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A HYBRID LINEAR COLLOCATION MULTISTEP SCHEME FOR SOLVING FIRST ORDER INITIAL VALUE PROBLEMS

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ABSTRACT. This paper proposes a five-step ninth order hybrid linear multistep method with three non-step points for the solution of first order initial value problems(IVPs). The main method and additional methods are obtained from the same continuous scheme derived via interpolation and collocation procedure. The methods are then applied in block form as simultaneous numerical integrators over non-overlapping intervals. The schemes are consistent, zero stable, convergent and accurate.

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1. INTRODUCTION

Ordinary differential equations are important tools in solving real-world problems and a wide variety of natural phenomena are modelled by ordinary differential equations. Ordinary differential equations have been applied to many problems in physics, engineering, biology, social science and so on. These equations have received much attention in last 20 years [18].

Collocation Methods are widely considered as ways of generating numerical solution to ordinary differential equation of the form;

$$y' = f(x, y),\ y(x_0) = y_0,\ x \in [x_0,\ b], \tag{1}$$

where prime indicates derivative with respect to x and f satisfies the Lipschitz condition of the existence and uniqueness of solution. The collocation method is dated as far back as 1965 when Lanczos [21] introduced the standard collocation method with some selected points.

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The solution of equation (1) has been discussed by various researchers such as [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 16, 17, 19, 20, 22, 23, 24, 25, 26, 27, 28, 29, 30]. In 1994, [28] showed that traditional Multistep Methods including the hybrid ones can be made continuous through the ideal of Multistep Collocation scheme as against the discrete schemes due to the fact that better global error can be estimated and approximation can equally be obtained at all interior points. The introduction of the continuous Collocation Method has been able to bridge the gap between the discrete collocation method and the conventional multistep method, thus it is possible to write the Linear Multistep Methods in form of some continuous collocation schemes. [1] developed a continuous new Butcher type three-step block hybrid multistep method for the solution of equation (1). The result obtained showed a class of discrete of order seven and error constants $C_8 = \left\{\frac{27}{777426}, \frac{155525}{1273724928}\right\}$. Also [8] derived a generalized form of 2-step continuous linear multistep schemes with hybrid points selected at four(4) difeferent points.

In this paper, we present a continuous five-step block method employing multistep collocation approach which produces a class of eight discrete scheme of order 9 and error constants as shown in the Table 1.

2. DERIVATION OF THE SCHEME

We seek a k-step multistep collocation polynomial $y(x)$ of the form:

$$\sum_{j=4}^{5} \alpha_j y_{n+j} = h \left[\sum_{j=0}^{5} \beta_j f_{n+j} + \beta_v f_{n+v} \right], \tag{2}$$

where α_j and β_j are coefficients and $v = \left\{\frac{13}{3}, \frac{9}{2}, \frac{14}{3}\right\}$ are hybrid points. In order to obtain equation (2), we proceed by seeking an approximation of the exact solution $y(x)$ by assuming a continuous solution of the form:

$$y(x) = \sum_{j=0}^{p+q-1} a_j \varphi(x), \tag{3}$$

such that $x \in [x_0, b]$, a_j are unknown coefficients and $\varphi(x)$ are polynomial basis function of degree $p + q - 1$, where the number of interpolation points p and the number of distinct collocation points q are respectively chosen to satisfy $1 \leq p < k$ and $q > 0$. The interger $k \geq 1$ denotes the step number of the method.

We construct a k-step continuous multistep method with $\varphi(x) = x^j$, $j = 0, 1, 2, \cdots, 9$, $p = 1$, $q = 9$, $k = 5$ by imposing the above condition, we have:

$$\sum_{j=0}^{9} j a_j x_{n+i}^{j-1} = f_{n+i},\ i = \left\{0, 1, 2, 3, 4, \frac{13}{3}, \frac{9}{2}, \frac{14}{3}, 5\right\}, \tag{4}$$

$$\sum_{j=0}^{9} a_j x_{n+i}^{j} = y_{n+i}, i = 4, \tag{5}$$

where n is the grid index.

