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AN APPLICATION OF EXTENDED DERIVATIVE ON PARTIAL SUMS OF CERTAIN ANALYTIC AND UNIVALENT FUNCTIONS

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ABSTRACT. In this paper the authors by means of a more generalized Salagean operator studied partial sums of certain class of analytic and univalent functions, lower bounds for these classes were determined.

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1. INTRODUCTION

Let A denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1}$$

which are analytic in the open unit disc $U = \{z : |z| < 1\}$. Furthermore, let S denote the class of all functions in A which are univalent in U. Let $S^*(\alpha)$ denote the subclass of A consisting of functions $f(z)$ which satisfy

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, (z \in U) \tag{2}$$

for some $\alpha(0 \leq \alpha < 1)$. A function $f(z)$ in $S^*(\alpha)$ is said to be starlike of order α in U. Also, let $K(\alpha)$ denote the subclass of A consisting of all functions $f(z)$ which satisfy

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha, (z \in U)$$

for some $\alpha(0 \leq \alpha < 1)$. A function $f(z)$ belonging to $K(\alpha)$ is said to be convex of order α in U. It is easy to see that $f(z) \in S^*(\alpha)$ if and only if $zf'(z) \in K(\alpha)$.

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Recently, Frasin [3] defined and studied the class $H_{\Psi}(c_k, \delta)$ consisting of functions $f(z)$ of the form (1) which satisfy the inequality,

$$\sum_{k=2}^{\infty} c_k |a_k| \leq \delta$$

where $\delta > 0$, and $\Psi(z)$ is analytic and univalent and a fixed function of the form

$$\Psi(z) = z + \sum_{k=2}^{\infty} c_k z^k, \quad (c_k \geq c_2 > 0, k \geq 2) . \tag{3}$$

In his investigation, he showed that for suitable choices of c_k and δ, the subclass of $H_{\Psi}(c_k, \delta)$ reduces to various known subclasses studied by various authors. Though after the presentation of his investigation in [3] some mispresentation of conditions were discussed and pointed out by some other recent authors who carried out further investigations in this direction,[see Powal and Dixit [7], Oladipo [8]].

To define the extended Salagean derivative operator which we shall employ in this work we recall from [9] that

$$\begin{aligned} \Omega^{\delta} f(z) &= \Gamma(2-\delta) z^{\delta} D^{\delta} f(z) \\ &= z + \sum_{k=2}^{\infty} \frac{\Gamma(k+1)\Gamma(2-\delta)}{\Gamma(k+1-\delta)} a_k z^k \end{aligned} \tag{4}$$

and also from [2] we recall that

$$I^n(\lambda, l) f(z) = I(\lambda, l)(I^{n-1}(\lambda, l) f(z))$$

$$= z + \sum_{k=2}^{\infty} \left(\frac{1 + \lambda(k-1) + l}{1+l} \right)^n a_k z^k, n \in N_0, \lambda \geq 0, l \geq 0. \tag{5}$$

Let $J_{\delta}^m(\lambda, l) : A \to A$ such that

$$J_{\delta}^m(\lambda, l) = z + \sum_{k=2}^{\infty} \left(\frac{\Gamma(k+1)\Gamma(2-\delta)}{\Gamma(k+1-\delta)} \right) \left(\frac{1 + \lambda(k-1) + l}{1+l} \right)^m a_k z^k \tag{6}$$

where

$$J_{\delta}^m(\lambda, l) = \Omega^{\delta} f(z) * I^n(\lambda, l) f(z) \tag{7}$$

and $\delta > 0$, $\lambda \geq 0$,$l \geq 0$, $m \in N_0$.

In this paper, we extend the results of [3],[4],[6],[7],[8] and some new results are embedded in our class.We now state and prove the following.

2. MAIN RESULTS

Theorem 2.1: If $f \in H_\Psi(c_k, \delta)$, then

$$(i) Re\left\{\frac{J_\delta^m(\lambda,l)f(z)}{J_\delta^m(\lambda,l)f_n(z)}\right\} \geq \frac{c_{n+1}-\phi_1\beta}{c_{n+1}}, \quad (z \in U) \tag{8}$$

and

$$(ii) Re\left\{\frac{J_\delta^m(\lambda,l)f_n(z)}{J_\delta^m(\lambda,l)f(z)}\right\} \geq \frac{c_{n+1}}{c_{n+1}+(\phi)_1\beta}, \quad (z \in U) \tag{9}$$

where

$$c_k \geq \begin{cases} \sigma\beta \quad if k = 2, 3... \\ \frac{\sigma c_{n+1}}{\phi_1} \quad if k = n+1, n+2, ... \end{cases}$$

and $\phi = \left(\frac{\Gamma(n+2)\Gamma(2-\delta)}{\Gamma(n+2-\delta)}\right)\left(\frac{1+\lambda n+l}{1+l}\right)^m$
while $\sigma = \left(\frac{\Gamma(k+1)\Gamma(2-\delta)}{\Gamma(k+1-\delta)}\right)\left(\frac{1+\lambda(k-1)+l}{1+l}\right)^m$.
The result (8) and (9) are sharp with the function given by

$$f(z) = z + \frac{\beta}{c_{n+1}} z^{n+1} \tag{10}$$

where $0 < \beta \leq \frac{c_{n+1}}{\phi_1}$.

