with property (P).
Then the sesquilinear form valued multifunction, $(t, X(t)) \to \mathbb{P}(t, X(t))(\eta,\xi)$

$$\begin{aligned}\mathbb{P}(t,X(t))(\eta,\xi) = (\mu E)(t,X(t))(\eta,\xi) + (\nu F)(t,X(t))(\eta,\xi)\\ + (\sigma G)(t,X(t))(\eta,\xi) + H(t,X(t))(\eta,\xi)\end{aligned}$$

admits a τ^+-continuous selection.
Proof: $\mathbb{P}$ is non-empty , since each of $\Psi \in \{\mu E, \nu F, \sigma G, H\}$ is non-empty.
Therefore, $\mathbb{P}$ is a non-empty lower semicontinuous sesquilinear form-valued multifunction.
We shall employ a similar procedure as in the proof of Theorem 3.2 in [5] to construct a τ^+-continuous ϵ-approximate selections P_ϵ of $\mathbb{P}$, hence by inductive hypothesis we obtain a τ^+-continuous selection P of $\mathbb{P}$.
Let $\epsilon > 0$ be fixed , since $X \to \mathbb{P}(t,X)(\eta,\xi)$ is lower semicontinuous , for every $X(t) \in \widetilde{\mathcal{A}}$, we choose point $y_{\eta\xi,X}(t) \in \mathbb{P}(t,X(t))(\eta,\xi)$ and neighbourhood U_X of $X(t)$ such that

$$\inf_{y_{\eta\xi,\mathbb{P}}(t)\in\mathbb{P}(t,X(t'))(\eta,\xi)} \mid y_{\eta\xi,X}(t) - y_{\eta\xi,\mathbb{P}}(t) \mid < \epsilon \ \forall\ X(t') \in U_X \tag{4}$$

Now , let $(V_\alpha)_{\alpha\in\beta^\epsilon}$ be a local finite open refinement of $(U_X)_{X(t)\in\widetilde{\mathcal{A}}}$, with $V_\alpha \subset U_{X_\alpha}$, and let $(W_\alpha)_{\alpha\in\beta^\epsilon}$ be another open refinement such that $cl(W_\alpha) \subset V_\alpha$ for all $\alpha \in \beta^\epsilon$. By property (P), for each α , we can choose a set Z_α , clopen w.r.t. τ^+, such that

$$cl(W_\alpha) \subset int(Z_\alpha) \subset cl(Z_\alpha) \subset V_\alpha \tag{5}$$

Then $(Z_\alpha)_\alpha$ is a local finite τ^+ clopen covering of $\widetilde{\mathcal{A}}$. Let $\preceq$ be a well-ordering of the set β^ϵ, define for each $\alpha \in \beta^\epsilon$,

$$\Omega^\epsilon_\alpha = Z_\alpha \setminus (\bigcup_{\lambda<\alpha} Z_\lambda)$$

Set $\mathcal{O}^\epsilon = (\Omega^\epsilon_\alpha)$, $\alpha \in \beta^\epsilon$. By well-ordering , every $x \in \widetilde{\mathcal{A}}$ belongs to exactly one set $\Omega^\epsilon_{\overline{\alpha}}$ where $\overline{\alpha} = \min\{\alpha \in \beta^\epsilon : x \in Z_\alpha\}$. Hence , $\mathcal{O}^\epsilon$ is a partition of $\widetilde{\mathcal{A}}$. Moreover, since Z_α is locally finite(wrt τ and therefore wrt τ^+), the sets $\bigcup_{\lambda<\alpha} Z_\lambda$ are τ^+ clopen. Hence $\mathcal{O}^\epsilon$ is a τ^+ clopen disjoint covering of $\widetilde{\mathcal{A}}$ such that, $\{cl(\Omega^\epsilon_\alpha)\}$ refines $(V_\alpha)_\alpha$.

By setting $y^{\epsilon}_{\eta\xi,\alpha} = y_{\eta\xi,X_\alpha}$ and $P_\epsilon(t, X(t))(\eta,\xi) = y_{\eta\xi,X_\alpha}$, $\forall\alpha \in \beta^\epsilon$ we have τ^+ continuous function P_ϵ, which by (4), satisfies

$$\inf_{y_{\eta\xi,\mathbb{P}}(t)\in\mathbb{P}(t,X(t))(\eta,\xi)} \mid P_\epsilon(t, X(t))(\eta,\xi) - y_{\eta\xi,\mathbb{P}}(t) \mid < \epsilon$$

Therefore , there exists an ϵ-approximate selection P_ϵ of $\mathbb{P}$.
Since ϵ was arbitrarily chosen ,thus we have a τ^+-continuous selection P of $\mathbb{P}$. $\Box$

Theorem 2: Suppose the following holds for an arbitrary $\eta,\xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, $\Psi \in \{\mu E, \nu F, \sigma G, H\}$:
(i) $t \to \Psi(t, X(t))(\eta,\xi)$ are measurable for all $X \in \widetilde{\mathcal{A}}$
(ii) $X \to \Psi(t, X(t))(\eta,\xi)$ are SD-lower semicontinuous with respect to a seminorm $\| \, . \, \|_{\eta\xi}$, for almost all $t \in I$
(iii) Ψ are integrably bounded, that is, there exists $L^{\Psi}_{\eta\xi}(t) \in L^1(I)$ such that, a.e. $t \in I$, for all $X \in \widetilde{\mathcal{A}}$,

$$\inf_{y\in\Psi(t,x)(\eta,\xi)} \mid y \mid \leq L^{\Psi}_{\eta\xi}(t).$$

Then the SD-lower semicontinuous quantum stochastic differential inclusions

$$\begin{gathered} \frac{d}{dt}\langle\eta, X(t)\xi\rangle \in \mathbb{P}(t, X(t))(\eta,\xi) \\ X(t_0) = x_0 \end{gathered} \tag{6}$$

has an adapted weakly absolutely continuous solution in the sense of Caratheodory.

