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HARDY TYPE INEQUALITIES FOR SUPERQUADRATIC AND SUBQUADRATIC FUNCTIONS

J. A. OGUNTUASE[1], E. O. ADELEKE AND P. DUROJAYE

ABSTRACT. Some new Hardy type inequalities for superquadratic and subquadratic functions are proved and discussed. We also derive a new class of refined Hardy-type inequalities involving a more general integral operators with a nonnegative kernel. The results obtained unify and extend several inequalities of Hardy-type for superquadratic and subquadratic functions known in the literature.

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1. INTRODUCTION

In a note published in 1920 G. H. Hardy [3] announced (without proof) (see also [5], [7], [8]) that if $p > 1$ and f is a nonnegative $p-$integrable function on $(0, \infty)$, then f is integrable over the interval $(0, x)$ for each positive x and that the inequality

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)dt\right)^p dx \leq \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)dx \tag{1.1}$$

holds, where the constant $(p/(p-1))^p$ on the right hand side of (1.1) is the best possible. Inequality (1.1) is today referred to as the classical Hardy's integral inequality and it has an interesting prehistory and history (see e.g. [5], [7], [8] and the references given there). Nowadays a well-known simple fact is that (1.1) can equivalently (via the substitution $f(x) = h(x^{1-\frac{1}{p}})x^{-\frac{1}{p}}$), be rewritten in the form

$$\int_0^\infty \left(\frac{1}{x}\int_0^x h(t)dt\right)^p \frac{dx}{x} \leq \int_0^\infty h^p(x)\frac{dx}{x}, \tag{1.2}$$

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[1]Corresponding author

and in this form it even holds with equality when $p = 1$. In this form we see that Hardy's inequality is a simple consequence of Jensen's inequality, but this was not discovered in the dramatic period when Hardy discovered and finally proved inequality (1.1) in his famous paper [4] in 1925 (see [7] and [8]).

The first refinement of Hardy's inequality (1.1) is due to Shum [15] and further developed using convexity argument by Imoru [6]. These Shum-Imoru results were recently complemented and further generalized by Persson and Oguntuase [14] using Hölder's and reversed Hölder's inequalities. In a recent paper, Oguntuase and Persson [11] used mainly the notion of superquadratic and subquadratic functions to obtain new refinements of Hardy inequality for $p \geq 2$, which also, hold in the reversed direction for $1 < p \leq 2$. The results in [11] are indeed surprising and in the breaking point $p = 2$ they even got equality (like some new Parseval formula for this operator) and this is completely different from the usual Hardy situation where the breaking point is $p = 1$ and no such can appear at this point. The multidimensional version of these results were recently obtained by Oguntuase et. al. [13] (see e.g. [9] and the references given there). Our aim is to study the positive function ϕ which may not necessarily be superquadratic for $p \geq 2$ and subquadratic for $1 < p \leq 2$, and such that $Ax^p \leq \phi(x) \leq Bx^p$ holds on $\mathbb{R}^+$ for some constants $A \leq B$. Furthermore, we shall derive a new class of refined Hardy-type inequalities involving a more general integral operator with a nonnegative kernel. This idea was first introduced by Oguntuase and Persson in [10] and further developed by Oguntuase et. al. in [12] for convex and concave functions. Our aim is to extend this idea to superquadratic and subquadratic functions introduced by Abramovich et. al. in [1], and also to functions that are not necessarily superquadratic and subquadratic.

2. PRELIMINARIES

In the sequel, we present the following definition and lemmas that will be used in the proof of our results in the next section.

Definition 2.1. [1, Definition 2.1] A function $\varphi : [0, \infty) \to \mathbb{R}$ is superquadratic provided that for all $x \geq 0$ there exists a constant $C_x \in \mathbb{R}$ such that

$$\varphi(y) - \varphi(x) - \varphi(|y - x|) \geq C_x (y - x)$$

for all $y \geq 0$. We say that φ is subquadratic if $-\varphi$ is superquadratic.

Lemma 2.2. [1, Theorem 2.3] *Let (Ω, μ) be a probability measure space. The inequality*

$$\varphi\left(\int_\Omega f(s)d\mu(s)\right) \leq \int_\Omega \varphi(f(s))d\mu(s) - \int_\Omega \varphi\left(\left|f(s) - \int_\Omega f(s)d\mu(s)\right|\right)d\mu \tag{2.1}$$

holds for all probability measures μ and all nonnegative $\mu-$integrable functions f if and only if φ is superquadratic. Moreover, (2.1) holds in the reversed direction if and only if φ is subquadratic.

