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STRONG CONVERGENCE OF A MODIFIED PICARD PROCESS TO A COMMOM FIXED POINT OF A FINITE FAMILY OF LIPSCHITZIAN HEMICONTRACTIVE MAPS

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ABSTRACT. Let K be a closed convex nonempty subset of a Hilbert space $H$and let the set of the common fixed points of a finite family of Lipschitzian hemicontractive maps from K into itself be non empty. Sufficient conditions for the strong convergence of the sequence of successive approximations generated by a Picard-like process to a common fixed point of the family are proved.

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1. INTRODUCTION

Let H be a Hilbert space and let K be a nonempty subset of H. K is said to be (sequentially) compact if every closed bounded sequence in K has a subsequence that converges in K. K is said to be boundedly compact if every bounded subset of K is compact. In finite dimensional spaces, closed subsets are boundedly compact. Given a subset S of K, we shall denote by $co(S)$ and $ccl(S)$ the convex hull and the closed convex hull of S respectively. If K is boundedly compact convex and S is bounded, then $co(S)$ and hence $ccl(S)$ are compact convex subsets of K.

A map $T : K \to E$ is said to be semi-compact if for any bounded sequence $\{x_n\} \subset K$ such that $\|x_n - Tx_n\| \to 0$ as $n \to \infty$ there exists a subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that x_{n_j} converges strongly to some $x^* \in K$ as $j \to \infty$. The map T is said to be demi-compact at $z \in E$ if for any bounded sequence $\{x_n\} \subset K$ such that $\|x_n - Tx_n\| \to z$ as $n \to \infty$ there exist a subsequence $\{x_{n_j}\} \subset \{x_n\}$ and a point $p \in K$ such that x_{n_j} converges strongly to p as $j \to \infty$. (Observe that if T is additionally continuous, then $p - Tp = z$).

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A nonlinear map $T : K \to E$ is said to be completely continuous if it maps bounded sets into relatively compact sets.Let T be a self-mapping on K. T is said to be Lipschitzian if $\exists\ L \geq 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\|, \qquad \forall\ x, y \in K. \tag{1}$$

If $L = 1$ then T is called *nonexpansive* and if $L < 1$ then the mapping T is called a contraction. The Mapping T with domain $D(T)$ and the range $R(T)$ in H is called pseudocontractive if $\forall\ x, y \in D(T)$,

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|(I - T)x - (I - T)y\|^2. \tag{2}$$

If (2) holds for all $x \in D(T)$and $y \in F(T)$(fixed point set of T), then T is said to be hemicontractive.

The class of pseudocontractive maps has been extensively studied (see e.g [1]-[6] and the references therein). It is clear that the important class of nonexpansive mappings is a subclass of the class of pseudocontractive maps.

In [4], Ishikawa introduced a new iteration method and proved that it converges strongly to a fixed point of a Lipschitz pseudocontractive map defined on a compact convex subset of a Hilbert space. In fact, he proved the following result.

Theorem 1 (Ishikawa [4])
Let K be a compact convex nonempty subset of a Hilbert space and let $T : K \to K$ be a Lipschitz Pseudocontractive map. Let $\{\alpha_n\}, \{\beta_n\}$ be real sequences satisfying the following conditions:

(1) $0 < \alpha_n \leq \beta_n < 1$.
(2) $\lim\limits_{n\to\infty} \beta_n = 0$.
(3) $\Sigma\alpha_n\beta_n = \infty$.

Then starting with an arbitrary $x_0 \in K$ the sequence $\{x_n\}$ of successive approximations defined by

$$x_{n+1} = \alpha_n T[\beta_n Tx_n + (1 - \beta_n)x_n] + (1 - \alpha_n)x_n;\ n \geq 0, \tag{3}$$

converges strongly to a fixed point of T.