Equations $(4-5)$ lead to system of $p+q$ equations which must be solved to obtain the coefficient a_j. We then obtain value of a_j's using Matlab Software package. The five step continuous Hybrid method is obtain by substituting these values of a_j's into equation (3). On evaluating the continuous scheme at points $x = \left\{x_{n+5},\ x_{n+\frac{14}{3}},\ x_{n+\frac{9}{2}},\ x_{n+\frac{13}{3}},\ x_{n+3},\ x_{n+2},\ x_{n+1},\ x_n\right\}$, we obtain the following eight discrete equations:

$$\begin{aligned} y_{n+5} - y_{n+4} = \ & h[\tfrac{-179}{165110400} f_n + \tfrac{73}{5174400} f_{n+1} - \tfrac{31}{313600} f_{n+2} \\ & + \tfrac{-359}{604800} f_{n+3} + \tfrac{1079}{13440} f_{n+4} + \tfrac{1496637}{2038400} f_{n+\frac{13}{3}} \\ & - \tfrac{61184}{99225} f_{n+\frac{9}{2}} + \tfrac{2458917}{3449600} f_{n+\frac{14}{3}} + \tfrac{1989}{22400} f_{n+5}], \end{aligned} \tag{6}$$

$$\begin{aligned} y_{n+\frac{14}{3}} - y_{n+4} = \ & h[\tfrac{22037}{33852791700} f_n - \tfrac{7253}{795685275} f_{n+1} + \tfrac{42467}{578680200} f_{n+2} - \\ & \tfrac{19853}{31000725} f_{n+3} + \tfrac{993749}{8266860} f_{n+4} + \tfrac{180667}{429975} f_{n+\frac{13}{3}} \\ & + \tfrac{7757824}{651015225} f_{n+\frac{9}{2}} + \tfrac{342733}{291060} f_{n+\frac{14}{3}} - \tfrac{29107}{10333575} f_{n+5}], \end{aligned} \tag{7}$$

$$\begin{aligned} y_{n+\frac{9}{2}} - y_{n+4} = \ & h[\tfrac{24863}{42268262400} f_n - \tfrac{32833}{3973939200} f_{n+1} + \tfrac{48323}{722534400} f_{n+2} - \\ & \tfrac{91493}{154828800} f_{n+3} + \tfrac{1220071}{10321920} f_{n+4} + \tfrac{231041241}{521830400} f_{n+\frac{13}{3}} \\ & - \tfrac{21389}{198450} f_{n+\frac{9}{2}} + \tfrac{43797591}{883097600} f_{n+\frac{14}{3}} - \tfrac{115447}{51609600} f_{n+5}], \end{aligned} \tag{8}$$

$$\begin{aligned} y_{n+\frac{13}{3}} - y_{n+4} = \ & h[\tfrac{690797}{108328933440} f_n - \tfrac{910757}{101847715200} f_{n+1} + \tfrac{1336457}{18517766400} f_{n+2} \\ & - \tfrac{2512217}{3968092800} f_{n+3} + \tfrac{31844549}{264539520} f_{n+4} + \tfrac{19955023}{55036800} f_{n+\frac{13}{3}} - \\ & \tfrac{134364928}{651015225} f_{n+\frac{9}{2}} + \tfrac{5577703}{93139200} f_{n+\frac{14}{3}} - \tfrac{3364243}{1322697600} f_{n+5}], \end{aligned} \tag{9}$$

$$\begin{aligned} y_{n+3}-y_{n+4}= \;& h[\tfrac{14669}{165110400}f_n-\tfrac{20869}{15523200}f_{n+1}+\tfrac{36329}{2822400}f_{n+2} \\ & -\tfrac{202169}{604800}f_{n+3}-\tfrac{100187}{40320}f_{n+4}+\tfrac{135557213}{2038400}f_{n+\frac{13}{3}} \\ & -\tfrac{745216}{99225}f_{n+\frac{9}{2}}+\tfrac{9737253}{3449600}f_{n+\frac{14}{3}}-\tfrac{31411}{201600}f_{n+5}], \end{aligned} \tag{10}$$

