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SUBORDINATION PROPERTIES ASSOCIATED WITH A CERTAIN SUBCLASS OF ANALYTIC FUNCTIONS DEFINED BY SALAGEAN DERIVATIVE

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ABSTRACT. In this paper, we discuss the subordination properties for functions of class $T_n^{\alpha^*}(\beta)$. A number of applications of the subordination properties are also considered.

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1. INTRODUCTION

Let A be the class of functions $g(z)$ analytic in the unit disk $U = \{z : |z| < 1\}$ and normalized by

$$g(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{1}$$

Note that,

$$g(z)^\alpha = z^\alpha + \sum_{k=2}^{\infty} a_k(\alpha) z^{\alpha+k-1}$$

by Binomial expansion.
The class $T_n^\alpha(\beta)$ was introduced and studied by Opoola [1] in 1994 as the class of functions $g(z)$ satisfying the condition

$$Re\left\{\frac{D^n g(z)^\alpha}{z^\alpha}\right\} > \beta.$$

Babalola and Opoola[2] in 2006, redefined the class $T_n^\alpha(\beta)$ as the class of analytic functions $g(z) \in A$ which satisfy

$$Re\left\{\frac{D^n g(z)^\alpha}{\alpha^n z^\alpha}\right\} > \beta. \tag{2}$$

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where $\alpha > 0$ is real, $0 \leq \beta < 1$ and $D^n (n \in N)$ is the Salagean derivative operator defined as:

$$D^n g(z) = D(D^{n-1} g(z)) = z(D^{n-1} g(z))'$$

with $D^0 g(z) = g(z)$ and the power in (2) meaning principal determination only.
We denote by $k(\alpha)$ the class of convex functions of order α i.e.

$$k(\alpha) = \left\{ f \in A : Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, \ z \in U \right\}$$

2. PRELIMINARY

We define some basic results which are relevant to our main result and also give certain fundamental definitions.

Definition 1: (Hadamard product or convolution)
Given two functions $f(z)$ and $g(z)$ where $g(z)$ is as defined in (1) and $f(z)$ is given by

$$f(z) = z + \sum_{k=2}^{\infty} b_k z^k,$$

the Hadamard product (or convolution) $g * f$ of $g(z)$ and $f(z)$ is defined by

$$(g * f)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (f * g)(z) \tag{3}$$

Definition 2: (Subordination Principle.)
Let $f(z)$ and $g(z)$ be analytic in the unit disk U. Then $g(z)$ is said to be subordinate to $f(z)$ in U denoted by

$$g(z) \prec f(z), \ z \in U,$$

if there exists a Schwarz function $w(z)$, analytic in U with $w(0) = 0$, $|w(z)| < 1$ such that

$$g(z) = f(w(z)), \ z \in U \tag{4}$$

In particular, if the function $f(z)$ is univalent in U, then $g(z)$ is subordinate to $f(z)$ if

$$g(0) = f(0), \ g(u) \subset f(u). \tag{5}$$

Definition 3: (Subordinating factor sequence)
A sequence $\{c_k\}_{k=1}^{\infty}$ of complex numbers is said to be a subordinating factor sequence if whenever $g(z)$ of the form (1) is analytic, univalent and convex in U, the subordination is given by

$$\sum_{k=1}^{\infty} a_k c_k z^k \prec g(z),\ z \in U,\ a_1 = 1.$$

Theorem 1: [3]
The sequence $\{c_k\}_{k=1}^{\infty}$ is a subordinating factor sequence if and only if

$$Re\left\{1 + 2\sum_{k=1}^{\infty} c_k z^k\right\} > 0,\ z \in U. \tag{6}$$

Theorem 2: [4]
If $g(z) \in A$ satisfies

$$\sum_{k=2}^{\infty} \alpha_0^n |a_k(\alpha)| \leq 1 - \beta \tag{7}$$

where $\alpha_0 = \frac{\alpha+k-1}{\alpha}, k = 2, 3, ...; 0 \leq \beta < 1,$ and $n \in \mathbb{N}_0$),then $g(z) \in T_n^{\alpha}(\beta)$
It is natural to consider the class $T_n^{\alpha^*}(\beta) \subset T_n^{\alpha}(\beta)$ such that

$$\begin{aligned} T_n^{\alpha^*}(\beta) = \Bigg\{ g \in A : \sum_{k=2}^{\infty} \alpha_0^n |a_k| \\ \leq 1 - \beta \Bigg\}. \end{aligned} \tag{8}$$

3. MAIN RESULT

Our main result in this paper is the the subordination result associated with the class $T_n^{\alpha^*}(\beta)$. Some applications of the main result which give important results of analytic functions are also investigated.