This result has been generalised in several ways by many authors see e.g. Li Qihou [5]who proved Theorem 1 in the case where T is a Lipschitz hemicontraction. Chidume and Moore [1] who proved Theorem 1 in the case where T is a continuous hemicontraction and for iteration process with errors, Ghosh and Debnath[3] who proved the convergence of a Picard-like process to a fixed point of a quasi-nonexpansive map.

Our purpose in this paper is to construct a Picard-like iteration process which converges strongly to a common fixed point of a finite family of Lipschitz hemicontractive self maps of a closed convex nonempty subset of a Hilbert space.

2. MAIN RESULT

We need the following lemma in this work.

Lemma 1: For any x, y, z in a Hilbert space H and a real number $\lambda \in [0, 1]$,

$$\begin{aligned} \|\lambda x + (1-\lambda)y - z\|^2 = \lambda\|x - z\|^2 + (1-\lambda)\|y - z\|^2 \\ - \lambda(1-\lambda)\|x - y\|^2. \end{aligned} \tag{4}$$

We define the following auxilliary maps:$\alpha, \beta \in (0, 1)$ constants;

$$S_{i\beta} = (1-\beta)I + \beta T_i;\ i \in \{1, 2, ..., N\}. \tag{5}$$

$$T_{i\alpha\beta} = (1-\alpha)I + \alpha T_i S_{i\beta}. \tag{6}$$

Theorem 1: Let H be a real Hilbert space and K a nonempty closed convex subset of H and let $\{T_i\}_{i=1}^{N}$ be a finite family of Lipschitzian(with constant $L_i > 0$) hemicontractive maps from K into itself such that $F = \bigcap_{i=1}^{N} F(T_i)$ is not empty. Let $L := \max_{1\le i\le N}\{L_i\}$, and choose $\alpha, \beta \in (0, 1)$ such that $0 < \alpha < \beta < \frac{1}{1+\sqrt{1+L^2}}$. Starting with an arbitrary $x_0 \in K$, define the iterative sequence $\{x_n\}$ by

$$x_{n+1} = T_{i\alpha\beta} x_n; \qquad n \ge 0 \qquad n + 1 \equiv i \mod N, \tag{7}$$

then

(1) $\{x_n\}$ is bounded.
(2) $\lim\limits_{n\to\infty} \|x_n - x^*\|$ exists for $x^* \in F$.
(3) $\forall i \in \{1, 2, ..., N\}.\ \lim\limits_{n\to\infty} \|x_n - T_i x_n\| = 0$.

Proof:
Let $x^* \in \bigcap_{i=1}^N F(T_i)$.
Now,

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|T_{i\alpha\beta}x_n - x^*\|^2 \\
&= \|(1-\alpha)x_n + \alpha T_i S_{i\beta}x_n - x^*\|^2 \\
&= (1-\alpha)\|x_n - x^*\|^2 + \alpha\|T_i S_{i\beta}x_n - x^*\|^2 \\
&\quad -\alpha(1-\alpha)\|T_i S_{i\beta}x_n - x_n\|^2 \\
&\leq (1-\alpha)\|x_n - x^*\|^2 + \alpha[\|S_{i\beta}x_n - x^*\|^2 \\
&\quad +\|S_{i\beta}x_n - T_i S_{i\beta}x_n\|^2] \\
&\quad -\alpha(1-\alpha)\|T_i S_{i\beta}x_n - x_n\|^2.
\end{aligned}$$

$$\begin{aligned}
\|S_{i\beta}x_n - x^*\|^2 &= \|(1-\beta)x_n + \beta T_i x_n - x^*\|^2 \\
&= (1-\beta)\|x_n - x^*\|^2 + \beta\|T_i x_n - x^*\|^2 \\
&\quad -\beta(1-\beta)\|T_i x_n - x_n\|^2 \\
&\leq (1-\beta)\|x_n - x^*\|^2 + \beta(\|x_n - x^*\|^2 \\
&\quad +\|x_n - T_i x_n\|^2) - \beta(1-\beta)\|T_i x_n - x_n\|^2 \\
&= \|x_n - x^*\|^2 + \beta^2\|x_n - T_i x_n\|^2.
\end{aligned}$$

