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UNCOUNTABLY INFINITE SOLUTIONS TO A CLASS OF NONLINEAR SINGULAR TWO-POINT EIGENVALUE BOUNDARY VALUE PROBLEMS

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ABSTRACT. We proved the existence of unique weak solutions in weighted Sobolev spaces, up to each of an arbitrarily selectable parameter $m \in (0, a]$, for the nonlinear singular second order two-point boundary value problems

$$u''(r) + \frac{a}{r}u'(r) + g(u(r)) = h(r), \ a \geq 1, \quad r \in (0, 1)$$
$$u(0) = u(1) = 0,$$

where $h \in L^2[(0, 1), r^m]$, for $1 < m \leq a$, and $g : \mathbb{R} \to \mathbb{R}$ is differentiable, with $0 \not\equiv |g'| \leq \gamma =$ constant. Our solutions are uncountably infinite, since the possible choices of the parameter $m \in (1, a]$ are uncountably infinite.

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1. INTRODUCTION

The following nonlinear second order two-point eigenvalue boundary value problems are considered:

$$u''(r) + \frac{a}{r}u'(r) + g(u(r)) = h(r), \ a \geq 1, \quad r \in (0, 1) \tag{1}$$
$$u(0) = u(1) = 0. \tag{2}$$

In applications, singular boundary value problems arise in the fields of boundary layer theory, gas dynamics, nonlinear optics, combustion, quantum mechanics, etc.; see for examples [1], [4], [15], [19], [24] and the literature in them.

(1) with the boundary conditions $u'(0) = 0, \ u(1) = A$ is solved numerically in [3], [14] for the case $a = 2$. The presence of the singular coefficient $\frac{a}{r}$ and the nonlinear term $g(u)$ motivates the numerical solutions. Similar or some other singular boundary value

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problems have been solved by various numerical methods. For a few of such works, we refer the reader to [3], [9], [12], [14], [16], [17], [23] and the literature cited in them.

Existence results for various classes of singular boundary value problems abound in the literature. For example, see [5], [8], [10], [21], and the literature cited in them. In none of these works, was the existence of weak or generalized solution to the respective singular boundary problems proved. For the examples of relatively few works on the existence of weak solutions for ordinary differential equation boundary value problems, the reader is referred to [6], [13], [20], [22]. Furthermore, existence results for singular ordinary differential equations in weighted Sobolev spaces appear to be scarce. Recently, we gave existence results, in weighted Sobolev's spaces, for some other class of nonlinear singular two-point boundary value problems in [18].

Inspired by previous works, we study, in the current work, the existence and uniqueness of weak solutions to the problem (1)-(2) in a weighted Sobolev space. We refer the reader to [2] and [11], for information on weighted Sobolev spaces. We assume that $g : \mathbb{R} \to \mathbb{R}$ is differentiable, with

$$0 \not\equiv |g'(v)| \leq \gamma \tag{3}$$

and $h(r) \in L^2[(0,1), r^m]$, for $1 < m \leq a$.

The following weighted Sobolev's spaces are used in this paper:

$$L^2[(0,1), r^m] := \{w : \ (0,1) \to \ R \ \big| \ \|w\|_{L^2[(0,1),r^m]} < \infty\} \tag{4}$$

$$\text{where } \|w\|_{L^2[(0,1),r^m]} = \sqrt{\int_0^1 r^m w^2 dr} \tag{5}$$

$$H^1[(0,1), r^m] := \{w : \ (0,1) \to \ R \ \big| \ \|w\|_{H^1[(0,1),r^m]} < \infty\} \tag{6}$$

$$\text{where } \|w\|_{H^1[(0,1),r^m]} = \sqrt{\int_0^1 r^m w^2 dr + \int_0^1 r^m w'^2 dr} \tag{7}$$