Lemma 2.3. *Let $b \in (0, \infty)$, $u : (0, b) \to \mathbb{R}$ be a weight function such that the function $x \to \frac{u(x)}{x^2}$ is locally integrable on $(0, b)$, and define the weight function v by*

$$v(t) = t\int_t^b \frac{u(x)}{x^2}dx, \quad t \in (0, b).$$

Let I be an interval in $\mathbb{R}$, $\phi : I \to \mathbb{R}$ and $f : (0, b) \to \mathbb{R}$ be integrable function such that $f(x) \in I$ $\forall x \in (0, b)$. If ϕ is superquadratic, then the following inequality holds:

$$\int_0^b u(x)\phi\left(\frac{1}{x}\int_0^x f(t)dt\right)\frac{dx}{x} + \int_0^b\int_t^b \phi\left(\left|f(t) - \frac{1}{x}\int_0^x f(t)dt\right|\right)\frac{u(x)}{x^2}dxdt \leq \int_0^b v(x)\phi(f(x))\frac{dx}{x}, \tag{2.2}$$

while the sign of the inequality (2.2) is reversed if ϕ is subquadratic.

Proof. Lemma 2.3 is an easy consequence of Jensen's inequality and Fubini's theorem (for details, see [11]). □

Remark 2.4. By putting $u(x) \equiv 1$ yields $v(x) = 1 - \frac{x}{b}$, and so inequality (2.2) becomes

$$\int_0^b \phi\left(\frac{1}{x}\int_0^x f(t)dt\right)\frac{dx}{x} + \int_0^b\int_t^b \phi\left(\left|f(t) - \frac{1}{x}\int_0^x f(t)dt\right|\right)\frac{dx}{x^2}dt \leq \int_0^b \left(1 - \frac{x}{b}\right)\phi(f(x))\frac{dx}{x}. \tag{2.3}$$

The sign of inequality (2.3) is reversed if ϕ is subquadratic.

Lemma 2.5. *Let $b \in [0, \infty)$, $u : (b, \infty) \to \mathbb{R}$ be locally integrable weight function on (b, ∞), and define the weight function v by*

$$v(t) = \frac{1}{t}\int_b^t u(x)dx, \quad t \in (b, \infty).$$

Suppose that I is an interval in $\mathbb{R}$, $\phi : I \to \mathbb{R}$, and that $f : (b, \infty) \to \mathbb{R}$ is an integrable function such that $f(x) \in I$ $\forall x \in (b, \infty)$. If ϕ is superquadratic, then

$$\begin{aligned} &\int_b^\infty u(x)\phi\left(x\int_x^\infty f(t)\frac{dt}{t^2}\right)\frac{dx}{x} \\ &+ \int_b^\infty \int_b^t \phi\left(\left|f(t) - x\int_x^\infty f(t)\frac{dt}{t^2}\right|\right)u(x)dx\frac{dt}{t^2} \\ &\leq \int_b^\infty v(x)\phi(f(x))\frac{dx}{x}, \end{aligned} \tag{2.4}$$

while the sign inequality (2.4) is reversed, if φ is subquadratic.

Proof. This is similar to the proof of Lemma 2.3 (for details, see [11]). □

Remark 2.6. By putting $u(x) \equiv 1$ yields $v(x) = 1 - \frac{x}{b}$, and so inequality (2.4) reads

$$\begin{aligned} &\int_b^\infty \phi\left(x\int_x^\infty f(t)\frac{dt}{t^2}\right)\frac{dx}{x} \\ &+ \int_b^\infty \int_b^t \phi\left(\left|f(t) - x\int_x^\infty f(s)\frac{ds}{s^2}\right|\right)dx \\ &\leq \int_b^\infty \left(1 - \frac{b}{x}\right)\phi(f(x))\frac{dx}{x}, \end{aligned} \tag{2.5}$$

while the sign of inequality (2.5) is reversed if φ is subquadratic.

3. SOME NEW REFINED HARDY TYPE INEQUALITIES

We now state and prove our results in this section which are the new refined Hardy type inequalities for nonnegative superquadratic functions. Our first result reads:

Theorem 3.1. *Let $p > 1$, $k > 1$, $b \in (0, \infty]$, and let the function f be locally integrable on $(0, b)$ such that*

$$0 < \int_0^b \left[1 - \left(\frac{x}{b}\right)^{\frac{k-1}{p}}\right] x^{p-k} f^p(x)dx < \infty.$$

If $\phi : [0,\infty) \to \mathbb{R}$ is superquadratic, such that $Ax^p \leq \phi(x) \leq Bx^p$ holds on $[0,\infty)$ for some constants $0 < A \leq B < \infty$, then for $p \geq 2$ and $b \in (0,\infty]$ the inequality

$$\begin{aligned}
&\int_0^b x^{-k}\phi\left(\int_0^x f(t)dt\right)dx \\
&+\frac{k-1}{p}\int_0^b\int_t^b \phi\left(\left|\frac{p}{k-1}\left(\frac{t}{x}\right)^{1-\frac{k-1}{p}} f(t)-\frac{1}{x}\int_0^x f(t)dt\right|\right) \\
&\times x^{p-k-\frac{k-1}{p}}t^{\frac{k-1}{p}-1}dxdt \\
&\leq \left(\frac{B}{A}\right)^2\left(\frac{p}{k-1}\right)^p\int_0^b\left[1-\left(\frac{x}{b}\right)^{\frac{k-1}{p}}\right]x^{p-k}\phi(f(x))dx \qquad (3.1)
\end{aligned}$$

holds for all nonnegative integrable function $f : (0,b) \to \mathbb{R}$. If ϕ is subquadratic, then the sign of inequality (3.1) is reversed.