Theorem 3: (A subordination result associated with the class $T_n^{\alpha^*}(\beta)$) Let $g(z) \in T_n^{\alpha^*}(\beta)$, then

$$\frac{(\alpha+1)^n}{2[\alpha^n(1-\beta) + (\alpha+1)^n]}(g * f)(z) \prec f(z) \tag{9}$$

where $\alpha > 0, \beta(0 \leq \beta < 1)$and $f(z) \in k(\alpha),\ z \in U$. And

$$Re(g(z)) > -\frac{\alpha^n(1-\beta)+(\alpha+1)^n}{(\alpha+1)^n}. \tag{10}$$

The constant factor

$$\frac{(\alpha+1)^n}{2[\alpha^n(1-\beta)+(\alpha+1)^n]} \tag{11}$$

cannot be replaced by a larger one. Next we proof the main theorem:

Proof of Main Result:
Let $g(z) \in T_n^{\alpha^*}(\beta)$, and suppose that

$$f(z) = z + \sum_{k=2}^{\infty} b_n z^k \in k(\alpha).$$

Then

$$\begin{aligned} &\frac{(\alpha+1)^n}{2[\alpha^n(1-\beta)+(\alpha+1)^n]}(g * f)(z) = \\ &\frac{(\alpha+1)^n}{2[\alpha^n(1-\beta)+(\alpha+1)^n]}\left(z + \sum_{n=2}^{\infty} a_k b_k z^k\right) \end{aligned} \tag{12}$$

Hence, by Definition 3 the subordination (9) Will hold true if,

$$\left\{\frac{(\alpha+1)^n}{2[\alpha^n(1-\beta)+(\alpha+1)^n]}a_k\right\}_{k=1}^{\infty} \tag{13}$$

is a subordinating factor sequence with $a_1 = 1$.
Therefore by theorem 1 it is sufficient to show that

$$Re\left\{1 + 2\sum_{k=1}^{\infty}\frac{(\alpha+1)^n}{2[\alpha^n(1-\beta)+(\alpha+1)^n]}a_k z^k\right\} > 0; (z \in U) \tag{14}$$

Now,

$$\begin{aligned} &Re\{1 + 2\sum_{k=1}^{\infty}\frac{(\alpha+1)^n}{2[\alpha^n(1-\beta)+(\alpha+1)^n]}a_k z^k\} \\ &= Re\{1 + \frac{(\alpha+1)^n}{\alpha^n(1-\beta)+(\alpha+1)^n}a_1 z + \frac{1}{\alpha^n(1-\beta)+(\alpha+1)^n\}} \\ &\times \sum_{k=2}^{\infty}(\alpha+1)^n |a_k| z^k\} \end{aligned} \tag{15}$$

But $\alpha_0 = \frac{\alpha+k-1}{\alpha}$; $k = 2, 3, ...$ such that $\alpha_0^n = (\frac{\alpha+1}{\alpha})^n$ for $k = 2$ and $(\alpha+1)^n = \alpha^n \alpha_0^n$
Thus,

$$\begin{aligned} &Re\{1 + 2\sum_{k=1}^{\infty} \frac{(\alpha+1)^n}{2[\alpha^n(1-\beta) + (\alpha+1)^n]} a_k z^k\} \\ &\geq \{1 - \frac{(\alpha+1)^n}{\alpha^n(1-\beta) + (\alpha+1)^n} r\} \\ &- \frac{1}{\alpha^n(1-\beta) + (\alpha+1)^n} \sum_{k=2}^{\infty} \alpha^n \alpha_0^n |a_k| z^k \end{aligned} \tag{16}$$