$H_0^1[(0,1), r^m]$ is the closure of $C_c^\infty(0,1)$ with respect to the norm $\|w\|_{H^1[(0,1),r^m]}$, where $\|w\|_{H_0^1[(0,1),r^m]} := \|w'\|_{L^2[(0,1),r^m]}$ (8)

$$L^2[(0,1), \frac{(m-1)(a-m)}{2m} r^{m-2}] :=$$

$$\{w : \ (0,1) \to \ R \ \big| \ \|w\|_{L^2[(0,1), \frac{(m-1)(a-m)}{2m} r^{m-2}]} < \infty\}, \text{ where} \tag{9}$$

$$\|w\|_{L^2[(0,1), \frac{(m-1)(a-m)}{2m} r^{m-2}]} = \sqrt{\int_0^1 \frac{(m-1)(a-m)}{2m} r^{m-2} w^2 dr} \tag{10}$$

$$X := L^2[(0,1), \frac{(m-1)(a-m)}{2m} r^{m-2}] \cap H_0^1[(0,1), r^m], \tag{11}$$

with the norm

$$\|w\|_X = \sqrt{\|w\|_{L^2[(0,1), \frac{(m-1)(a-m)}{2m} r^{m-2}]} + \|w\|_{H_0^1[(0,1), r^m]}} \tag{12}$$

We shall use the following equivalent divergence form of (1)-(2):

$$-(r^m u')' - (a-m) r^{m-1} u' = r^m [g(u(r)) - h(r)], \ r \in (0,1) \tag{13}$$

$$u(0) = u(1) = 0 \tag{14}$$

where $1 < m \leq a$.

Definition: 1. *A solution $u \in X$ is called a weak solution of the boundary value problem (1)-(2) provided*

$$-\int_0^1 r^m u' v' dr + (a-m) \int_0^1 r^{m-1} u' v dr = \int_0^1 r^m [h - g(u)] v dr, \tag{15}$$

for each $v \in X$.

We arrange the rest of the paper as follows: In Section 2, we proved existence and uniqueness of solutions to an auxiliary linear problem, utilized in establishing existence and uniqueness of solutions to the nonlinear problem in Section 3. The existence and uniqueness of solution was proved by applying the Banach's fixed point theorem. Our solutions are uncountably infinite, since the possible choices of the parameter $m \in (0, a]$ are uncountably infinite.

2. AUXILIARY LINEAR PROBLEM

Consider the following linear singular eigenvalue boundary problem:

$$-(r^m u')' - (a-m) r^{m-1} u' = r^m [g(s(r)) - h(r)], \ r \in (0,1) \tag{16}$$

$$u(0) = u(1) = 0, \tag{17}$$

with $1 < m \leq a$, where $s, \ h, \in L^2[(0,1), r^m]$ and $g(s)$ are known functions of r.

We first prove the following lemma.

Lemma: 1. *(A Sobolev's embedding). Let $u \in H_0^1[(0,1), r^m]$. Then we have the estimate*

$$\|u\|_{L^2[(0,1), r^m]} \leq \frac{2}{m+1} \|u\|_{H_0^1[(0,1), r^m]}. \tag{18}$$

Proof: Using integration by parts, we have 8

$$\int_0^1 r^m u^2 dr = \frac{r^{m+1}}{m+1}u^2\Big|_0^1 - \int_0^1 \frac{r^{m+1}}{m+1}(u^2)' dr$$
$$= -\frac{2}{m+1}\int_0^1 r r^m u u' dr \leq \frac{2}{m+1}\int_0^1 r^m |u||u'| dr, \text{ (since } r \leq 1)$$
$$\leq \frac{2}{m+1}\Big(\int_0^1 r^m u^2 dr\Big)^{\frac{1}{2}}\Big(\int_0^1 r^m u'^2 dr\Big)^{\frac{1}{2}} \text{ (by Hölder's inequality)}$$
$$\leq \frac{1}{2}\int_0^1 r^m u^2 dx + \frac{4}{2(m+1)^2}\int_0^1 r^m u'^2 dx, \tag{19}$$

by Cauchy's inequality. Simplifying (19), we easily deduce (18).

Theorem: 1. *(A priori estimates). Let u be a solution of (16)-(17). Then, $u \in X$ and we have the estimate*

$$\|u\|_X \leq \sqrt{\frac{2}{m}}\alpha\Big(\|s\|_{L^2[(0,1),r^m]} + \|h\|_{L^2[(0,1),r^m]} + 1\Big), \tag{20}$$

where

$$\alpha := \max\{1, \gamma, |g(0)|\}. \tag{21}$$

Proof: We split the proof in three steps.
Step 1. Multiply (16) by u, integrate by parts and use (17) to get

$$\int_0^1 r^m u'^2 dr - \int_0^1 (a-m) r^{m-1} u' u dr = \int_0^1 r^m u(g(s)-h) dr \tag{22}$$

