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APPROXIMATE ANALYTICAL SOLUTIONS FOR PIPE FLOW OF A THIRD GRADE FLUID WITH VARIABLE MODELS OF VISCOSITIES AND HEAT GENERATION/ ABSORPTION

O. J. JAYEOBA[1] AND S. S. OKOYA

ABSTRACT. We study a one-dimensional heat generation and viscous dissipation model of a third grade fluid in a cylinder involving viscosities Reynold's and a Vogel's. The analysis is based on regular perturbation technique. Approximate analytical expressions are constructed for the dimensionless velocity and temperature fields. The heat transfer model is also numerically simulated. The present numerical solutions agree very well with the previous finite difference scheme for special cases. The new analytical solutions are compared with numerical integration and with relative error of 5 % for all physical parameters, an excellent agreement is observed. In particular, in the absence of heat generation/absorption case $\delta = 0$, we recover earlier known analytical results. Then, if $\delta \neq 0$, the effects of several dimensionless parameters on the heat transfer characteristics are reported graphically to elucidate special features of the solutions.

Keywords and phrases: Third grade fluid, Heat generation, Temperature dependent viscosity models, pipe flow, perturbation method
2010 Mathematical Subject Classification: 76A05, 80A20, 80A35

1. INTRODUCTION

Several aspects of steady laminar Non-Newtonian fluids have been studied extensively during the pass few decades. The interest in such problems stems from their important applications, such as the scenario of preheating coal-water mixture, polymer melts and many emulsions. Due to the complexities of such fluids, there are many mathematical models governing non-Newtonian fluids. However, the fluids of differential type have received much attention (see [5] for a review). For problems involving heat transfer, one of the

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[1]Corresponding author

most interesting fluids of differential type that has received some attention in the last few years is the third-grade fluid.

Heat transfer problem of third-grade fluids without heat source term has been studied by several authors; Fosdick and Rajagopal [9] performed a complete thermodynamic analysis of the constitutive equations for the third grade fluid involving heat transfer process. The flow and heat transfer analysis of a thermodynamically compatible third grade fluid has received some attention in recent times. Massoudi and Christe [14] studied numerically the flow of a third grade fluid in a pipe without heat source where the shear viscosity was assumed to be temperature dependent. In [15], the authors extended the analysis by studying the flow of such a fluid in a pipe when the temperature of the pipe is assumed to be higher than the temperature of the fluid. Approximate solutions using regular perturbation series method for the same model with temperature dependent viscosities have been established by Yurusoy and Pakdemirli [29], Yurusoy et al. [30], Nadeem et al. [18] and Jaiyeoba [11]. Furthermore, related studies are also given by Ellahi and Afzal [6], Ellahi et al. [7], Erdogan and Imrak [8], Hayat et al. [10] Nadeem and Ali [17], Nadeem et al. [19] and Nadeem et al. [20].

Analytical and numerical studies of linear heat source/sink term in flow system have been for the most part confined to plane two-dimensional flows, see for examples Aziz and Na [2], Chinyoka [3], Liu [12], Molla [16] Olajuwon [22], Reddy and Narasimhan [24], Siddheshwar and Mahabaleswar [26], Sivasankaran et al. [27], and the references contained there-in. Some one-dimensional problems have quite recently been considered. Arslanturk [1] employed Adomian decomposition method to evaluate the efficiency of straight fins temperature-dependent thermal conductivity and to also determine the temperature distribution within the fin, Costa and Sandberg [4] developed a mathematical model to estimate burn rate, temperature profiles and positions of a natural smoldering log while Makinde [13] employed a novel hybrid numerical-analytical scheme based on a special type of Hermite-Pade approximants to examined the flow of a variable viscosity optically thin fluid through a channel with isothermal walls.

Motivated by the industrial applications of Non-Newtonian fluids in preheated coal-water mixture [28] and previous approximate analytical studies, this paper aims to find analytic approximations to the coupled equations of dimensionless velocity and temperature for the steady flow of a third grade fluid in cylindrical coordinate. We use the artificial small parameters methods [29] and present results for Reynold's and Vogel's model viscosities. A heat source is included in the problem as an extension of [15] and [29]. Comparison between the set of analytical and numerical solutions is given and discussed while the competing effects of the emerging parameters are discussed with the help of graphical illustrations.

2. PHYSICAL MODEL AND DERIVATION

An infinitely long cylinder is considered with the steady incompressible flow of a third grade fluid. The physical configuration considered is as shown in Fig. 1.

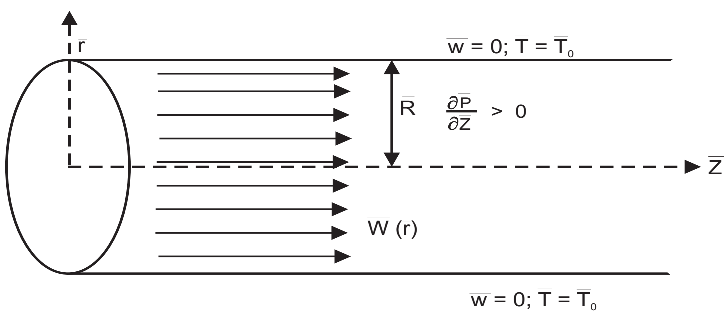

Fig. 1. Physical model and coordinate system.

The equations for the velocity and the temperature, given by Massoudi and Christe [15] as well as Yurusoy and Pakdemirli [29], may be extended to incorporate a source term (see, [22]) given by

$$\frac{1}{\bar{r}}\frac{d}{d\bar{r}}\left(\bar{r}(2\alpha_1+\alpha_2)\left[\frac{d\bar{w}}{d\bar{r}}\right]^2\right)=\frac{\partial\bar{p}}{\partial\bar{r}}, \tag{1}$$

$$0=\frac{\partial\bar{p}}{\partial\phi}, \tag{2}$$

$$\frac{1}{\bar{r}}\frac{d}{d\bar{r}}\left(\bar{r}\bar{\mu}\frac{d\bar{w}}{d\bar{r}}\right)+\frac{1}{\bar{r}}\frac{d}{d\bar{r}}\left(2\bar{r}\beta_3\left[\frac{d\bar{w}}{d\bar{r}}\right]^3\right)=\frac{\partial\bar{p}}{\partial\bar{z}}, \tag{3}$$