$$\begin{aligned} y_{n+2}-y_{n+4}= \;& h[\tfrac{-247}{396900}f_n+\tfrac{43}{3675}f_{n+1}-\tfrac{3509}{9800}f_{n+2}-\tfrac{6701}{4725}f_{n+3} \\ & +\tfrac{871}{420}f_{n+4}-\tfrac{13851}{1225}f_{n+\frac{13}{3}}+\tfrac{1466368}{99225}f_{n+\frac{9}{2}} \\ & -\tfrac{60507}{9800}f_{n+\frac{14}{3}}+\tfrac{69}{175}f_{n+5}], \end{aligned} \tag{11}$$

$$\begin{aligned} y_{n+1}-y_{n+4}= \;& h[\tfrac{32951}{6115200}f_n-\tfrac{568893}{1724800}f_{n+1}-\tfrac{459807}{313600}f_{n+2} \\ & +\tfrac{1189}{22400}f_{n+3}-\tfrac{50499}{4480}f_{n+4} \\ & +\tfrac{92569149}{2038400}f_{n+\frac{13}{3}}-\tfrac{212224}{3675}f_{n+\frac{9}{2}} \\ & +\tfrac{82333989}{3449600}f_{n+\frac{14}{3}}-\tfrac{34107}{22400}f_{n+5}], \end{aligned} \tag{12}$$

$$\begin{aligned} y_{n}-y_{n+4}= \;& h[\tfrac{-351518}{1289925}f_n-\tfrac{212552}{121275}f_{n+1}+\tfrac{10016}{11025}f_{n+2} \\ & -\tfrac{31672}{4725}f_{n+3}+\tfrac{19454}{315}f_{n+4}-\tfrac{4391496}{15925}f_{n+\frac{13}{3}} \\ & +\tfrac{35618816}{99225}f_{n+\frac{9}{2}}-\tfrac{2035368}{13475}f_{n+\frac{14}{3}}+\tfrac{15592}{1575}f_{n+5}]. \end{aligned} \tag{13}$$

3. ANALYSIS OF THE SCHEME

In this section, we discuss the local truncation error and order, consistency and zero stability of the scheme generated .

3.1 LOCAL TRUNCATION ERROR AND ORDER

Following [11] and [19], we define the local truncation error associated with equation (2) to be linear difference operator L as

$$L[y(x),h]=\sum_{j=0}^{k}\left\{\alpha_j y(x_{x+j})-h\beta_j y(x_{n+j})\right\}. \tag{14}$$

Assuming that $y(x)$ is sufficiently deferentiable, we can expand equation (14) as a Taylor series about the point x to obtain the expression

$$L[y(x),h]=C_0y(x_n)+C_1hy'(x_n)+\cdots+C_qh^qy^q(x)+\cdots, \tag{15}$$

where the constant coefficient C_q, $q = 0,\ 1,\ \cdots$, are given as

$$C_0 = \sum_{j=0}^{k} \alpha_j, \tag{16}$$

$$C_1 = \sum_{j=0}^{k} j\alpha_j, \tag{17}$$

$$C_q = \frac{1}{q!}[\sum_{j=0}^{k} j\alpha_j - q(q-1)(\sum_{j=0}^{k} j^{q-1}\beta_j + \sum_{j=0}^{k} v^{q-1}\beta_{vj})], \tag{18}$$

where $v \notin \{0, 1, 2, \cdots, n\}$.
According to [15], we say that the method (2) has order p if $C_0 = C_1 = \cdots = C_{p-1} = C_p$ and $C_{p+1} \neq 0$. Therefore C_{p+1} is the error constant and $C_{p+1}h^{p+1}y^{p+1}(x_n)$ is the principal local truncation error at the point x_n. Thus we can write the local truncation error (LTE) of the method of order p as

$$\text{LTE} = C_{p+1}h^{p+1}y^{p+1}(x_n) + O(h^{p+2}). \tag{19}$$

It is established from our calculation that the Hybrid Linear Collocation Multistep Scheme (HLCMS) have high order and relatively small error constant as displayed in the Table 1 below.