We next obtain an equivalent of the second term on the left side of (22) by integration by parts and applying (17):

$$-(a-m)\int_0^1 r^{m-1} u' u dr = -\frac{a-m}{2}\int_0^1 r^{m-1}(u^2)' dr$$
$$= \frac{(a-m)(m-1)}{2}\int_0^1 r^{m-2} u^2 dr. \tag{23}$$

Using (23) in (22) gives

$$\int_0^1 r^m u'^2 dr + \frac{(m-1)(a-m)}{2}\int_0^1 r^{m-2}u^2 dr = \int_0^1 r^m u(g(s)-h) dr \tag{24}$$

Step 2. Using (24), Hölder inequality, and Lemma 1 we estimate:

$$\begin{aligned}
&\int_0^1 r^m u'^2 dr + \frac{(m-1)(a-m)}{2} \int_0^1 r^{m-2} u^2 dr \\
&\leq \Big(\int_0^1 r^m u^2 dr\Big)^{\frac{1}{2}} \Big[\Big(\int_0^1 r^m h^2 dr\Big)^{\frac{1}{2}} + \Big(\int_0^1 r^m g(s)^2 dr\Big)^{\frac{1}{2}}\Big] \\
&\leq \frac{2}{m+1}\Big(\int_0^1 r^m u'^2 dr\Big)^{\frac{1}{2}} \Big[\Big(\int_0^1 r^m h^2 dr\Big)^{\frac{1}{2}} + \Big(\int_0^1 r^m g(s)^2 dr\Big)^{\frac{1}{2}}\Big] \\
&\leq \frac{2}{(m+1)}\Big[2\epsilon \int_0^1 r^m u'^2 dr + \frac{1}{4\epsilon}\Big(\int_0^1 r^m h^2 dr + \int_0^1 r^m g^2 dr\Big)\Big], \quad (25)
\end{aligned}$$

by Cauchy's inequality with ϵ. Choosing $\epsilon = \frac{1}{4}$ and simplifying we deduce

$$\begin{aligned}
&\int_0^1 r^m u'^2 dr + \frac{(m-1)(a-m)}{2m} \int_0^1 r^{m-2} u^2 dr \\
&\leq \frac{2}{m}\Big(\int_0^1 r^m h^2 dr + \int_0^1 r^m g^2 dr\Big). \quad (26)
\end{aligned}$$

Step 3. Using (3), we estimate

$$\begin{aligned}
&\int_0^1 r^m |g(s)|^2 dr = \int_0^1 r^m |\int_0^s g'(\xi) d\xi + g(0)|^2 dr \\
&\leq \int_0^1 r^m (\gamma|s| + |g(0)|)^2 dr \\
&\leq \beta^2 (\|s\|_{L^2[(0,1),r^m]} + 1)^2, \text{ where } \beta := \max\{\gamma, |g(0)|\}. \quad (27)
\end{aligned}$$

The conclusion of the theorem follows by substituting (27) into (26) and simplifying.

Definition: 2. *(i) The Bilinear form $B[.,.]$ associated with the elliptic operator L defined by (16) is*

$$B[u,v] := \int_0^1 r^m u'v' dr - (a-m) \int_0^1 r^{m-1} u'v dr, \quad (28)$$

for $u, v \in X$. (ii) $u \in X$ is said to be a weak solution of the boundary value problem (16)-(17) provided

$$B[u,v] = (g(s) - h, v)_{r^m} \quad (29)$$

for all $v \in X$, where $(.,.)_{r^m}$ is the inner product in $L^2[(0,1), r^m]$ defined by $(u,v)_{r^m} := \int_0^1 r^m uv dx$.

Theorem: 2. *$B[.,.]$ satisfies precisely the hypotheses of the Lax-Milgram Theorem. That is, there exists constants $\lambda, \sigma > 0$ such that*

(i) $|B[u,v]| \leq \lambda \|u\|_X \|v\|_X$
and (ii) $\sigma \|u\|_X^2 \leq B[u,u]$
for all $u, v \in X$.