$$K\left(\frac{1}{\bar{r}}\frac{d}{d\bar{r}}\left(\bar{r}\frac{d\bar{T}}{d\bar{r}}\right)\right)+\bar{\mu}\left(\frac{d\bar{w}}{d\bar{r}}\right)^2+2\beta_3\left(\frac{d\bar{w}}{d\bar{r}}\right)^4 +\bar{Q}\bar{C}_0(\bar{T}-\bar{T}_0)=0, \tag{4}$$

where all symbols are defined in the Nomenclature. The source term $\bar{Q}$ represents the heat generation when $\bar{Q}>0$ and the heat absorption term when $\bar{Q} < 0$. Here equation (3) is to be integrated for a given $\partial\bar{p}/\partial\bar{z}$ and once the flow field is determined, the actual pressure field can be obtained from equations (1) and (3). The appropriate boundary conditions to solve equations (3) and (4) are

$$\bar{w}(\bar{R}) = \bar{T}(\bar{R}) = 0, \ \frac{d\bar{w}}{d\bar{r}}(0) = \frac{d\bar{T}}{d\bar{r}}(0) = 0. \tag{5}$$

The corresponding dimensionless equations for equations (3)-(5) leads to the following:

$$\frac{d\mu}{dr}\frac{dw}{dr} + \frac{\mu}{r}\left(\frac{dw}{dr} + r\frac{d^2w}{dr^2}\right) + \frac{\Lambda}{r}\left(\frac{dw}{dr}\right)^2\left(\frac{dw}{dr} + 3r\frac{d^2w}{dr^2}\right) = C, \tag{6}$$

$$\frac{d^2\theta}{dr^2} + \frac{1}{r}\frac{d\theta}{dr} + \Gamma\left(\frac{dw}{dr}\right)^2\left(\mu + \Lambda\left(\frac{dw}{dr}\right)^2\right) + \delta\theta = 0, \tag{7}$$

$$w(1) = \theta(1) = 0, \text{ and } \frac{dw}{dr}(0) = \frac{d\theta}{dr}(0) = 0. \tag{8}$$

The viscosity μ is assumed to be a function of temperature and the form of the equations (6) and (7) depend on the viscosity model. Here we present the results for two different viscosity models: Reynold's and Vogel's models (see e.g., Massoudi and Christe [15], Nadeem and Ali [17], Okoya [21]) and Pardemirli and Yilbas [23]:

$$\bar{\mu}(\bar{T}) = \begin{cases} \bar{\mu}_0\exp(-\bar{M}(\bar{T} - \bar{T}_0)), & \text{Reynold's model case,} \\ \bar{\mu}_0\exp(a/(b + \bar{T}_0)), & \text{Vogel's model case.} \end{cases} \tag{9}$$

It is well known that Reynold's viscosity decreases/increases with increasing temperature for liquids/gas whenever M is positive/ negative. When M is large, then the effect of variable viscosity can be neglected. The corresponding non-dimensional form of the above equation is

$$\mu = \begin{cases} \exp(-\gamma\theta), & \text{Reynold's model case,} \\ \exp(A/(B + \theta) - \bar{T}_0), & \text{Vogel's model case.} \end{cases} \tag{10}$$

The coupled nonlinear ordinary differential equations (6) and (7), with the boundary conditions (8), can be solved in principle by several methods, the perturbation technique being a systematic, efficient and powerful tool. Here, as well in [29], [30], we shall use the regular perturbation technique to determine flow field and thermal distribution.

3. ANALYTICAL SOLUTIONS

In this section the regular perturbation series solutions will be determined for the dimensionless velocity and temperature by using Reynold's and Vogel's models of viscosity.

3.1 *Case I*: For Reynolds' model viscosity

Taking the Maclaurin's series expansion of the exponential term, we can express that

$$\mu = \exp(-\gamma\theta) = 1 - \gamma\theta + O(\gamma^2). \tag{11}$$

Here, we shall select

$$\Lambda = \epsilon M,\ \ \delta = \epsilon N,\ \ \gamma = \epsilon P, \tag{12}$$

where ϵ is the artificial small parameter. Series solutions of equations (6)-(8) and conditions (10) may be obtained by using perturbation method treating ϵ as a perturbation parameter. Solutions are then sought in the form

$$w = w_0 + \epsilon w_1 + O(\epsilon^2), \tag{13}$$

$$\theta = \theta_0 + \epsilon\theta_1 + O(\epsilon^2). \tag{14}$$

Substituting the solutions series in equation (13) and (14) into equations (6)-(8) and (10a) and collecting the coefficients of like powers of ϵ, we obtained the following governing equations

Leading terms:

$$O(\epsilon^0):\ r\frac{d^2w_0}{dr^2} + \frac{dw_0}{dr} = Cr, \tag{15}$$

$$O(\epsilon^0):\ r\frac{d^2\theta_0}{dr^2} + \frac{d\theta_0}{dr} + r\Gamma\left(\frac{dw_0}{dr}\right)^2 = 0, \tag{16}$$

$$w_0(1) = \theta_0(1) = \frac{dw_0}{dr}(0) = \frac{d\theta_0}{dr}(0) = 0. \tag{17}$$

Correction terms:

$$\begin{aligned} O(\epsilon):\ & r\frac{d^2w_1}{dr^2} - P\frac{\theta_0}{dr}\frac{dw_0}{dr} - rP\theta_0\frac{d^2w_0}{dr^2} - rP\frac{d\theta_0}{dr}\frac{dw_0}{dr} = \\ & -M\left(\frac{dw_0}{dr}\right)^2\left[\frac{dw_0}{dr} + 3r\frac{d^2w_0}{dr^2}\right], \end{aligned} \tag{18}$$

$$\begin{aligned} O(\epsilon):\ & r\frac{d^2\theta_1}{dr^2} + \frac{d\theta_1}{dr} + r\Gamma\left(\frac{dw_0}{dr}\right)^2\left(M\left(\frac{dw_0}{dr}\right)^2 - P\theta_0\right) \\ & +2r\Gamma\frac{dw_0}{dr}\frac{dw_1}{dr} + rN\theta_0 = 0, \end{aligned} \tag{19}$$

$$w_1(1) = \theta_1(1) = \frac{dw_1}{dr}(0) = \frac{d\theta_1}{dr}(0) = 0. \tag{20}$$