Proof. This follows from Lemma 2.3 by choosing the weight function $u(x) \equiv 1$, that is, by using inequality (2.3). First, consider the case $p \geq 2$, we have by (2.3) that

$$\begin{aligned}
&\int_0^b \phi\left(\frac{1}{x}\int_0^x f(t)dt\right)\frac{dx}{x}+\int_0^b\int_t^b\phi\left(\left|f(t)-\frac{1}{x}\int_0^x f(t)dt\right|\right)\frac{dx}{x^2}dt \\
&\leq \int_0^b\left(1-\frac{x}{b}\right)\phi(f(x))\frac{dx}{x}. \qquad (3.2)
\end{aligned}$$

Denote the first and second terms on the left hand side of (3.2) by I_1 and I_2 and the right hand side by I_3, respectively. Replace the parameter b by $a = b^{\frac{k-1}{p}}$ and choose for f the function $x \mapsto f(x^{\frac{p}{k-1}})x^{\frac{p}{k-1}-1}$. Thereafter, use the substitutions $y = x^{\frac{p}{k-1}}$ and $s = t^{\frac{p}{k-1}}$. Then

$$\begin{aligned}
I_1 &= \int_0^a \phi\left(\frac{1}{x}\int_0^x f(t^{\frac{p}{k-1}})t^{\frac{p}{k-1}-1}dt\right)\frac{dx}{x} \\
&\geq A\int_0^a\left(\frac{1}{x}\int_0^x f(t^{\frac{p}{k-1}})t^{\frac{p}{k-1}-1}dt\right)^p\frac{dx}{x} \\
&= A\left(\frac{k-1}{p}\right)^p\int_0^a x^{-p-1}\left(\int_0^{x^{\frac{p}{k-1}}} f(s)ds\right)^p dx
\end{aligned}$$

$$
\begin{aligned}
&\geq \frac{A}{B}\left(\frac{k-1}{p}\right)^{p}\int_0^a x^{-p-1}\phi\left(\int_0^{x^{\frac{p}{k-1}}} f(s)ds\right)dx\\
&= \frac{A}{B}\left(\frac{k-1}{p}\right)^{p+1}\int_0^b \phi\left(\int_0^y f(s)ds\right)y^{-k}dy, \qquad (3.3)
\end{aligned}
$$

$$
\begin{aligned}
I_2 &= \int_0^a\int_t^a \phi\left(\left|f(t^{\frac{p}{k-1}})t^{\frac{p}{k-1}-1} - \frac{1}{x}\int_0^x f(t^{\frac{p}{k-1}})t^{\frac{p}{k-1}-1}dt\right|\right)\frac{dx}{x^2}dt\\
&\geq A\int_0^a\int_t^a \left|f(t^{\frac{p}{k-1}})t^{\frac{p}{k-1}-1} - \frac{1}{x}\int_0^x f(t^{\frac{p}{k-1}})t^{\frac{p}{k-1}-1}dt\right|^p\frac{dx}{x^2}dt\\
&= A\left(\frac{k-1}{p}\right)^{p+1}\int_0^b\int_{s^{\frac{k-1}{p}}}^a \left|\frac{p}{k-1}f(s)s^{1-\frac{k-1}{p}} - \frac{1}{x}\int_0^{x^{\frac{p}{k-1}}} f(s)ds\right|^p\\
&\times \frac{dx}{x^2}s^{\frac{k-1}{p}-1}ds\\
&= A\left(\frac{k-1}{p}\right)^{p+2}\int_0^b\int_s^b \left|\frac{p}{k-1}f(s)s^{1-\frac{k-1}{p}} - \frac{1}{y^{\frac{k-1}{p}}}\int_0^y f(s)ds\right|^p\\
&\times y^{\frac{1-k}{p}-1}s^{\frac{k-1}{p}-1}dyds\\
&= A\left(\frac{k-1}{p}\right)^{p+2}\int_0^b\int_s^b \left|\frac{p}{k-1}f(s)\left(\frac{s}{y}\right)^{1-\frac{k-1}{p}} - \frac{1}{y}\int_0^y f(s)ds\right|^p\\
&\times y^{p-k-\frac{k-1}{p}}s^{\frac{k-1}{p}-1}dyds\\
&\geq \frac{A}{B}\left(\frac{k-1}{p}\right)^{p+2}\int_0^b\int_s^b \phi\left(\left|\frac{p}{k-1}f(s)\left(\frac{s}{y}\right)^{1-\frac{k-1}{p}}\right.\right.\\
&\left.\left.-\frac{1}{y}\int_0^y f(s)ds\right|\right)y^{p-k-\frac{k-1}{p}}s^{\frac{k-1}{p}-1}dyds\\
&= \frac{A}{B}\left(\frac{k-1}{p}\right)^{p+2}\int_0^b\int_s^b \phi\left(\left|\frac{p}{k-1}f(s)\left(\frac{s}{y}\right)^{1-\frac{k-1}{p}}\right.\right.\\
&\left.\left.-\frac{1}{y}\int_0^y f(s)ds\right|\right)y^{p-k-\frac{k-1}{p}}s^{\frac{k-1}{p}-1}dyds \qquad (3.4)
\end{aligned}
$$