Since $\alpha_0^n > 0$,we have that

$$\begin{aligned} &Re\{1 + \sum_{k=1}^{\infty} \frac{(\alpha+1)^n}{\alpha^n(1-\beta) + (\alpha+1)^n} a_k z^k\} \\ &\geq 1 - \frac{(\alpha+1)^n}{\alpha^n(1-\beta) + (\alpha+1)^n} r - \frac{\alpha^n}{\alpha^n(1-\beta) + (\alpha+1)^n} \\ &\times \sum_{k=2}^{\infty} \alpha_0^n |a_k| z^k, \quad 0 < r < 1. \end{aligned} \tag{17}$$

By (7)

$$\sum_{k=2}^{\infty} \alpha_0^n |a_k| \leq 1 - \beta$$

Hence,

$$\begin{aligned} &Re\left\{1 + \sum_{k=1}^{\infty} \frac{(\alpha+1)^n}{\alpha^n(1-\beta) + (\alpha+1)^n} a_k z^k\right\} \\ &> 1 - \frac{(\alpha+1)^n}{\alpha^n(1-\beta) + (\alpha+1)^n} r - \frac{\alpha^n(1-\beta)}{\alpha^n(1-\beta) + (\alpha+1)^n} r \\ &= 1 - \{\frac{(\alpha+1)^n}{\alpha^n(1-\beta) + (\alpha+1)^n} + \frac{\alpha^n(1-\beta)}{\alpha^n(1-\beta) + (\alpha+1)^n}\} r \\ &= 1 - \frac{(\alpha+1)^n + \alpha^n(1-\beta)}{\alpha^n(1-\beta) + (\alpha+1)^n} r = 1 - r > 0; \ (|z| = r < 1). \end{aligned} \tag{18}$$

Therefore we obtain that,

$$Re\left\{1 + 2\sum_{k=1}^{\infty} \frac{(\alpha+1)^n}{2[\alpha^n(1-\beta) + (\alpha+1)^n]} a_k z^k\right\} > 0$$

which is (14) that is required to be established.
We now show that

$$Re(g(z)) > -\frac{\alpha^n(1-\beta)+(\alpha+1)^n}{(\alpha+1)^n}.$$

Now taking

$$f(z) = \frac{z}{1-z} \in K(\alpha),$$

(9) becomes

$$\frac{(\alpha+1)^n}{2[\alpha^n(1-\beta)+(\alpha+1)^n]}g(z) \prec \frac{z}{1-z}$$

Therefore,

$$Re\{\frac{(\alpha+1)^n}{2[\alpha^n(1-\beta)+(\alpha+1)^n]}\}g(z) > -\frac{1}{2} \tag{19}$$

Since

$$Re(\frac{z}{1-z}) > -\frac{1}{2}, \quad |z| < r \tag{20}$$

which implies that

$$\frac{(\alpha+1)^n}{2[\alpha^n(1-\beta)+(\alpha+1)^n]}Re(g(z)) > -\frac{1}{2} \tag{21}$$

Hence, we have

$$Re(g(z)) > -\frac{\alpha^n(1-\beta)+(\alpha+1)^n}{(\alpha+1)^n}.$$

which is (10).
To show the sharpness of the constant factor

$$\frac{(\alpha+1)^n}{2[\alpha^n(1-\beta)+(\alpha+1)^n]},$$

we consider the function:

$$f_1(z) = z - \frac{\alpha^n(1-\beta)}{(\alpha+1)^n}z^2 = \frac{z(\alpha+1)^n - \alpha^n(1-\beta)z^2}{(\alpha+1)^n}. \tag{22}$$

Then applying Theorem 3 when $f(z) = \dfrac{z}{1-z}$ and $g(z) = g_1(z)$ we have

$$\frac{z(\alpha+1)^n - \alpha^n(1-\beta)z^2}{2[\alpha^n(1-\beta)+(\alpha+1)^n]} \prec \frac{z}{1-z} \tag{23}$$

Using the fact that

$$|Re\ z| \leq |z| \tag{24}$$

we now show that

$$\mathrm{Min}\{Re\frac{z(\alpha+1)^n-\alpha^n(1-\beta)z^2}{2[\alpha^n(1-\beta)+(\alpha+1)^n]}\}=-\frac{1}{2},\quad (z\in U). \tag{25}$$