Table 1. Order and Error constant of the scheme

Equation number	Order	Error Constant
6	9	$-1.727800174 \times 10^{-7}$
7	9	$0.8158198007 \times 10^{-7}$
8	9	$0.7291577588 \times 10^{-7}$
9	9	$0.7948622829 \times 10^{-7}$
10	9	$0.9042154242 \times 10^{-5}$
11	9	$-0.4255200022 \times 10^{-4}$
12	9	$0.2086679579 \times 10^{-3}$
13	9	$-0.1722049588 \times 10^{-2}$

3.2 CONSISTENCY

A linear multistep method (2) is consistent if:
(i) the order $p \geq 1$, (ii)$\sum_{j=0}^{k} \alpha_j = 0$, (iii)$\sum_{j=0}^{k} j\alpha_j = \sum_{j=0}^{k} \beta_j$, (iv)$\rho(1) = 0$, $\rho'(1) = \sigma(1)$,
where ρ and σ are the first and second characteristic polynomials of equation (2) [19].
Applying this definition to the schemes $(6 - 13)$, they were found to be consistent.

3.3. ZERO STABILITY

According to [19, 20], a linear multistep method of the form (2) is said to be Zero stable if no roots of the first characteristic polynomial $\rho(r)$ has modulus greater than one, and if every root of the modulus one is simple (i.e., not multiple).
Also, applying this definition to the schemes $(6-13)$, they were found to be Zero stable.

4. IMPLEMMENTATION OF THE SCHEME

The derived scheme is implemented more efficiently by combining the hybrid linear multistep method as simultaneous intergrator for IVPs without requiring starting values and predictors. We proceed by explicitly obtaining initial conditions at x_{n+5}, $n = 0, 5, \cdots, N-5$ using the computed values $y(x_{n+5})$=y_{n+5} over subintervals $[x_0,\ x_5], \cdots, [x_{N-5},\ x_N]$. For instance, using equations $(6-13)$, and with $n = 0$, $\mu = 0$. $(y_1,\ y_2,\ y_3,\ y_4,\ y_{\frac{13}{3}},\ y_{\frac{9}{2}},\ y_{\frac{14}{3}},\ y_5)^T$, are simultaneously obtained over the sub-interval $[x_0,\ x_5]$, as y_0 is known from the IVP (1). Also for $n = 5$, $\mu = 1$,
$(y_6,\ y_7,\ y_8,\ y_9,\ y_{\frac{28}{3}},\ y_{\frac{19}{2}},\ y_{\frac{29}{3}},\ y_{10})^T$ are simultaneosly obtained over the sub-interval $[x_5,\ x_{10}]$, as y_5 is known from the previous block, where T is the transpose and so on. Hence, the sub-interval do not over-lap and the solution in this manner are more accurate than those obtain in the conventional fashion. For linear problems, we solved equation (1) directly from the start with Gaussian elimination using partial pivoting and for non linear problems, we use a modified Newton-Raphson method.