It is straightforward to show that the solutions of equations (15)-(17) are given as

$$w_0 = -\frac{C}{4}(1 - r^2), \tag{21}$$

$$\theta_0 = \frac{\Gamma C^2}{64}(1 - r^4). \tag{22}$$

Substitute these solutions into equations (18) and (19) and integrate the resulting equations twice subject to equation (20) one obtains

$$w_1 = \frac{MC^3}{32}\left(1 - r^4\right) - \frac{P\Gamma C^3}{768}\left(2 - 3r^2 + r^6\right), \tag{23}$$

$$\theta_1 = -\frac{M\Gamma C^4}{576}(1 - r^6) + \frac{N\Gamma C^2}{2304}(8 + 9r^2 - r^6) + \frac{P\Gamma^2 C^4}{16384}\left(3 - 4r^4 + r^8\right). \tag{24}$$

Substituting solutions at each order to the expansions and changing back to the original parameters, the equations finally give

$$w = \frac{-C}{4}(1 - r^2) + \frac{\Lambda}{32}C^3(1 - r^4) - \frac{\gamma\Gamma C^3}{768}(2 - 3r^2 + r^6), \tag{25}$$

$$\theta = \frac{\Gamma C^2}{64}(1 - r^4) - \frac{\Gamma\Lambda C^4}{576}(1 - r^6) + \frac{\gamma\Gamma^2 C^4}{16384}(3 - 4r^4 + r^8) + \frac{\delta\Gamma C^2}{2304}\left(8 + 9r^2 - r^6\right). \tag{26}$$

3.2 *Case II*: For Vogel's model viscosity

With the use of Maclaurin's theorem, we can write

$$\mu = \exp\left(\frac{A}{B + \theta} - T_0\right) = \exp\left(\frac{A}{B} - T_0\right)\left(1 - \epsilon\frac{A\theta}{B^2} + O(\epsilon^2)\right). \tag{27}$$

Setting

$$\Lambda = \epsilon M,\ \ \delta = \epsilon N,\ \ \Gamma = \epsilon Q. \tag{28}$$

Due to the nonlinear nature of the velocity and temperature field equations in (6)-(8) and (10b), it is convenient to form a power series expansion both in the parameter ϵ, i.e.,

$$w = w_0 + \epsilon w_1 + O(\epsilon^2), \tag{29}$$

$$\theta = \epsilon\theta_0 + \epsilon^2\theta_1 + O(\epsilon^3). \tag{30}$$

Substituting these form of expansions, equating coefficients of these powers, one obtains

Leading terms

$$O(\epsilon^0): \quad r\frac{d^2w_0}{dr^2}+\frac{dw_0}{dr}=C^*r, \tag{31}$$

$$O(\epsilon): \quad r\frac{d^2\theta_0}{dr^2}+\frac{d\theta_0}{dr}+rQ\frac{C}{C^*}\left(\frac{dw_0}{dr}\right)^2=0. \tag{32}$$

$$w_0(1)=\theta_0(1)=\frac{dw_0}{dr}(0)=\frac{d\theta_0}{dr}(0)=0. \tag{33}$$

Correction terms

$$\begin{aligned} O(\epsilon): r\frac{d^2w_1}{dr^2}+\frac{dw_1}{dr} &= \frac{A}{B^2}\theta_0\left(r\frac{d^2w_0}{dr^2}+\frac{dw_0}{dr}\right)+\frac{A}{B^2}r\frac{d\theta_0}{dr}\frac{dw_0}{dr} \\ &\quad -M\frac{C^*}{C}\left(\frac{dw_0}{dr}\right)^2\left(3r\frac{d^2w_0}{dr^2}+\frac{dw_0}{dr}\right), \end{aligned} \tag{34}$$

$$\begin{aligned} O(\epsilon^2): r\frac{d^2\theta_1}{dr^2}+\frac{d\theta_1}{dr} &= rQ\frac{C}{C^*}\frac{A}{B^2}\theta_0\left(\frac{dw_0}{dr}\right)^2-2rQ\frac{C}{C^*}\frac{dw_0}{dr}\frac{dw_1}{dr} \\ &\quad -rN\theta_0, \end{aligned} \tag{35}$$

$$w_0(1)=\theta_0(1)=\frac{dw_0}{dr}(0)=\frac{d\theta_0}{dr}(0)=0. \tag{36}$$

where C^* is defined as

$$C^*=\frac{C}{\exp(A/B-\bar{T}_0)}. \tag{37}$$

By simple integration, the solutions of (31)-(33) are

$$w_0=-\frac{C^*}{4}(1-r^2), \tag{38}$$

$$\theta_0=\frac{QCC^*}{64}(1-r^4). \tag{39}$$

Substituting these solutions into equation (34) and (35), one has

$$w_1=\frac{QC^{*4}}{32C}(1-r^4)-\frac{AQCC^{*2}}{768B^2}(2-3r^2+r^6,) \tag{40}$$

$$\begin{aligned} \theta_1=-\frac{QMC^{*4}}{576}(1-r^6)+\frac{AM^2C^2C^{*2}}{16384B^2}(3-4r^4+r^8)+\frac{\delta QCC^*}{2304}(8 \\ +9r^2-r^6) \end{aligned} \tag{41}$$

The solution of the problem in the original parameters are

$$w=-\frac{C^*}{4}(1-r^2)+\frac{\Lambda C^{*4}}{32C}(1-r^4)-\frac{A}{B^2}\frac{\Gamma CC^{*2}}{768}(2-3r^2+r^6), \tag{42}$$

$$\theta = \frac{\Gamma C C^*}{64}(1-r^4) - \frac{\Gamma \Lambda C^{*4}}{576}(1-r^6) + \frac{A}{B^2}\frac{\Gamma^2 C^2 C^{*2}}{16384}(3-4r^4+r^8) + \frac{\delta \Gamma C C^*}{2304}(8+9r^2-r^6). \quad (43)$$

Although the restriction to small δ is made for mathematical convenience, it is also a limit of practical relevance (see Seddeek and Aboeldahab [25]). At this point, it is worth noting that the dimensionless velocities for the models are the same with Yurusoy and Pakdemirli [29]. Since the velocities distribution have been reported there, we will look only at the dimensionless temperature. The energy equation is different from Yurusoy and Pakdemirli [29] due to the terms associated with heat generation/absorption including the parameters δ. It is very important to state that the first three terms of equations (26) and (43) are in full agreement with that obtained in [29]. In the next section, the perturbation solutions in the two cases will be contrasted with those generated numerically by the software package Maple Version 12.