and

$$
\begin{aligned}
I_3 &= \int_0^a\left(1-\frac{x}{a}\right)\phi\left(f(x^{\frac{p}{k-1}})x^{p\left(\frac{p}{k-1}-1\right)}\right)\frac{dx}{x}\\
&\leq B\int_0^a\left(1-\frac{x}{a}\right)f^p(x^{\frac{p}{k-1}})x^{p\left(\frac{p}{k-1}-1\right)}\frac{dx}{x}
\end{aligned}
$$

$$
\begin{aligned}
&= B\left(\frac{k-1}{p}\right)\int_0^b\left(1-\left[\frac{y}{b}\right]^{\frac{k-1}{p}}\right)y^{p-k}f^p(y)dy\\
&\leq \frac{B}{A}\left(\frac{k-1}{p}\right)\int_0^b\left(1-\left[\frac{y}{b}\right]^{\frac{k-1}{p}}\right)y^{p-k}\phi(f(y))dy. \qquad (3.5)
\end{aligned}
$$

The proof of (3.1) follows by combining (3.3)-(3.5). (*ii*) The proof of the case $1 < p \leq 2$ is similar and the only difference is that in this case all the inequalities signs are reversed. □

Remark 3.2. In the special case $\phi(x) = x^p$ and $A = B = 1$, then Theorem 3.1 reduces to Theorem 3.1 in [11].

When $A = B = 1$, $\phi(x) = x^p$ and $k = p$, yields the following result:

Corollary 3.3. *Let $p > 1$, $b \in (0, \infty]$, and let the function f be locally integrable on $(0, b)$ such that*

$$
0 < \int_0^\infty \left[1-\left(\frac{x}{b}\right)^{\frac{p-1}{p}}\right] f^p(x)dx < \infty.
$$

If $\phi : [0, \infty) \to \mathbb{R}$ is superquadratic, then for $p \geq 2$

$$
\begin{aligned}
&\int_0^b\left(\frac{1}{x}\int_0^x f(t)dt\right)^p dx\\
&+\frac{p-1}{p}\int_0^b\int_t^b\left|\frac{p}{p-1}\left(\frac{t}{x}\right)^{1-\frac{p-1}{p}} f(t)-\frac{1}{x}\int_0^x f(t)dt\right|^p\\
&\times x^{-\frac{p-1}{p}}t^{\frac{p-1}{p}-1}dxdt\\
&\leq\left(\frac{p}{p-1}\right)^p\int_0^b\left[1-\left(\frac{x}{b}\right)^{\frac{p-1}{p}}\right]f^p(x)dx \qquad (3.6)
\end{aligned}
$$

holds for all nonnegative integrable function $f : (0, b) \to \mathbb{R}$. If $1 < p \leq 2$, then the sign of inequality (3.6) is reversed.

Proof. This follows directly from Theorem 3.1. □

By using Corollary 3.3 with $b = \infty$ we obtain the following result:

Example 3.4. Let $0 < b \leq \infty$ and let the function f be locally integrable on $(0, b)$ such that $0 < \int_0^b f^p(x)dx < \infty$. If $p \geq 2$, then

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)dt\right)^p dx$$
$$+\frac{p-1}{p}\int_0^\infty\int_t^\infty \left|\frac{p}{p-1}\left(\frac{t}{x}\right)^{\frac{1}{p}} f(t) - \frac{1}{x}\int_0^x f(t)dt\right|^p x^{-\frac{k-1}{p}} t^{\frac{-1}{p}} dxdt$$
$$\leq \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x))dx. \tag{3.7}$$

The inequality sign is reversed if $1 < p \leq 2$.

Remark 3.5. The case $p = 2$ and $b = \infty$ in Example 3.4 reduces to Remark 4.2 in [11].