We summarize the process as follows: The given domain π_N is partition such that
$\pi_N : a = x_0 < x_1 < x_2 \cdots x_n < x_{n+1} < \cdots x_N = b$, and set $h = x_{n+1} - x_n$, $n = 0, 1, \cdots, N-1$.
Step 1: Choose N, for $k = 5$, $h = \frac{(b-a)}{N}$, the number of blocks $\Gamma = \frac{N}{k}$. Using equations $(6-13)$, $n = 0$, $\mu = 0$, the values of $(y_1,\ y_2,\ y_3,\ y_4,\ y_{\frac{13}{3}},\ y_{\frac{9}{2}},\ y_{\frac{14}{3}},\ y_5)^T$ are simultaneously obtained over the sub-interval $[x_0,\ x_5]$, as y_0 is known from IVP (1).
Step 2: For $n = 5$, $\mu = 1$, the values of $(y_6,\ y_7,\ y_8,\ y_9,\ y_{\frac{28}{3}},\ y_{\frac{19}{2}},\ y_{\frac{29}{3}}$ $,\ y_{10})^T$ are simultaneously obtained over the sub-interval $[x_5,\ x_{10}]$, as y_5 is known from the previous block.
Step 3: The process is continued for $n = 10, \cdots, N-5$ and $\mu =$

$2, \cdots, \Gamma$ to obtain approximate solution to equation (1) on sub-intervals $[x_1,\, x_{10}] \cdots [x_{N-5},\, x_N]$.

4.1. NUMERICAL ILLUSTRATIONS

We now illustrate the self starting schemes $(6-13)$ with examples listed below. All calculations and program are carried out with the aid of MATLAB software. We present results obtain in tabular form where YEX is the exact solution, YN is the numerical solution and ER = YEX-YN is the error.

Problem 1:

We consider the initial value problem given by

$$y' = x - y,\; y(0) = 0, \tag{20}$$

whose exact solution is $y(x) = x + e^{-x} - 1$.

Table 2a. Result for problem 1, with $h = 0.1$

X	YEX	YN	ER
0.0	0.00	0.00	0.00
0.1	0.004837418035960	0.004837418035828	0.13201×10^{-14}
0.2	0.018730753077982	0.018730753077878	0.10348×10^{-14}
0.3	0.040818220681718	0.040818220681623	9.52016×10^{-14}
0.4	0.070320046035639	0.070320046035553	8.65280×10^{-14}
0.5	0.106530659712633	0.106530659712555	7.81458×10^{-14}
0.6	0.148811636094027	0.148811636093879	0.14738×10^{-14}
0.7	0.196585303791410	0.196585303791286	0.12312×10^{-14}
0.8	0.249328964117222	0.249328964117108	0.1135×10^{-14}
0.9	0.306569659740599	0.306569659740497	0.10242×10^{-14}
1.0	0.367879441171442	0.367879441171350	9.25926×10^{-14}

Table 2b. Comparison of Errors for problem 1

X	Areo et al(2008)	Proposed scheme
0.1	0.0	1.3×10^{-15}
0.2	0.0	1.0×10^{-15}
0.3	6.0×10^{-10}	9.5×10^{-14}
0.4	2.0×10^{-11}	8.7×10^{-14}
0.5	7.0×10^{-10}	7.8×10^{-14}
0.6	1.0×10^{-10}	1.5×10^{-15}
0.7	8.0×10^{-10}	1.2×10^{-15}
0.8	2.0×10^{-10}	1.1×10^{-15}
0.9	9.0×10^{-10}	1.0×10^{-15}
1.0	4.0×10^{-10}	9.2×10^{-14}

Problem 2:
We next consider the initial value problem given by

$$\cos(x)y' + \sin(x)y = 2\cos^3(x)\sin(x) - 1,\ 0 \le x < \frac{\pi}{2},$$
$$y(\frac{\pi}{4}) = 3\sqrt{2}, \qquad (21)$$

whose exact solution is given by $y(x) = -\frac{1}{2}\cos(x)\cos(2x) - \sin(x) + 7\cos(x)$.