Now,

$$\begin{aligned}
&\left|Re\frac{z(\alpha+1)^n-\alpha^n(1-\beta)z^2}{2[\alpha^n(1-\beta)+(\alpha+1)^n]}\right|\le\left|\frac{z(\alpha+1)^n-\alpha^n(1-\beta)z^2}{2[\alpha^n(1-\beta)+(\alpha+1)^n]}\right|\\
&=\frac{|z[(\alpha+1)^n-\alpha^n(1-\beta)z]|}{|2[\alpha^n(1-\beta)+(\alpha+1)^n]|}\le\frac{|z||(\alpha+1)^n-\alpha^n(1-\beta)z|}{2[\alpha^n(1-\beta)+(\alpha+1)^n]}\\
&\le\frac{|(\alpha+1)^n-\alpha^n(1-\beta)z|}{2[\alpha^n(1-\beta)+(\alpha+1)^n]}\le\frac{|(\alpha+1)^n+\alpha^n(1-\beta)z|}{2[\alpha^n(1-\beta)+(\alpha+1)^n]}\\
&\le\frac{(\alpha+1)^n+\alpha^n(1-\beta)}{2[\alpha^n(1-\beta)+(\alpha+1)^n]}=\frac{1}{2},\ (|z|=1).
\end{aligned} \tag{26}$$

This implies that,

$$\left|\{Re\frac{z(\alpha+1)^n-\alpha^n(1-\beta)z^2}{2[\alpha^n(1-\beta)+(\alpha+1)^n]}\}\right|\le\frac{1}{2}$$

i.e,

$$-\frac{1}{2}\le Re\frac{z(\alpha+1)^n-\alpha^n(1-\beta)z^2}{2[\alpha^n(1-\beta)+(\alpha+1)^n]}\le\frac{1}{2}. \tag{27}$$

Hence,

$$\mathrm{Min}\{Re\frac{z(\alpha+1)^n-\alpha^n(1-\beta)z^2}{2[\alpha^n(1-\beta)+(\alpha+1)^n]}\}f_1(z)\}=-\frac{1}{2}\ (z\in U).$$

Which complete the proof of Theorem (3)

4. SOME APPLICATIONS

Taking $n=0$,in Theorem 1.3, we obtain the following:

Corollary 1. *If the function $g(z)$ defined by (1.1) is in $T_n^{\alpha^*}(\beta)$, then*

$$\frac{1}{2(2-\beta)}(g*f)(z)\prec f(z) \tag{28}$$

$$(z\in U;0\le\beta<1,f\in K(\alpha)).$$

In particular,

$$Re(g(z))>-(2-\beta)\ (z\in U) \tag{29}$$

The constant factor

$$\frac{1}{2(2-\beta)}$$

cannot be replaced by any larger one.

Remark 1. *If $\beta = \frac{5}{6}$ in the corollary* 1*, we obtain the results of Aouf etal* [4].

Taking $n = 1$ and $\beta = 0$ in Theorem 1.3; we obtain the following:

Corollary 2. *If the function $g(z)$ defined by* (1.1) *is in $T_n^{\alpha^*}(\beta)$, then*

$$\frac{\alpha+1}{2(2\alpha+1)}(g*f)(z) \prec f(z) \tag{30}$$

$$(z \in U; \alpha > 0, f \in K(\alpha)).$$

In particular,

$$Re(g(z)) > -\frac{2\alpha+1}{\alpha+1} \tag{31}$$

The constant factor

$$\frac{\alpha+1}{2(2\alpha+1)}$$

cannot be replaced by any larger one.

Remark 2. *By putting $\alpha = 1$ and $\alpha = \frac{1}{3}$ in corollary* 2*, we obtain the results of Selvaraj and Karthikeyan* [5].

Taking $n = 0$, and $\beta = 0$, in Theorem 1.3 we obtain the following:

Corollary 3. *If the function $g(z)$ defined by* (1.1) *is in $T_n^{\alpha^*}(\beta)$, then*

$$\frac{1}{4}(g*f)(z) \prec f(z) \tag{32}$$

$$(z \in U, g \in K(\alpha)).$$

In particular,

$$Re(g(z)) > -2 \ (z \in U) \tag{33}$$

The constant factor $\frac{1}{4}$ cannot be replaced by any larger one.

Remark 3. *Our result in corollary* 3*, is equivalent to the result of Rosihan, Ali, etal*[6].

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