Proof: The proof is split in two steps.
Step 1. We have

$$
\begin{aligned}
&|B[u,v]| = |\int_0^1 r^m u'v'dr - (a-m)\int_0^1 r^{m-1}u'vdr| \\
&= |\int_0^1 r^m u'v'dr - \\
&\sqrt{\frac{2m(a-m)}{m-1}} \int_0^1 (r^{\frac{m}{2}}u') \left(\sqrt{\frac{(m-1)(a-m)}{2m}} r^{\frac{m-2}{2}} v \right) dr| \\
&\leq \Big(\int_0^1 r^m u'^2 dr \Big)^{\frac{1}{2}} \Big(\int_0^1 r^m v'^2 dr \Big)^{\frac{1}{2}} + \\
&\sqrt{\frac{2m(a-m)}{m-1}} \Big(\int_0^1 r^m u'^2 dr \Big)^{\frac{1}{2}} \Big(\int_0^1 \frac{(m-1)(a-m)}{2m} r^{m-2} v^2 dr \Big)^{\frac{1}{2}} \\
&\leq \lambda \|u\|_X \|v\|_X \qquad (30)
\end{aligned}
$$

for some appropriate constant $\lambda > 0$.
Step 2. Furthermore, we readily check

$$
\begin{aligned}
\sigma \|u\|_X^2 \leq B[u,u] &= \int_0^1 r^m u'^2 dr - (a-m)\int_0^1 r^{m-1} \Big(\frac{u^2}{2}\Big)' dr \\
&= \int_0^1 r^m u'^2 dr + \frac{(m-1)(a-m)}{2} \int_0^1 r^{m-2} u^2 dr \qquad (31)
\end{aligned}
$$

for appropriate constant $\sigma > 0$.

Theorem: 3. *There exist unique weak solutions* $u \in X$ *of the linear boundary value problem (16)-(17).*

Proof: Notice that (27) and the hypothesis on h imply that

$$g(s) - h \in L^2[(0,1), r^m].$$

Now fix $g(s) - h \in L^2[(0,1), r^m]$ and set $\langle g(s) - h, v\rangle := (g(s) - h, v)_{r^m}$ (where $\langle \ , \ \rangle$ is the pairing of $L^2[(0,1), r^m]$ with its dual). This is a bounded linear functional on $L^2[(0,1), r^m]$ and hence on X.

We apply the Lax-Milgram Theorem (see [7]) to find a unique function $u \in X$ satisfying

$$B[u,v] = \langle g(s) - h, v\rangle$$

for all $v \in X$. Consequently, u is the unique weak solution of (16)-(17).

3. MAIN RESULT

Theorem: 4. *Let* $\sqrt{2/m}\gamma < 1$, *then there exists unique weak solutions* $u \in X$ *to (1)-(2).*

Proof: We split the proof in six steps.
Step 1. The fixed point argument to (1)-(2) is

$$-(r^m w')' - (a-m)r^{m-1}w' = r^m(g(u(r)) - h(r)) \tag{32}$$

$$w(0) = w(1) = 0. \tag{33}$$

Define a mapping

$$A : X \to X, \tag{34}$$

by setting $A[u] = w$ whenever w is derived from u via (32)-(33). We claim that the mapping A is a strict contraction if $\sqrt{2/m}\gamma < 1$.
Step 2.Choose $u, \tilde{u} \in X$, and define $A[u] = w,\ A[\tilde{u}] = \tilde{w}$. Hence for two solutions $w, \tilde{w}$ of (32)-(33) we have that

$$-(r^m(w-\tilde{w})')' - (a-m)r^{m-1}(w-\tilde{w})' = r^m(g(u(r)) - g(\tilde{u}(r))) \tag{35}$$

$$(w-\tilde{w})(0) = (w-\tilde{w})(1) = 0. \tag{36}$$

Using (35)-(36), we have an analogous estimate to (26) in the proof of Theorem 1, viz:

$$\begin{aligned}\|w-\tilde{w}\|_X &\le \sqrt{\frac{2}{m}}\|g(u) - g(\tilde{u})\|_{L^2[(0,1),r^m]}\\ &\le \sqrt{\frac{2}{m}}\|\int_{\tilde{u}}^{u} g'(\xi)d\xi\|_{L^2[(0,1),r^m]}\\ &\le \sqrt{\frac{2}{m}}\gamma\|u-\tilde{u}\|_{L^2[(0,1),r^m]} \quad \text{(using (3))}\end{aligned} \tag{37}$$

We now use (37) and our definition of the mapping A to deduce

$$\|A[u] - A[\tilde{u}]\|_X = \|w-\tilde{w}\|_X \le \sqrt{\frac{2}{m}}\gamma\|u-\tilde{u}\|_X. \tag{38}$$

Therefore the mapping A is a strict contraction for $\sqrt{\frac{2}{m}}\gamma < 1$. Thus, by Banach's fixed point theorem (see [7]), A has a unique fixed point in X.