Also,

$$\begin{aligned}
\|S_{i\beta}x_n - T_i S_{i\beta}x_n\|^2 &= \|(1-\beta)x_n + \beta T_i x_n - T_i S_{i\beta}x_n\|^2 \\
&= (1-\beta)\|x_n - T_i S_{i\beta}x_n\|^2 \\
&\quad +\beta\|T_i x_n - T_i S_{i\beta}x_n\|^2 \\
&\quad -\beta(1-\beta)\|T_i x_n - x_n\|^2.
\end{aligned}$$

But

$$\begin{aligned}
\|T_i x_n - T_i S_{i\beta}x_n\|^2 &\leq L_i^2\|x_n - S_{i\beta}x_n\|^2 \\
&= L_i^2\|x_n - (1-\beta)x_n - \beta T_i x_n\|^2 \\
&= \beta^2 L_i^2\|x_n - T_i x_n\|^2 \\
&\leq \beta^2 L^2\|x_n - T_i x_n\|^2
\end{aligned}$$

So that

$$\begin{aligned}
\|S_{i\beta}x_n - T_i S_{i\beta}x_n\|^2 &\leq (1-\beta)\|x_n - T_i S_{i\beta}x_n\|^2 \\
&\quad +\beta^3 L^2\|x_n - T_i x_n\|^2 \\
&\quad -\beta(1-\beta)\|T_i x_n - x_n\|^2 \\
&= (\beta^3 L^2 + \beta^2 - \beta)\|x_n - T_i x_n\|^2 \\
&\quad +(1-\beta)\|x_n - T_i S_{i\beta}x_n\|^2.
\end{aligned}$$

Hence

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\le (1-\alpha)\|x_n - x^*\|^2 + \alpha[\|x_n - x^*\|^2 \\
&\quad +\beta^2\|x_n - T_i x_n\|^2 \\
&\quad +(\beta^3 L^2 + \beta^2 - \beta)\|x_n - T_i x_n\|^2 \\
&\quad +(1-\beta)\|x_n - T_i S_{i\beta} x_n\|^2] \\
&\quad -\alpha(1-\alpha)\|x_n - T_i S_{i\beta} x_n\|^2. \\
&= \|x_n - x^*\|^2 - \alpha\beta(1 - 2\beta - \beta^2 L^2)\|x_n - T_i x_n\|^2 \\
&\quad -\alpha(\beta - \alpha)\|x_n - T_i S_{i\beta} x_n\|^2 \\
&\le \|x_n - x^*\|^2 - \alpha\beta(1 - 2\beta - \beta^2 L^2)\|x_n - T_i x_n\|^2.
\end{aligned}$$

Thus,

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\le \|x_n - x^*\|^2 - c\alpha\beta\|x_n - T_i x_n\|^2 \\
&\le \|x_n - x^*\|^2,
\end{aligned}$$

where $c := 1 - 2\beta - \beta^2 L^2 > 0$. Thus, conditions (1)and (2) above hold.

Now $\forall i \in \{1, 2, ..., N\}$,

$$c\alpha\beta\|x_n - T_i x_n\|^2 \le \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2. \tag{8}$$

Hence, summing above equation from $n = 0$ and observing that the right hand side telescopes, we have

$$c\alpha\beta \sum_{n\ge 0} \|x_n - T_i x_n\|^2 \le \|x_0 - x^*\|^2 < \infty; \forall i \in \{1, ..., N\}.$$

Thus,

$$\sum_{n\ge 0} \|x_n - T_i x_n\|^2 < \infty; \forall i \in \{1, ..., N\}.$$

Which implies that

$$\lim_{n\to\infty} \|x_n - T_i x_n\| = 0; \forall i \in \{1, ..., N\}. \tag{9}$$

This completes the proof.