In the next result we state the dual of Theorem 3.1 as follows:

Theorem 3.6. *Let $p > 1$, $k < 1$, $b \in [0, \infty)$, and let the function f be locally integrable on (b, ∞) and such that*

$$0 < \int_b^\infty \left[1 - \left(\frac{b}{x}\right)^{\frac{1-k}{p}}\right] x^{p-k}\phi(f(x))dx < \infty.$$

If $\phi : [b, \infty) \to \mathbb{R}$ is superquadratic, such that $Ax^p \leq \phi(x) \leq Bx^p$ holds on $[b, \infty)$ for some constants $0 < A \leq B < \infty$, then for $p \geq 2$ the inequality

$$\int_b^\infty x^{-k}\phi\left(\int_x^\infty f(t)dt\right)dx$$
$$+\left(\frac{1-k}{p}\right)\int_b^\infty\int_b^s \phi\left(\left|\frac{p}{1-k}f(t)\left(\frac{t}{x}\right)^{1+\frac{1-k}{p}} - \frac{1}{x}\int_y^\infty f(t)dt\right|\right)$$
$$\times x^{\frac{1-k}{p}+p-k} t^{\frac{k-1}{p}-1} dxdt$$
$$\leq \left(\frac{B}{A}\right)^2 \left(\frac{1-k}{p}\right)^p \int_b^\infty \left[1 - \left(\frac{b}{x}\right)^{\frac{1-k}{p}}\right] x^{p-k}\phi(f(x))dx \tag{3.8}$$

holds for all nonnegative integrable function $f : (b, \infty) \to \mathbb{R}$. The inequality sign is reversed if $1 < p \leq 2$.

Proof. By applying Lemma 2.5 with the weight function $u(x) \equiv 1$, we find for $p \geq 2$ that

$$\int_b^\infty \left(x \int_x^\infty f(t) \frac{dt}{t^2} \right)^p \frac{dx}{x} + \int_b^\infty \int_b^t \left| f(t) - x \int_x^\infty f(t) \frac{dt}{t^2} \right|^p dx \frac{dt}{t^2} \leq \int_b^\infty \left(1 - \frac{b}{x}\right) f^p(x) \frac{dx}{x}. \tag{3.9}$$

Again, denote the first and second terms on the left hand side of (3.9) by I_1 and I_2 and the right hand side by I_3, respectively. Then, in (3.9) replace the parameter b by $a = b^{\frac{1-k}{p}}$ and the function f by $g(x) = f(x^{\frac{p}{1-k}}) x^{\frac{p}{1-k}+1}$. Thereafter, use the substitutions $y = x^{\frac{p}{1-k}}$ and $s = t^{\frac{p}{1-k}}$. The rest of the proof is similar to the proof of Theorem 3.1 and so the details are omitted. The proof of the case $1 < p \leq 2$ is similar and the only difference is that in this case all the inequalities signs are reversed. □

Remark 3.7. In the special case $\phi(x) = x^p$ and $A = B = 1$, then Theorem 3.6 reduces to Theorem 3.2 in [11].

By using Theorem 3.6 with $\phi(x) = x^p$, $A = B = 1$ and $b = 0$ we obtain the following result:

Example 3.8. Let $0 \leq b < \infty$ and let the function f be locally integrable on $[b, \infty)$ such that

$$0 < \int_0^\infty f^p(x) dx < \infty.$$

If $p \geq 2$, then

$$\int_0^\infty x^{-k} \left(\int_x^\infty f(t) dt \right)^p dx + \left(\frac{1-k}{p} \right) \int_0^\infty \int_0^s \left| \frac{p}{1-k} f(t) \left(\frac{t}{x} \right)^{1+\frac{1-k}{p}} - \frac{1}{x} \int_y^\infty f(t) dt \right|^p \times x^{\frac{1-k}{p}+p-k} t^{\frac{k-1}{p}-1} dx dt \leq \left(\frac{1-k}{p} \right)^p \int_0^\infty x^{p-k} f^p(x) dx. \tag{3.10}$$

The inequality sign is reversed if $1 < p \leq 2$.

4. REFINED HARDY TYPE INEQUALITIES WITH KERNELS

Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be measure spaces with positive $\sigma-$finite measures. Let the weight function $u : \Omega_1 \to \mathbb{R}$ be a measurable function and $k : \Omega_1 \times \Omega_2 :\to \mathbb{R}$ be measurable, nonnegative such that

$$K(x) = \int_{\Omega_2} k(x, y) d\mu_2(y) > 0, \quad x \in \Omega_1. \tag{4.1}$$

Suppose $I \subseteq \mathbb{R}$ and $\Phi : I \to \mathbb{R}$ and $f : \Omega_2 \to \mathbb{R}$ is measurable function with values in I, and the general integral operators $A_k f$ is defined by

$$A_k f(x) = \frac{1}{K(x)} \int_{\Omega_2} k(x, y) f(y) d\mu_2(y), \qquad x \in \Omega_1. \tag{4.2}$$

In the sequel, we state and prove new refined Hardy type inequalities with a general integral operator $A_k f$ and the function ϕ which may not necessarily be superquadratic under certain assumptions. Our first result in this direction reads:

Theorem 4.1. *Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be measure spaces with positive $\sigma-$finite measures and $v : \Omega_1 \to \mathbb{R}$ be a measurable function defined by*

$$v(y) = \int_{\Omega 1} \frac{k(x, y)}{K(x)} u(x) d\mu_1(x) < \infty, \qquad y \in \Omega_2 \tag{4.3}$$

where $u : \Omega_1 \to \mathbb{R}$ is a nonnegative weight function such that the function $x \to u(x)\frac{k(x,y)}{K(x)}$ is integrable on Ω_1 for each $y \in \Omega_2$ and $k : \Omega_1 \times \Omega_2 :\to \mathbb{R}$ is a nonnegative function such that K is as defined in (4.1). If ψ is a superquadratic function on an interval $I \subseteq \mathbb{R}$ and $\Phi : I \to \mathbb{R}$ is an function satisfying

$$A\psi(x) \leq \Phi(x) \leq B\psi(x) \tag{4.4}$$

on I for some constants A, B, such that $0 < A \leq B < \infty$. Then, the inequality

$$\int_{\Omega_1} u(x)\Phi(A_k f(x))d\mu_1(x)$$
$$+\int_{\Omega_2}\int_{\Omega 1} \Phi\left(\left|f(y) - A_k f(x)\right| d\mu_2(y)\right) \frac{k(x,y)}{K(x)} u(x)d\mu_1(x)d\mu_2(y)$$
$$\leq \frac{B}{A}\int_{\Omega_2} v(y)\Phi(f(y))d\mu_2(y), \tag{4.5}$$

holds for all measurable functions $f : \Omega_2 \to \mathbb{R}$ with values in I, where $A_k f(x)$ is as defined in (4.2).

Proof. By using assumption (4.4), Lemma 2.2 and Fubini's theorem to the first term in (4.5) we obtain

$$\int_{\Omega_1} u(x)\Phi(A_k f(x))d\mu_1(x)$$
$$\leq B\int_{\Omega_1} u(x)\psi\left(\frac{1}{K(x)}\int_{\Omega_2} k(x,y)f(y)d\mu_2(y)\right)d\mu_1(x) \tag{4.6}$$
$$\leq B\int_{\Omega_1} \frac{u(x)}{K(x)}\left(\int_{\Omega_2} k(x,y)\psi(f(y))d\mu_2(y)\right)d\mu_1(x)$$
$$- B\int_{\Omega_1} \frac{u(x)}{K(x)}\left(\int_{\Omega_2} k(x,y)\psi\left(\left|f(y) - A_k f(x)\right|\right)d\mu_2(y)\right)d\mu_1(x)$$
$$= B\int_{\Omega_2} \psi(f(y))\left(\int_{\Omega_1} \frac{k(x,y)}{K(x)} u(x)d\mu_1(x)\right)d\mu_2(y)$$
$$- B\int_{\Omega_2}\int_{\Omega 1} \psi\left(\left|f(y) - A_k f(x)\right|\right)\frac{k(x,y)}{K(x)} u(x)d\mu_1(x)d\mu_2(y)$$
$$= B\int_{\Omega_2} \psi(f(y))v(y)d\mu_2(y)$$
$$- B\int_{\Omega_2}\int_{\Omega 1} \psi\left(\left|f(y) - A_k f(x)\right|\right)\frac{k(x,y)}{K(x)} u(x)d\mu_1(x)d\mu_2(y) \tag{4.7}$$

Hence, inequality (4.7) can be re-written as

$$\int_{\Omega_1} u(x)\Phi(A_k f(x))d\mu_1(x)$$
$$+ B\int_{\Omega_2}\int_{\Omega 1} \psi\left(\left|f(y) - A_k f(x)\right|\right)\frac{k(x,y)}{K(x)} u(x)d\mu_1(x)d\mu_2(y)$$
$$\leq B\int_{\Omega_2} \psi(f(y))v(y)d\mu_2(y).$$

Again, by using assumption (4.4) we obtain that

$$\int_{\Omega_1} u(x)\Phi(A_k f(x))d\mu_1(x)$$
$$+\int_{\Omega_2}\int_{\Omega 1}\Phi\left(|f(y)-A_k f(x)|\right)\frac{k(x,y)}{K(x)}u(x)d\mu_1(x)d\mu_2(y)$$
$$\le \frac{B}{A}\int_{\Omega_2}\Phi(f(y))v(y)d\mu_2(y),$$

and the proof is complete. □

Remark 4.2. The special case $A = B = 1$ in Theorem 4.1 yields Theomem 3.1 in [2].

As an easy application of Theorem 4.1 we have the following results:

Theorem 4.3. *Let* $0 < b \le \infty$ *and* $k : (0,b)\times(0,b) \to \mathbb{R}$ *be a nonnegative measurable function such that*

$$K(x) = \int_0^x k(x,y)dy > 0, \quad x \in (0,b). \tag{4.8}$$

If $u : (0,b) \to \mathbb{R}$ *is a weight function such that* $x \to \frac{k(x,y)}{K(x)}.\frac{u(x)}{x}$ *is integrable on* (y,b) *for each fixed* $y \in (0,b)$ *and* $w : (0,b) \to \mathbb{R}$ *be a function defined by*

$$w(y) = y\int_y^b \frac{k(x,y)}{K(x)}u(x)\frac{dx}{x}. \tag{4.9}$$

If ψ *is a superquadratic function on an interval* $I \subseteq \mathbb{R}$ *and* $\phi : I \to \mathbb{R}$ *is a function satisfying*