4. THE NUMERICAL SOLUTION METHODOLOGY

The non-dimensional forms of the coupled equations (6) and (7) with the conditions (8) are solved numerically. First, we note that equations (6) and (7) contain a singularity for $r = 0$. The procedure to handle this problem numerically involves using L'Hospital rule by which we have the new equations

$$\frac{d\mu}{dr}\frac{dw}{dr} + 2\mu\left(\frac{d^2w}{dr^2}\right) + 6\Lambda\left(\frac{dw}{dr}\right)^2\frac{d^2w}{dr^2} = C, \quad (44)$$

$$2\frac{d^2\theta}{dr^2} + \Gamma\left(\frac{dw}{dr}\right)^2\left(\mu + \Lambda\left(\frac{dw}{dr}\right)^2\right) + \delta\theta = 0, \quad (45)$$

which are valid near $r = 0$. It is evident that equations (44) and (45) did not contain a singularity. The boundary conditions at $r = 1$ are specified in a straightforward manner. Equations (6), (7), (10), (44) and (45) with the conditions given in equation (8) were solved numerically using Maple solver boundary value problem designed for the solution of boundary value problems. In practice, the neighbourhood of the axis of the pipe to start the integration is chosen sufficiently small to the order of 10^{-3}.

Furthermore, the present numerical code is also validated with published numerical results for the two viscosity models at different parameter conditions when heat generation/absorption is absent i.e.

$\delta = 0$. For this special case, the numerical solution agree very well with the finite-difference results presented in Massoudi and Christe [15] with the difference being less than 10^{-3}.

5. THE RELATIONSHIP BETWEEN NUMERICAL AND PERTURBATION SOLUTIONS

For the purpose of comparison we tabulate the mid plane temperature distribution of the pipe $\theta\ (0) = \theta_{max}$ for the obtained numerical solution (N) and the corresponding perturbation solution (P) while the relative errors (R.E.) are also computed.

Table 1. The relation between the perturbation and numerical results for Reynold's model when $\Gamma = \delta = \gamma = 1$.

$\lvert C\rvert$	$\Lambda = 1$			Λ	$C = -1$		
	θ_{max}(P)	θ_{max}(N)	R.E.		θ_{max}(P)	θ_{max}(N)	R.E.
0.25	0.00187	0.00123	3.76%	0	0.01928	0.02014	4.27%
0.5	0.00468	0.00485	3.64%	0.5	0.01841	0.01915	3.86%
0.75	0.01025	0.01066	3.84%	1	0.01754	0.01842	4.75%
1	0.01754	0.01842	4.75%	1.1	0.01737	0.01829	5.04%
2	0.05154	0.06507	20.8%	1.5	0.01668	0.01783	6.48%

Table 2. The relation between the perturbation and numerical results for Reynold's model when $\Lambda = \gamma = -C = 1$.

Γ	$\delta = 1$			δ	$\Gamma = 1$		
	θ_{max}(P)	θ_{max}(N)	R.E.		θ_{max}(P)	θ_{max}(N)	R.E.
0	0.0	0.0		-0.4	0.01268	0.01328	4.49 %
1	0.01754	0.01842	4.75 %	-0.2	0.01338	0.01386	3.47 %
3	0.05373	0.05645	4.82 %	0	0.01407	0.01448	2.84 %
5	0.09138	0.09619	5.00 %	0.5	0.01581	0.01625	2.74 %
6	0.11076	0.11674	5.13 %	1	0.01754	0.01842	4.75 %

Table 3. Relation between the perturbation and numerical results for Reynold's model when $\Gamma = \gamma = -C = \delta = 1$.

γ	θ_{max}(P)	θ_{max}(N)	R.E.
-6	0.01626	0.01717	5.28 %
-4	0.01663	0.01750	5.00 %
0	0.01736	0.01823	4.75 %
5	0.01828	0.01934	5.00 %
6	0.01846	0.01946	5.13 %

Table 4. Relation between perturbation and numerical results for Vogel's model with $-C = \bar{T}_0 = \Gamma = \Lambda = \delta = 1$.

A	$B = 1$			B	$A = 1$		
	θ_{max}(P)	θ_{max}(N)	R.E.		θ_{max}(P)	θ_{max}(N)	R.E.
1	0.01754	0.01842	4.73%	0.25	0.00096	0.00100	4.18%
2	0.00704	0.00734	4.04%	0.5	0.00709	0.00741	4.33%
3	0.00259	0.00270	4.01%	0.75	0.01339	0.01392	3.87%
4	0.00095	0.00099	3.93%	1	0.01754	0.01842	4.73%
5	0.00035	0.00036	3.88%	1.25	0.01964	0.02134	7.98%

Table 5. Relation between perturbation and numerical results for Vogel's model with $A = B = \Lambda = \Gamma = \delta = 1$.

$\|C\|$	$T_0 = 1$			T_0	C = -1		
	θ_{max}(P)	θ_{max}(N)	R.E		θ_{max}(P)	θ_{max}(N)	R.E.
0.25	0.00119	0.00123	3.76%	0	0.00702	0.00730	3.90%
0.5	0.00468	0.00485	3.64%	0.25	0.00897	0.00933	3.84%
0.75	0.01025	0.01066	3.84%	0.5	0.01141	0.01186	3.75%
1.0	0.01754	0.01842	4.73%	1.0	0.01754	0.01842	4.73%
1.25	0.02605	0.02790	6.64%	1.1	0.01874	0.01991	5.86%

Table 6. Relation between perturbation and numerical results for Vogel's model with $A = B = -C = \bar{T}_0 = \delta = 1$.

Λ	$\Gamma = 1$			Γ	$\Lambda = 1$		
	θ_{max} (P)	θ_{max} (N)	R.E.		θ_{max} (P)	θ_{max} (N)	R.E
0	0.01928	0.02014	4.25%	0	0.00017	0.00018	4.75%
0.25	0.01885	0.01960	3.84%	5	0.09138	0.09582	4.63%
0.5	0.01841	0.01915	3.85%	10	0.19192	0.20068	4.36%
1	0.01754	0.01842	4.73%	15	0.30161	0.31381	3.89%
1.25	0.01711	0.01811	5.51%	20	0.42046	0.43427	3.18%

Table 7. Relation between perturbation and numerical results for Vogel's model with $A = B = -C = \Lambda = \Gamma = 1$.