Proof: To prove (i) part, we define the function $\omega(z)$ by

$$\frac{1+\omega(z)}{1-\omega(z)} = \frac{c_{n+1}}{\phi_1\beta}\left[\frac{J^m\delta(\lambda,l)f(z)}{J^m\delta(\lambda,l)f_n(z)} - \left(\frac{c_{n+1}-\phi_1\beta}{c_{n+1}}\right)\right]$$

$$\frac{1+\sum_{k=2}^{n}\sigma a_k z^{k-1} + \frac{c_{n+1}}{\phi_1\beta}\sum_{k=n+1}^{\infty}\sigma a_k z^{k-1}}{1+\sum_{k=2}^{n}\sigma a_k z^{k-1}} \tag{11}$$

It suffices to show that $|\omega(z)| \leq 1$. Now, from (9) we can write

$$\omega(z) = \frac{\frac{c_{n+1}}{\phi^m\beta}\sum_{k=n+1}^{\infty}\sigma a_k z^{k-1}}{2+2\sum_{k=2}^{n}\sigma a_k z^{k-1} + \frac{c_{n+1}}{\phi\beta}\sum_{k=n+1}^{\infty}\sigma a_k z^{k-1}}$$

Hence, we obtain

$$|\omega(z)| \leq \frac{\frac{c_{n+1}}{\phi_1\beta}\sum_{k=n+1}^{\infty}\sigma|a_k|}{2-2\sum_{k=2}^{n}\sigma|a_k| - \frac{c_{n+1}}{\phi_1\beta}\sum_{k=n+1}^{\infty}\sigma|a_k|}$$

Since $|\omega(z)| \leq 1$, we have

$$\frac{c_{n+1}}{\phi_1\beta}\sum_{k=n+1}^{\infty}\sigma|a_k| \leq 2-2\sum_{k=2}^{n}\sigma|a_k| - \frac{c_{n+1}}{\phi_1\beta}\sum_{k=n+1}^{\infty}\sigma|a_k|$$

If

$$2\frac{c_{n+1}}{\phi_1\beta}\sum_{k=n+1}^{\infty}\sigma|a_k| \leq 2-2\sum_{k=2}^{n}\sigma|a_k|$$

Or equivalently

$$\sum_{k=2}^{n}\sigma|a_k| + \frac{c_{n+1}}{\phi_1\beta}\sum_{k=n+1}^{\infty}\sigma|a_k| \leq 1 \tag{12}$$

It suffices to show that L.H.S is bounded above by $\sum_{k=2}^{n}\frac{c_k}{\beta}|a_k|$ which is equivalent to

$$\sum_{k=2}^{n}\left(\frac{c_k-\beta\sigma}{\beta}\right)|a_k| + \sum_{k=n+1}^{\infty}\left(\frac{\phi_1 c_k - c_k\sigma}{\phi_1\beta}\right)|a_k| \geq 0. \tag{13}$$

To see that the function given by (11) gives the sharp result. We observe that for $z = re^{\frac{i\pi}{n}}$.

$$\frac{J_\delta^m(\lambda,l)f(z)}{J_\delta^m(\lambda,l)f_n(z)} = 1+\frac{\beta}{c_{n+1}}\phi_1 z^n \to 1-\frac{\beta}{c_{n+1}}\phi_1 = \frac{c_{n+1}-\beta\phi_1}{c_{n+1}}$$

where $r\to 1^-$.
To prove the (ii) part of the Theorem,

$$\frac{1+\omega(z)}{1-\omega(z)} = \frac{c_{n+1}+\phi_1\beta}{\phi_1\beta}\left[\frac{J_\delta^m(\lambda,l)f(z)}{J_\delta^m(\lambda,l)f_n(z)} - \left(\frac{c_{n+1}}{c_{n+1}+\phi_1\beta}\right)\right]$$

$$\frac{1+\sum_{k=2}^{n}\sigma a_k z^{k-1} - \frac{c_{n+1}}{\phi_1\beta}\sum_{k=n+1}^{\infty}\sigma a_k z^{k-1}}{1+\sum_{k=2}^{n}\sigma a_k z^{k-1}}$$

where

$$|\omega(z)| \leq \frac{\left(\frac{c_{n+1}+\phi_1\beta}{\phi\beta}\right)\sum_{k=n+1}^{\infty}\sigma|a_k|}{2-2\sum_{k=2}^{n}\sigma|a_k| - \frac{c_{n+1}-\phi_1\beta}{\phi_1\beta}\sum_{k=n+1}^{\infty}\sigma|a_k|} \leq 1.$$

This last inequality is equivalent to

$$\sum_{k=2}^{n}\sigma|a_k| + \frac{c_{n+1}}{\phi_1\beta}\sum_{k=n+1}^{\infty}\sigma|a_k| \leq 1.$$