Proof: Since for arbitrary $\eta,\xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, $\Psi \in \mu E, \nu F, \sigma G, H$ are SD-lower semicontinuous then $\mathbb{P}(t,x)(\eta,\xi)$ is SD-lower semicontinuous, $\forall x \in \widetilde{\mathcal{A}}$, a.e. $t \in I$. The sequence of disjoint compact sets $J_n = \bigcap_\Psi J^{\Psi}_n$ and $meas(I \setminus \cup_{n\in\mathbb{N}} J_n) = 0$ such that $\mathbb{P}(.,.)(\eta,\xi)$ restricted to $\Omega_n = J_n \times \widetilde{\mathcal{A}}$ is lower semicontinuous, with respect to $\| \, . \, \|_{\eta\xi}$.
Also, suppose $L_{\eta\xi} = 5 \max L^{\Psi}_{\eta\xi}(t)$, then a.e. $t \in I$,

$$\inf_{y\in\mathbb{P}(t,x)(\eta,\xi)} \mid y \mid \leq L_{\eta\xi}(t),$$

for all $X \in \widetilde{\mathcal{A}}$
For each $n \geq 1$, we can apply Theorem (1) and obtain τ^+-continuous selections $P_n \in \mathbb{P}$.
For an arbitrary selection g from $\mathbb{P}$, if we define

$$P(t, X)(\eta,\xi) = \begin{cases} P_n(t, X)(\eta,\xi) & \text{if } t \in J_n, \\ g(t, X)(\eta,\xi) & \text{if } t \notin \cup_{n\in\mathbb{N}} J_n \end{cases}$$

then P is a τ^+-continuous selection of $\mathbb{P}$, such that $\mid P(t,x)(\eta,\xi) \mid \leq L_{\eta\xi}(t) < L_{n,\eta\xi}$, for every $(t,X) \in I \times \widetilde{\mathcal{A}}$, $\eta,\xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$.
Then by applying Lusin's property to each bound of $L_{n,\eta\xi}$, $n \in \mathbb{N}$ the set of solutions of τ^+-continuous quantum stochastic differential equations is the solution set of (6) in the sense of Caratheodory. □

REFERENCES

[1] J.P. Aubin and A. Cellina, *Differential Inclusions*, Springer- Verlag, Berlin, 1984.

[2] E.O. Ayoola, Continuous selections of solution sets of Lipschitzian quantum stochastic differential inclusions.*Int. J. Theor.Phys.*43, 10, 2041-2059, 2004.

[3] E.O. Ayoola, Topological properties of solution sets of Lipschitzian quantum stochastic differential inclusions.*Acta Appl.Math*, 100, 15-37, 2008.

[4] A. Bressan and A. Cortesi, Directionally continuous selections in Banach spaces, *Non linear Analysis. Theory, Method and Applications* 13, 987-992, 1989.

[5] A. Bressan and G. Colombo, Selections and representations of multifunctions in paracompact spaces *Studia Mathematical* 102, 209-216, 1992.

[6] A. Cernea and V. Staicu, Directionally continuous selections and nonconvex evolution inclusions *Set-Valued Analysis* 11, 9-20, 2003.

[7] K. Deimling, *Non linear Functional Analysis*, Springer-Verlag, Berlin, 1985.

[8] K. Deimling, *Multivalued differential equations*, Walter de Gruyter , 1992.

[9] G.O.S. Ekhaguere, Lipschitzian quantum stochastic differential inclusions *Int. J. Theor. Phys.*, 31, 11, 2003-2034, 1992.

[10] G.O.S. Ekhaguere, Quantum stochastic evolutions *Int. J. Theor. Phys.*, 35, 9, 1909-1946, 1996.

[11] G.O.S. Ekhaguere, Topological solutions of Non commutative stochastic differential equations *Stoch. Anal. and Appl.* 25, 9, 961-993, 2007.

[12] A. Guichardet, *Symmetric Hilbert spaces and related topics* Lecture Notes in Mathematics, 261, Springer-Verlag, Berlin, 1972.

[13] R.L. Hudson and K.R. Parthasarathy, Quantum Ito's formula and stochastic evolutions*Comm. Math. Phys.*, 93, 3, 301-323, 1984.

[14] A. Kucia, Scorza-Dragoni type theorems ,*Fundamental Mathematical* 138, 197-203, 1991.

[15] M.O. Ogundiran and E.O. Ayoola, Mayer problem for Quantum stochastic control *J. Math. Phys.*51, 1, 023521-023521-8, 2010.

[16] M.O. Ogundiran and E.O. Ayoola, Directionally continuous quantum stochastic differential equations *Far East J. Appl. Maths.* 57, 1, 33-48, 2011.

[17] M. Reed and B. Simon, *Methods of Modern Mathematical Physics I. Functional Analysis*, 2nd edn., Academic press, New York-London, 1980.

[18] A. Tolstonogov, *Differential inclusions in a Banach space*, Mathematics and its applications, Kluwer Academics Publishers, New York, 2000.

DEPARTMENT OF MATHEMATICS, OBAFEMI AWOLOWO UNIVERSITY, ILE - IFE, NIGERIA
E-mail address: `adeolu74113@yahoo.com`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF IBADAN, IBADAN, NIGERIA
E-mail address: `eoayoola@googlemail.com`