δ	θ_{max} (PR)	θ_{max} (NR)	Rel. error
-0.4	0.01268	0.01328	4.48 %
-0.2	0.01338	0.01386	3.46 %
0.0	0.01407	0.01448	2.81 %
0.5	0.01581	0.01625	2.73 %
1.0	0.01754	0.01842	4.73 %

Values are generated in Tables 1-3 for Reynold's model viscosity. In Table 1, column 1-4, the variation of parameter C is captured. It is obvious that for $|C| \leq 1$ the perturbation and numerical solutions exhibit good agreement with relative error less than 5 % when $\Lambda = \Gamma = \lambda = \delta = 1$. Table 1, column 5-8, the variation of the non-Newtonian parameter (Λ) is considered. For $\Lambda \leq 1$ the relative error is less than 5 %. In Table 2, column 1-4, variation of Γ is considered. Analytical results are in very good agreement with the numerical results since the relative error is less than or equal to 5 % for $\Gamma \leq 5$. In Table 2, column 5-8, variation of the heat generation/absorption parameter δ is investigated. The perturbation solutions are in very good agreement to relative error of 5 % when $-0.4 \leq \delta \leq 1$. In Table 3, the range for 5 % relative error for γ is $-4 \leq \gamma \leq 5$.

The physical quantities of principle interest according to the Vogel's model viscosity are A, B and the term $\bar{T}_0$ in the place of γ in the Reynold's case. Variation of relative error as a function of A could be analyzed through Table 4, columns 1-4. It is seen from the table that there is a good agreement between perturbation and numerical results as the relative error decreases as A increases. For the variation of B in Table 4, columns 5-8 the relative error is greater than 5 % for $B > 1$. Table 5, columns 1-4, the variation of parameter C is considered and found to be remarkably similar in error values to the Reynold's case. For $\bar{T}_0 \leq 1$, the relative error is less than 5 % in Table 5, columns 5-8. The variation of Λ is investigated in Table 6, column 1-4 and it is seen that for $\Lambda \leq 1$, the relative error is less than 5 %. Table 6, columns 5-8, it is seen that there is a good agreement between perturbation and numerical results as the relative error decreases as Γ increases. Finally, the variation of heat generation/absorption parameter δ is shown in Table 7. For $-0.4 \leq \delta \leq 1$ the results are in good agreement with relative error of 5 %. Based on the data reported in Tables 1-7, the perturbation method is adequate for solutions of the present problem.

Knowing the validity of the perturbation solutions we proceed to capture the effect of emerging parameters of interest on the solutions of the energy equations.

6. RESULTS AND DISCUSSION

The analytical solutions (26) and (43) for the dimensionless temperature distributions are plotted against the pipe radius. The temperature distributions for Reynold's viscosity model are displayed in Figs. 1-5 while Figs. 6-12 are for Vogel's viscosity model. In these figures, the variation of the embedding parameters are taken into account.

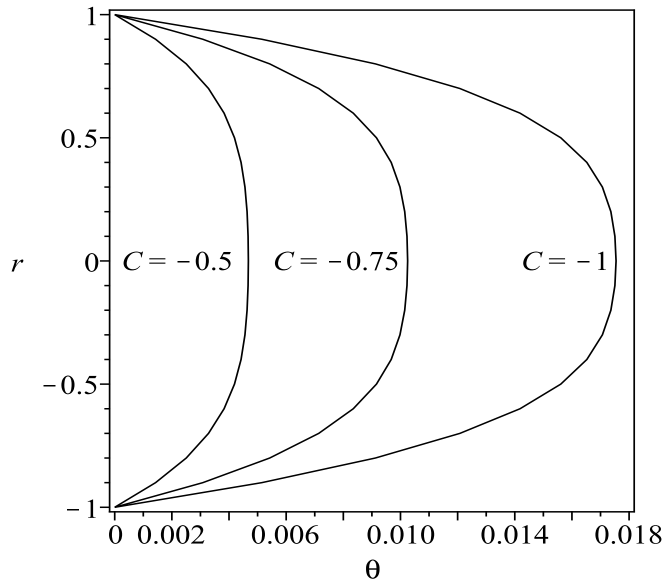

Fig. 2. Radius r versus temperature θ of Reynold's viscosity model for values of C when $\Gamma = \Lambda = \gamma = \delta = 1$.

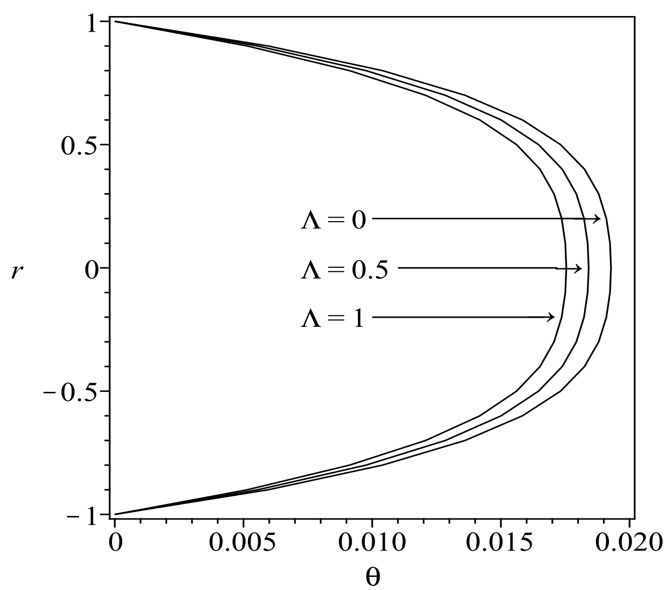

Fig. 3. Radius r versus temperature θ of Reynold's viscosity model for values of Λ when $\Gamma = \gamma = \delta = -C = 1$.

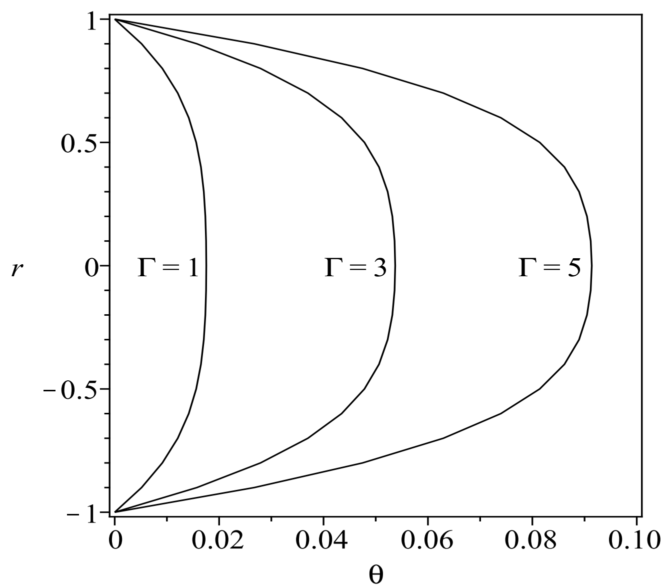

Fig. 4. Radius r versus temperature θ of Reynold's viscosity model for values of Γ when $\Lambda = \gamma = \delta = -C = 1$.