$$A\psi(x) \le \phi(x) \le B\psi(x) \quad \forall x \in I, \tag{4.10}$$

then the inequality

$$\int_0^b u(x)\phi(A_k f(x))\frac{dx}{x}$$
$$+\int_0^b\int_y^b \phi(|f(y)-A_k f(x)|)\frac{k(x,y)}{K(x)}u(x)\frac{dx}{x}dy$$
$$\le \frac{B}{A}\int_0^b w(y)\phi(f(y))\frac{dy}{y}. \tag{4.11}$$

holds for all measurable function $f : (0,b) \to \mathbb{R}$ *with values in* I *and* $A_k f$ *is defined by*

$$A_k f(x) = \frac{1}{K(x)}\int_0^x k(x,y)f(y)dy, \quad x \in (0,b). \tag{4.12}$$

Proof. This follows directly from Theorem 4.1. To see this, set $\Omega_1 = \Omega_2 = (0, b)$, $0 < b \leq \infty$, $u(x)$ by $\frac{u(x)}{x}$, replace $d\mu_1(x)$ and $d\mu_2(y)$ by the Lebesgue measures dx and dy respectively, then we obtain inequality (4.11). Moreover, (4.3) and (4.2) reduce to (4.9) and (4.12) respectively. In addition, $w(y) = yv(y)$, $y \in (0, b)$. $\square$

Note that by applying Theorem 4.3 with $k(x, y) = 1$ we obtain:

Example 4.4. Let $0 < b \leq \infty$ and $u : (0, b) \to \mathbb{R}$ be a weight function such that $x \to \frac{u(x)}{x^2}$ is integrable on (y, b) and $w(y) = y \int_y^b \frac{u(x)}{x^2} dx$. If ψ is a superquadratic function on $I \subseteq \mathbb{R}$ and $\phi : I \to \mathbb{R}$ is a function satisfying (4.4) for all $x \in I$, then

$$\begin{aligned} &\int_0^b u(x)\phi\left(\frac{1}{x}\int_0^x f(y)dy\right)\frac{dx}{x} \\ &+ \int_0^b \int_y^b \phi\left(\left|f(y) - \frac{1}{x}\int_0^x f(y)dy\right|\right)\frac{u(x)}{x^2}dxdy \\ &\leq \frac{B}{A}\int_0^b w(y)\phi(f(y))\frac{dy}{y}, \end{aligned} \tag{4.13}$$

holds for all measurable functions $f : (0, b) \to \mathbb{R}$. The inequality sign of (4.13) is reversed if ϕ is subquadratic.

Remark 4.5. We note that for $A = B = 1$, Example 4.4 coincides with Proposition 2.1 in [11].

Remark 4.6. Note also that by setting $u(x) = 1$, $A = B = 1$ and $\phi(x) = x^p$, $p \geq 2$ (respectively $1 < p \leq 2$), then Example 4.4 implies a result in [11, Example 4.3].

5. FURTHER RESULTS AND REMARKS

As an easy consequence of Theorem 3.1 we obtain the following result:

Corollary 5.1. *Let $p > 1$, $k > 1$, $b \in (0, \infty]$, and let the function f be locally integrable on $(0, b)$ such that*

$$0 < \int_0^b \left[1 - \left(\frac{x}{b}\right)^{\frac{k-1}{p}}\right] x^{p-k} f^p(x) dx < \infty.$$

If $\phi : [0, \infty) \to \mathbb{R}$ is superquadratic, such that $Ax^p \leq \phi(x) \leq Bx^p$ holds on $[0, \infty)$ for some constants $0 < A \leq B < \infty$, then for $p \geq 2$

and $b \in (0,\infty]$ the inequality

$$\int_0^b x^{-k}\phi\left(\int_0^x f(t)dt\right)dx$$
$$+\frac{k-1}{p}\int_0^b\int_t^b \phi\left(\left|\frac{p}{k-1}\left(\frac{t}{x}\right)^{1-\frac{k-1}{p}} f(t)-\frac{1}{x}\int_0^x f(t)dt\right|\right)$$
$$\times x^{p-k-\frac{k-1}{p}} t^{\frac{k-1}{p}-1} dxdt$$
$$\leq \frac{B^3}{A^2}\left(\frac{p}{k-1}\right)^p \int_0^b \left[1-\left(\frac{x}{b}\right)^{\frac{k-1}{p}}\right] x^{p-k} f^p(x)dx \tag{5.1}$$

holds for all nonnegative integrable function $f:(0,b)\to\mathbb{R}$.

If ϕ is subquadratic, then the sign of inequality (3.1) is reversed.