Step 3. Write $u_0 = u(0)$, and for $k = 0, 1, 2...$, inductively define $w = u_{k+1} \in X$ to be the unique weak solution of the linear boundary value problem

$$-(r^m u'_{k+1})' - (a-m)r^{m-1}u'_{k+1} = r^m(g(u_k) - h(r)) \quad a \geq 2 \quad (39)$$

$$u_{k+1}(0) = u_{k+1}(1) = 0 \quad (40)$$

Notice that our definition of u_{k+1} as the unique weak solution of (39)-(40) is justified by Theorem 3. By the definition of the mappings A, we have (for $k = 0, 1, 2, ...$), using (39)-(40), that

$$u_{k+1} = A[u_k] \quad (41)$$

Since A has a fixed point in X, there exists $u \in X$ such that

$$\lim_{k\to\infty} u_{k+1} = \lim_{k\to\infty} A[u_k] = A[u] = u \quad (42)$$

Step 4.We use (27) and Lemma 1 to deduce

$$\|g(u_k)\|_{L^2[(0,1),r^m]} \leq \sqrt{\frac{2}{m}}\beta\left(\|u'_k\|_{L^2[(0,1),r^m]} + 1\right) \quad (43)$$

Using (42), we can take the limit on the right side of (43) to deduce

$$\sup_k \|g(u_k)\|_{L^2[(0,1),r^m]} < \infty. \quad (44)$$

(44) implies the existence of a subsequence $\{g(u_{k_j})\}_{j=1}^{\infty}$ which converges weakly in $L^2[(0,1),r^m]$ to $g(u)$ in $L^2[(0,1),r^m]$.
Step 5. We next verify that u is a weak solution of (1)-(2). Fix $v \in X$. Using (39)-(40), we have

$$-\int_0^1 r^m u'_{k+1}v'dr + (a-m)\int_0^1 r^{m-1}u'_{k+1}vdr = \int_0^1 r^m[h(t) - g(u_k)]vdr. \quad (45)$$

Passage to limit is not immediately apparent in the second term of the left side of (45). Notice that

$$(a-m)\int_0^1 r^{m-1}u'_{k+1}vdr = \sqrt{\frac{2m(a-m)}{m-1}}\int_0^1 \left(r^{\frac{m}{2}}u_{k+1}\right)\left(\sqrt{\frac{(m-1)(a-m)}{2m}}r^{\frac{m-2}{2}}v\right)dr (46)$$

Hence, letting $k \to \infty$ in (45) yields (15) as desired. Step 6. Suppose that $\sqrt{2/m}\gamma < 1$, but that there exist two solutions $u, \bar{u} \in$

X. Then $A[u] = u$ and $A[\bar{u}] = \bar{u}$. Hence

$$\|u - \bar{u}\|_X = \|A[u] - A[\bar{u}]\|_X \leq \sqrt{\frac{2}{m}}\gamma\|u - \bar{u}\|_X \quad \text{(using (38))}, \quad (47)$$

so that $u = \bar{u}$.

4. ILLUSTRATIVE EXAMPLE

Consider the boundary value problem

$$u'' + \frac{a}{r}u' = \frac{1}{\sqrt{r}} - \tan^{-1}\frac{u}{2}, \ a \geq 1, \ r \in (0,1) \quad (48)$$

$$u(0) = u(1) = 0 \quad (49)$$

Here, we have $h = \frac{1}{\sqrt{r}} \in L^2[(0,1), r^m]$ (for $1 < m \leq a$) and $0 \not\equiv |g'(u)| = |\frac{2}{4+u^2}| \leq \gamma = \frac{1}{2}$. Notice that for this problem,

$$\sqrt{\frac{2}{m}}\gamma = \frac{1}{2}\sqrt{\frac{2}{m}} \leq \frac{1}{\sqrt{2}} < 1$$

Hence, by Theorem 4, there exist unique weak solutions $u \in X$ to the singular boundary value problems (48)-(49).

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