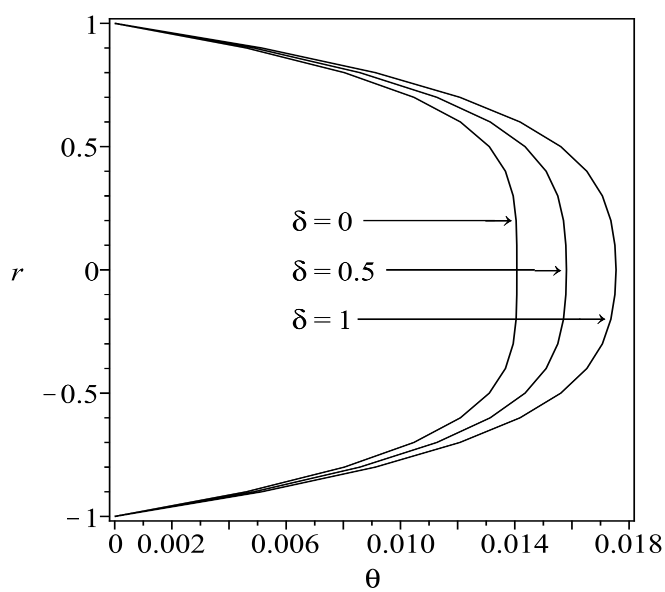

Fig. 5. Temperature distribution of Reynold's viscosity model for different values of δ with $\Gamma = \Lambda = \gamma = -C = 1$.

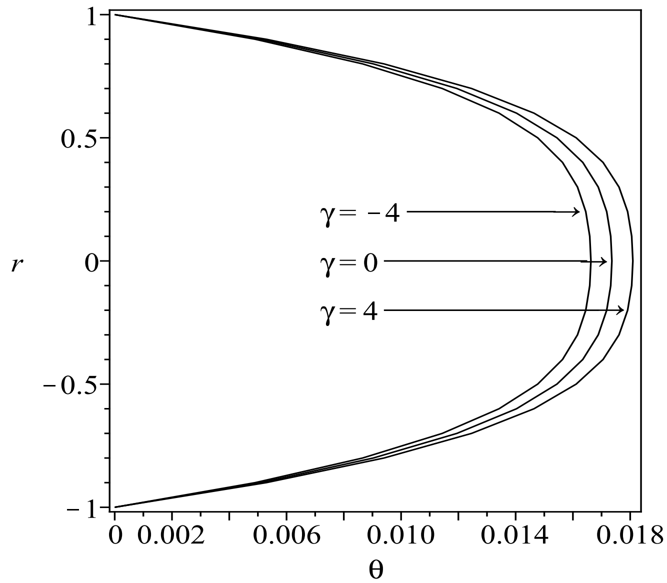

Fig. 6. Temperature distribution of Reynold's viscosity model for different values of γ with $\Gamma = \Lambda = \delta = -C = 1$.

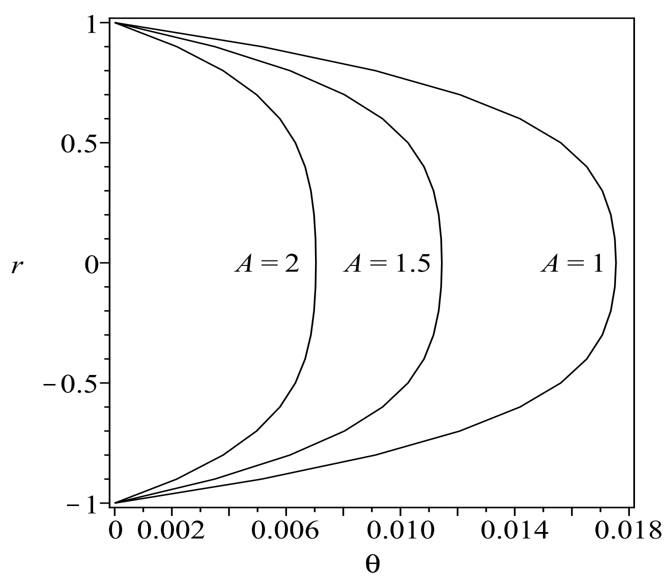

Fig. 7. Effect of the parameter A on graph of radius r with the temperature θ of Vogel's viscosity model with $B = -C = \bar{T}_0 = \Gamma = \Lambda = \delta = 1$.

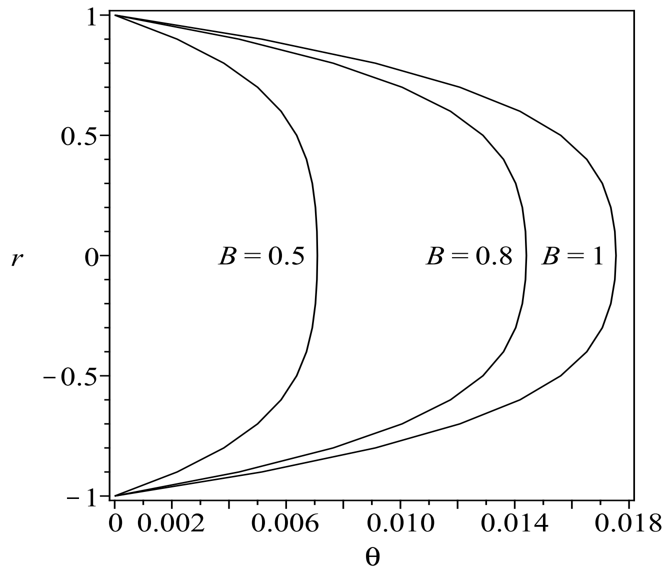

Fig. 8. Effect of the parameter B on variation of radius r with the temperature θ of Vogel's viscosity model with $A = -C = \bar{T}_0 = \Gamma = \Lambda = \delta = 1$.