Remark 1: Suppose that $\{x_n\}$ has a convergent subsequence $\{x_{n_j}\}$. Let $x_{n_j} \to p$ as $j \to \infty$. Since $x_n - T_i x_n \to 0$ as $n \to \infty$ $\forall$ $i \in \{1, 2, ..., N\}$. it implies that $x_{n_j} - T_i x_{n_j} \to 0$ as $j \to \infty$ $\forall$ $i \in I$ and that $T_i x_{n_j} \to T_i p$ as $j \to \infty$ $\forall$ i by continuity of T_i. So, $\|p - T_i p\| = \lim_{n\to\infty} \|x_{n_j} - T_i x_{n_j}\| = 0, \forall i$ implying that $p \in F$. Thus, $\|x_{n+1} - p\| \le \|x_n - p\|$ and $\lim_{n\to\infty} \|x_n - p\|$ exists but $\lim_{n\to\infty} \|x_{n_j} - p\| = 0$, so $\lim_{n\to\infty} \|x_n - p\| = 0$. Hence $x_n \to x^*(= p)$ as $n \to \infty$.

Conditions under which $\{x_n\}$ has a convergent subsequence include

(1) T_i is completely continuous $\forall i \in \{1, ..., N\}$.
(2) T_i is demicompact $\forall i \in \{1, ..., N\}$.
(3) T_i is semicompact for some $i \in \{1, ..., N\}$.
(4) K is compact.
(5) K is boundedly compact.

A map $A : H \to H$ is said to be accretive or monotone if $\forall x, y \in H$ and $\forall \alpha > 0$

$$\|x - y + \alpha(Ax - Ay)\| \geq \|x - y\|. \tag{10}$$

If the above holds $\forall x \in H$ and $\forall y \in Z(A) := \{w \in H | Aw = 0\}$ (the zero set of A), then A is said to be quasi-accretive. It is easy to see that T is hemicontractive if and only if $A = I - T$ is quasi-accretive. Now, let

$$G_{i\beta} = I - \beta A_i;\ i \in \{1, ..., N\}. \tag{11}$$

$$A_{i\alpha\beta} = I - \alpha\beta A_i - \alpha A_i G_{i\beta}. \tag{12}$$

We have the following theorem as an easy corollary to Theorem 2.1.

Theorem 2: Let H be a real Hilbert space and let $A_i : H \to H; i \in \{1, 2, ..., N\}$ be a finite family of L_i-Lipschitzian quasi-accretive maps such that the simultaneous nonlinear equations $A_i x = 0;\ i \in \{1, 2, ..., N\}$ have a solution $x^* \in H$. Let $L := \max_{1 \leq i \leq N}\{L_i\}$, and choose $\alpha, \beta \in (0, 1)$ such that $0 < \alpha < \beta < \frac{1}{1+\sqrt{1+L^2}}$. Starting with an arbitrary $x_o \in H$ define the iterative sequence $\{x_n\}$ by

$$x_{n+1} = A_{i\alpha\beta} x_n;\quad n \geq 0;\ n + 1 \equiv i \mod N. \tag{13}$$

then

(1) $\{x_n\}$ is bounded.

item $\lim\limits_{n\to\infty} \|x_n - x^*\|$ exists $\forall x^* \in \bigcap\limits_{i=1}^{N} Z(A_i)$.

(2) $\lim\limits_{n\to\infty} \|A_i x_n\| = 0;\ \ \forall i \in \{1, 2, ..., N\}$.

Proof:
Let $T_i = I - A_i$. Then T_i is a Lipschitzian hemicontraction. Further,

$$\begin{aligned} S_{i\beta} &= (1 - \beta)I + \beta T_i = I - \beta(I - T_i) = I - \beta A_i = G_{i\beta}. \\ T_{i\alpha\beta} &= (1 - \alpha)I + \alpha T_i S_{i\beta} = I - \alpha(I - G_{i\beta}) - \alpha A_i G_{i\beta} \\ &= I - \alpha\beta A_i - \alpha A_i G_{i\beta} = A_{i\alpha\beta}. \end{aligned}$$

Thus, Theorem 2.1 applies and we have the stated results.

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