Finally, equality holds in (10) for the function $f(z)$ given by (11).
This completes the proof of Theorem 2.1.
The following corollaries are obtained by varying various choices of parameters involved.
Taking $l=0$ in Theorem 2.1, we have

Corollary A: If $f \in H_\Psi(c_k, \delta,)$, then

$$(i) Re\left\{\frac{J_\delta^m(\lambda)f(z)}{J_\delta^m(\lambda)f_n(z)}\right\} \geq \frac{c_{n+1} - \phi_2\beta}{c_{n+1}}, \quad (z \in U) \tag{14}$$

and

$$(ii) Re\left\{\frac{J_\delta^m(\lambda)f_n(z)}{J_\delta^m(\lambda)f(z)}\right\} \geq \frac{c_{n+1}}{c_{n+1} + \phi_2\beta}, \quad (z \in U) \tag{15}$$

where

$$c_k \geq \begin{cases} \sigma_1\beta & if k = 2, 3... \\ \frac{\sigma_1 c_{n+1}}{\phi_2} & if k = n+1, n+2, ... \end{cases}$$

and $\phi_2 = \left(\frac{\Gamma(n+2)\Gamma(2-\delta)}{\Gamma(n+2-\delta)}\right)(1+\lambda n)^m$
while $\sigma_1 = \left(\frac{\Gamma(k+1)\Gamma(2-\delta)}{\Gamma(k+1-\delta)}\right)(1+\lambda(k-1))^m$
which serves as a new generalization in this direction and also the class of functions studied by Al-oboudi [1].
Setting $\lambda = 1, l = 0$ in Theorem 2.1, we obtain

Corollary B: If $f \in H_\Psi(c_k, \delta)$, then

$$(i) Re\left\{\frac{J_\delta^m f(z)}{J_\delta^m f_n(z)}\right\} \geq \frac{c_{n+1} - \phi_3\beta}{c_{n+1}}, \quad (z \in U) \tag{16}$$

and

$$(ii) Re\left\{\frac{J^m \delta f_n(z)}{J_\delta^m f(z)}\right\} \geq \frac{c_{n+1}}{c_{n+1} + \phi_3\beta}, \quad (z \in U) \tag{17}$$

where

$$c_k \geq \begin{cases} \sigma_2\beta & if k = 2, 3... \\ \frac{\sigma_2 c_{n+1}}{\phi_3} & if k = n+1, n+2, ... \end{cases}$$

and $\phi_3 = \left(\frac{\Gamma(n+2)\Gamma(2-\delta)}{\Gamma(n+2-\delta)}\right)(1+n)^m$, $\sigma_2 = \left(\frac{\Gamma(k+1)\Gamma(2-\delta)}{\Gamma(k+1-\delta)}\right)(k)^m$
which is the class of functions studied by Salagean [5].
Putting $m = 0, \delta = 0, \lambda = 1, l = 0$ in Theorem 2.1, we obtain

Corollary C: If $f \in H_\Psi(c_k, \delta)$, then

$$(i) Re\left\{\frac{f(z)}{f_n(z)}\right\} \geq \frac{c_{n+1} - \Gamma(2)\beta}{c_{n+1}}, \quad (z \in U) \tag{18}$$

and

$$(ii) Re\left\{\frac{f_n(z)}{f(z)}\right\} \geq \frac{c_{n+1}}{c_{n+1} + \Gamma(2)\beta}, \quad (z \in U) \tag{19}$$

where

$$c_k \geq \begin{cases} \Gamma(2)\beta & if k = 2, 3... \\ c_{n+1} & if k = n+1, n+2, ... \end{cases}$$

and $\phi_4 = \Gamma(2)$, $\sigma_3 = \Gamma(2)$ Putting $m = 1, \delta = 1, \lambda = 1, l = 0$ in Theorem 2.1, we obtain

Corollary D: If $f \in H_\Psi(c_k, \delta)$, then

$$(i)Re\left\{\frac{f'(z)}{f'_n(z)}\right\} \geq \frac{c_{n+1} - \Phi_5\beta}{c_{n+1}}, \quad (z \in U) \tag{20}$$

and

$$(ii)Re\left\{\frac{f'_n(z)}{f'(z)}\right\} \geq \frac{c_{n+1}}{c_{n+1} + \Phi_5\beta}, \quad (z \in U) \tag{21}$$

where

$$c_k \geq \begin{cases} \sigma_4\beta \quad if \quad k = 2, 3... \\ \frac{\sigma_4 c_{n+1}}{\Phi_5} \quad if \quad k = n+1, n+2, ... \end{cases}$$

and $\phi_5 = \frac{\Gamma(n+2)\Gamma(1)}{\Gamma(n+1)}(n+1)$, $\sigma_4 = \frac{\Gamma(k+1)\Gamma(1)}{\Gamma(k)}k$

Remark: By still varying some various choices of parameter involved many result will be generated.

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