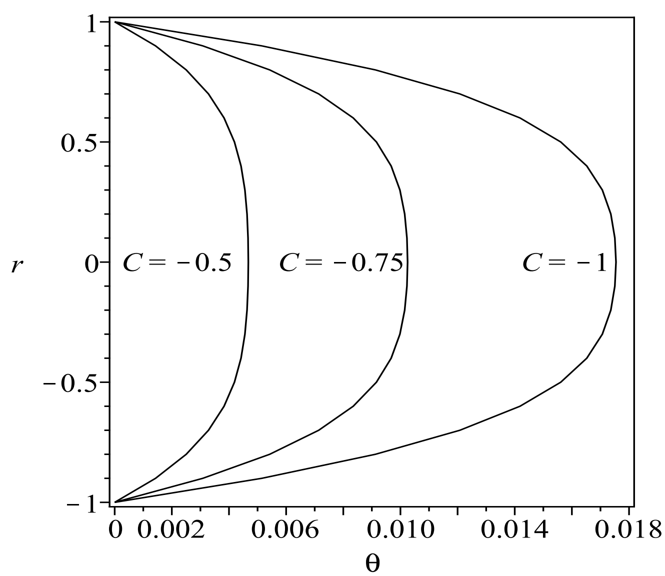

Fig. 9. Effect of the dimensionless pressure gradient C on the curve of radius r with the temperature θ of Vogel's viscosity model with $A = B = \bar{T}_0 = \Gamma = \Lambda = \delta = 1$.

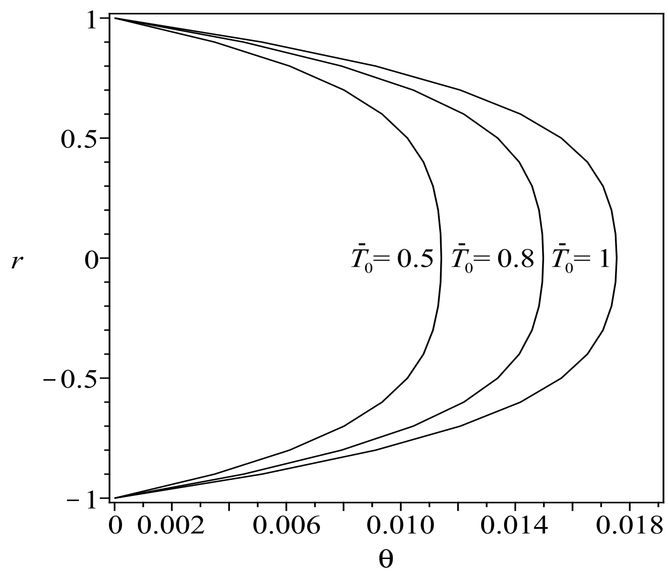

Fig. 10. Effect of initial temperature $\bar{T}_0$ on the picture of radius r on the temperature distribution θ of Vogel's viscosity model with $A = B = -C = \Gamma = \Lambda = \delta = 1$.

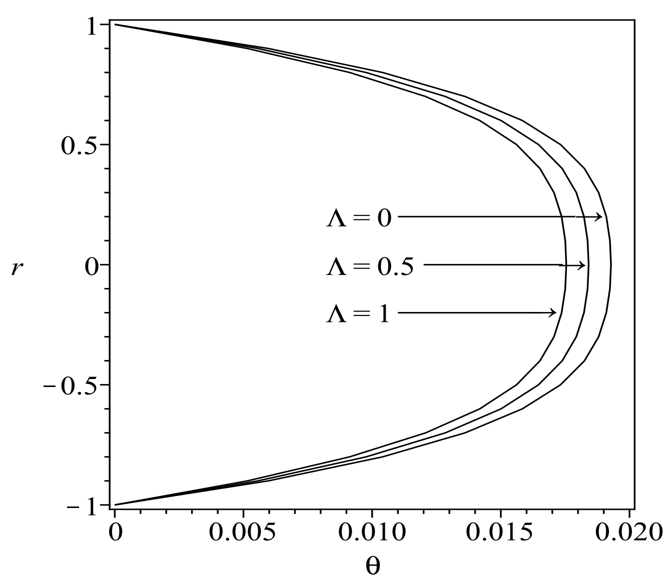

Fig. 11. Effect of Non-Newtonian parameter Λ on the plot of radius r versus the temperature distribution θ of Vogel's viscosity model with $A = B = -C = \bar{T}_0 = \Gamma = \delta = 1$.

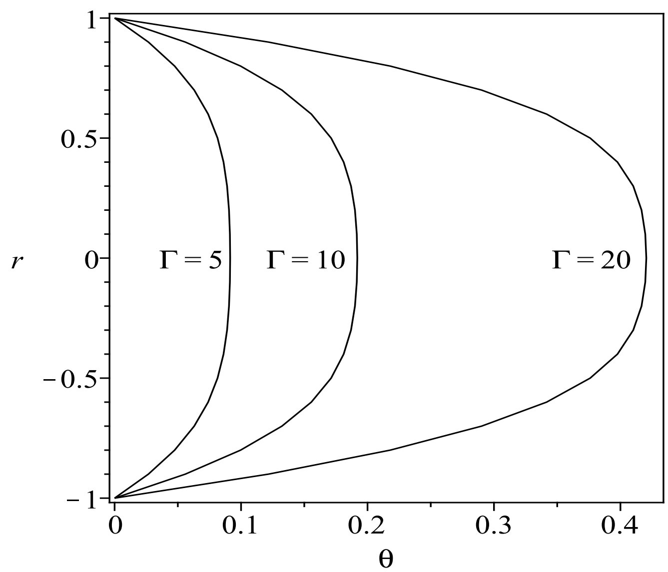

Fig. 12. Effect of viscous dissipation parameter Γ on the dependence of radius r on the temperature distribution θ of Vogel's viscosity model with $A = B = -C = \bar{T}_0 = \Lambda = \delta = 1$.