By using Corollary 5.1 with $A=B=1$, $\phi(x)=x^p$ we obtain:

Example 5.2. Let $0<b\leq\infty$ and let the function f be locally integrable on $(0,b)$ such that

$$0<\int_0^b \left[1-\left(\frac{x}{b}\right)^{\frac{k-1}{p}}\right] x^{p-k} f^p(x)dx<\infty.$$

If $p\geq 2$, then

$$\int_0^b x^{-k}\left(\int_0^x f(t)dt\right)^p dx$$
$$+\frac{k-1}{p}\int_0^b\int_t^b \left|\frac{p}{k-1}\left(\frac{t}{x}\right)^{1-\frac{k-1}{p}} f(t)-\frac{1}{x}\int_0^x f(t)dt\right|^p$$
$$x^{p-k-\frac{k-1}{p}} t^{\frac{k-1}{p}-1} dxdt$$
$$\leq \left(\frac{p}{k-1}\right)^p \int_0^b \left[1-\left(\frac{x}{b}\right)^{\frac{k-1}{p}}\right] x^{p-k} f^p(x)dx \tag{5.2}$$

The inequality is reversed if $1<p\leq 2$.

Remark 5.3. Note that by letting $k=p$ in Example 5.2 we obtain Example 4.3 in [11].

Also as a consequence of Theorem 3.6 we have

Corollary 5.4. *Let $p>1$, $k<1$, $b\in[0,\infty)$, and let the function f be locally integrable on (b,∞) and such that*

$$0<\int_b^\infty \left[1-\left(\frac{b}{x}\right)^{\frac{1-k}{p}}\right] x^{p-k}\phi(f(x))dx<\infty.$$

If $\phi : [b, \infty) \to \mathbb{R}$ is superquadratic, such that $Ax^p \leq \phi(x) \leq Bx^p$ holds on $[b, \infty)$ for some constants $0 < A \leq B < \infty$, then for $p \geq 2$ the inequality

$$\begin{aligned}
&\int_b^\infty x^{-k} \phi\left(\int_x^\infty f(t)dt\right) dx \\
&+ \left(\frac{1-k}{p}\right) \int_b^\infty \int_b^s \phi\left(\left|\frac{p}{1-k} f(t) \left(\frac{t}{x}\right)^{1+\frac{1-k}{p}} - \frac{1}{x}\int_y^\infty f(t)dt\right|\right) \\
&\times x^{\frac{1-k}{p}+p-k} t^{\frac{k-1}{p}-1} dxdt \\
&\leq \frac{B^3}{A^2} \left(\frac{1-k}{p}\right)^p \int_b^\infty \left[1 - \left(\frac{b}{x}\right)^{\frac{1-k}{p}}\right] x^{p-k} f^p(x) dx
\end{aligned} \tag{5.3}$$

holds for all nonnegative integrable function $f : (b, \infty) \to \mathbb{R}$.
The inequality sign is reversed if $1 < p \leq 2$.

Observe that by putting $A = B = 1$, $\phi(x) = x^p$ in Corollary 5.4 we obtain the following result.

Corollary 5.5. *Let $0 \leq b < \infty$, $k < 1$, and let the function f be locally integrable on (b, ∞) and such that*

$$0 < \int_b^\infty \left[1 - \left(\frac{b}{x}\right)^{\frac{1-k}{p}}\right] x^{p-k} \phi(f(x)) dx < \infty.$$

If $p \geq 2$, then

$$\begin{aligned}
&\int_b^\infty x^{-k} \left(\int_x^\infty f(t)dt\right)^p dx \\
&+ \left(\frac{1-k}{p}\right) \int_b^\infty \int_b^s \left|\frac{p}{1-k} f(t) \left(\frac{t}{x}\right)^{1+\frac{1-k}{p}} - \frac{1}{x}\int_y^\infty f(t)dt\right|^p \\
&\times x^{\frac{1-k}{p}+p-k} t^{\frac{k-1}{p}-1} dxdt \\
&\leq \left(\frac{1-k}{p}\right)^p \int_b^\infty \left[1 - \left(\frac{b}{x}\right)^{\frac{1-k}{p}}\right] x^{p-k} f^p(x) dx
\end{aligned} \tag{5.4}$$

holds for all nonnegative integrable function $f : (b, \infty) \to \mathbb{R}$.
The inequality sign is reversed if $1 < p \leq 2$.

Remark 5.6. The results obtained in this paper are the refinements of the one-dimensional analogues of some recent results in [13]. In particular, this opens the possibility to also generalize some of these results in the direction pointed out in this paper. However, it is far from obvious how the elementary technique pointed out in this paper can be generalized, for example, to a multidimensional situation.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF AGRICULTURE, P. M. B. 2240, ABEOKUTA, OGUN STATE, NIGERIA
E-mail address: `oguntuase@yahoo.com`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF AGRICULTURE, P. M. B. 2240, ABEOKUTA, OGUN STATE, NIGERIA
E-mail address: `yemi376@yahoo.com`

DEPARTMENT OF MATHEMATICS AND STATISTICS, FEDERAL POLYTECHNIC, P. M. B. 50, ILARO, OGUN STATE, NIGERIA
E-mail address: `opidi2000@gmail.com`