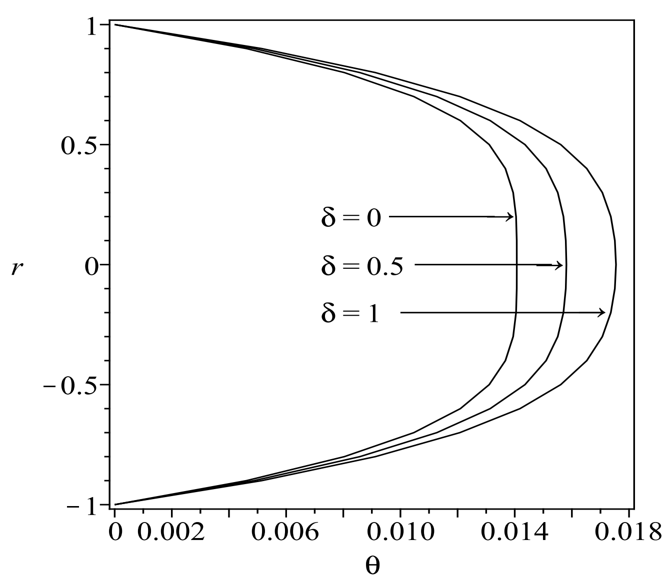

Fig. 13. Effect of heat generation parameter δ on the profile of radius r with temperature distribution θ of Vogel's viscosity model with $A = B = -C = \bar{T}_0 = \Gamma = \Lambda = 1$.

Figs. 2-6 show solutions obtained for the dimensionless temperature distributions for the Reynold's viscosity model, while Figs. 7-13 are for Vogel's model.

Fig. 2 indicates the effects of pressure drop C on the temperature distribution. As C becomes more negative, the maximum temperature of the fluid (at the center of the pipe) increases. Fig. 3 shows the importance of the non-Newtonian parameter Λ with the case $\Lambda = 0$ corresponding to a Newtonian fluid. It is found that, as Λ increases, the temperature decreases. The temperature distribution for different values of the viscous dissipation parameter Γ is given in Fig. 4. Viscous dissipation tends to increase the dimensionless temperature because of irreversible conversion of mechanical energy to thermal energy. This holds true for the present case as well as the previous one. Fig. 5 examines the influence of heat generation parameter δ. The special case without the effect of heat generation, i.e. for $\delta = 0$ is the least temperature distribution. From the figure, it can be glanced how the presence of heat generation inside the fluid influences the overall temperature distribution. As the amount of heat generation increases, the maximum temperature of the fluid increases gradually. This is due to the fact that the heat generation mechanism creates a layer of hot fluid and at some level when δ is relatively large the resulting temperature of fluid finally exceeds the least temperature distribution. Fig. 6 depicts the effect of the viscosity variational number γ. A similar behavior to δ is observed for γ with $\gamma = 0$ corresponding to the constant viscosity case.

Figs. 7-13 show the dependence of temperature distributions on the parameters of Vogel's viscosity model. Figs. 7 and 11 demonstrate that, as A, C and Λ increase, the value of the dimensionless temperature decreases while Figs. 8, 10, 12 and 13 show the opposite influence on the temperature as the parameters B, $\bar{T}_0$, Γ, δ increase.

7. CONCLUDING REMARKS

An analytical procedure is presented in this paper for studying the heat characteristics of a laminar third-grade fluid flow in a cylindrical pipe. The steady state one-dimensional heat transfer model is formulated and analytical solutions are developed for the Reynold's and Vogel's viscosity models with heat generation term. The heat transfer model is also solved numerically. The present numerical solutions for special cases are found to be in excellent agreement with previous ones obtained by finite difference method. The new regular perturbation solutions of the temperature profiles are compared with numerical integration and with relative error of 5 % for all physical parameters, an excellent agreement is found. Subsequently, the obtained analytical temperature solutions are used to investigated the effects of the heat transfer characteristics which has provided useful and relevant information for handling and processing of coal-based slurries.

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NOMENCLATURE

a and b	constants from dimensional Vogel's viscosity model
M	constant from dimensional Reynold's viscosity model
$A = a/(\bar{T}_0\beta)$	dimensionless constant from Vogel's viscosity model
$B = (b + \bar{T}_0)/(\bar{T}_0\beta)$	dimensionless constant from Vogel's viscosity model
$\bar{C}_0$	initial concentration of the reactant species
$C = (\bar{R}^2/\bar{\mu}_0^e\bar{w}_0)(\partial\bar{P}/\partial\bar{z})$	pressure gradient parameter
K	constant thermal conductivity
$\partial\bar{p}/\partial\bar{r}$	pressure gradient along the normal to the pipe axis
$\partial\bar{p}/\partial\bar{z}$	pressure gradient in the axial direction
$\partial\bar{p}/\partial\phi$	pressure gradient in rotational direction
$\bar{Q}$	heat generation constant
$\bar{r}$	dimensional perpendicular distance from pipe axis
$r = \bar{r}/\bar{R}$	dimensionless perpendicular distance from pipe axis
$\bar{R}$	radius of the pipe
$\bar{T}_0$	the initial temperature
$\bar{w}(\bar{r})$	dimensional velocity component in the $\bar{z}$ axis
$w = \bar{w}/\bar{w}_0$	dimensionless velocity component in the $\bar{z}$ axis
$\bar{w}_0$	dimensional reference velocity
$\bar{z}$	axis of the cylinder
Greek symbols	
α_1, α_2 and β_3	constant material coefficients
$\beta = R\bar{T}_0/E$	activation energy
$\gamma = \bar{M}\beta\bar{T}_0$	Reynold's viscosity variational parameter
$\bar{\mu}$	dynamic shear viscosity
$\mu = \bar{\mu}/\bar{\mu}_0^e$	dimensionless viscosity
$\bar{\mu}_0^e = \{\bar{\mu}_0 \text{ or } \bar{\mu}_*\}$	
$\bar{\mu}_* = \bar{\mu}_0\exp(\bar{T}_0)$	
ϕ	rotational direction
$\Gamma = 4\bar{\mu}_0^e\bar{w}_0^2/(k\bar{T}_0\beta)$	viscous heating parameter
$\theta = (\bar{T} - \bar{T}_0)E/(R\bar{T}_0^2)$	dimensionless temperature excess
$\Lambda = \beta_3\bar{w}_0^2/(\bar{\mu}_0^e\bar{r}_0^2)$	non-Newtonian material parameter of the fluid
$\delta = QEA_0\bar{R}^2C_0/(RK\bar{T}_0^2)$	heat generation parameter

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DEPARTMENT OF MATHEMATICS, OBAFEMI AWOLOWO UNIVERSITY, ILE - IFE, 220005, NIGERIA

E-mail address: `ola2jayeoba@yahoo.com.com`

DEPARTMENT OF MATHEMATICS, OBAFEMI AWOLOWO UNIVERSITY, ILE - IFE, 220005, NIGERIA

E-mail address: `ssokoya2003@yahoo.it`