Table 3a. Result for problem 2, when $h = \pi/100$

X	YEX	YN	ER
0	6.500000000000000	6.500000000000000	0.00
$\frac{\pi}{100}$	6.466368032239790	6.466368032239765	2.48690×10^{-14}
$\frac{\pi}{10}$	5.963668177544220	5.963668177544102	1.18128×10^{-13}
$\frac{19\pi}{100}$	5.075246308850297	5.075246308850111	1.86517×10^{-13}
$\frac{\pi}{4}$	4.242640687119286	4.242640687119247	3.90798×10^{-14}
$\frac{7\pi}{25}$	3.751175358038833	3.751175358039095	2.61568×10^{-13}
$\frac{37\pi}{100}$	1.998213828339344	1.998213828339672	3.27738×10^{-13}
$\frac{23\pi}{50}$	-0.054084236584255	-0.054084236583958	2.9667×10^{-13}
$\frac{47\pi}{100}$	-0.290583073193301	-0.290583073193028	2.73614×10^{-13}
$\frac{12\pi}{25}$	-0.527345392958974	-0.527345392958757	2.16604×10^{-13}
$\frac{49\pi}{100}$	-0.763956858258736	-0.763956858258543	1.93289×10^{-13}
$\frac{\pi}{2}$	-1.000000000000000	-0.998923631803195	1.07637×10^{-03}

Table 3b. Comparison of Errors for problem 2

X	Areo et al(2008)	Proposed scheme
$\frac{\pi}{100}$	9.1×10^{-05}	2.4×10^{-14}
$\frac{\pi}{10}$	1.1×10^{-04}	1.2×10^{-13}
$\frac{19\pi}{100}$	1.2×10^{-04}	1.9×10^{-13}
$\frac{\pi}{4}$	1.3×10^{-04}	3.9×10^{-14}
$\frac{7\pi}{25}$	1.4×10^{-04}	2.6×10^{-13}
$\frac{37\pi}{100}$	1.4×10^{-04}	3.3×10^{-13}
$\frac{23\pi}{50}$	1.5×10^{-04}	3.0×10^{-13}
$\frac{47\pi}{100}$	1.5×10^{-04}	2.7×10^{-13}
$\frac{12\pi}{25}$	1.5×10^{-04}	2.2×10^{-13}
$\frac{49\pi}{100}$	1.6×10^{-04}	1.9×10^{-13}
$\frac{\pi}{2}$	1.6×10^{-01}	1.1×10^{-03}

Problem 3:

We also consider the initial value problem given by

$$y' = -y, \; y(0) = 1, \tag{22}$$

whose exact solution is $y(x) = e^{-x}$.

Table 4a. Result for problem 3, with $h = 0.1$

X	YEX	YN	ER
0.0	1.000000000000000	1.000000000000000	0.00
0.1	0.904837418035960	0.904837418035829	1.30340×10^{-13}
0.2	0.818730753077982	0.818730753077880	1.01807×10^{-13}
0.3	0.740818220681718	0.740818220681623	9.43689×10^{-14}
0.4	0.670320046035639	0.670320046035554	8.54872×10^{-14}
0.5	0.606530659712633	0.606530659712556	7.71605×10^{-14}
0.6	0.548811636094027	0.548811636093877	1.49880×10^{-13}
0.7	0.496585303791410	0.496585303791283	1.26399×10^{-13}
0.8	0.449328964117222	0.449328964117107	1.14575×10^{-13}
0.9	0.406569659740599	0.406569659740498	1.01308×10^{-13}
1.0	0.367879441171442	0.367879441171349	9.29811×10^{-14}

Table 4b. Comparison of Errors for problem 3

X	Areo et al(2008)	Proposed scheme
0.1	2.1×10^{-10}	1.3×10^{-13}
0.2	2.2×10^{-10}	1.0×10^{-13}
0.3	6.0×10^{-10}	9.4×10^{-14}
0.4	1.0×10^{-10}	8.5×10^{-14}
0.5	4.0×10^{-09}	7.7×10^{-14}
0.6	7.0×10^{-10}	1.5×10^{-13}
0.7	1.5×10^{-09}	1.3×10^{-13}
0.8	7.0×10^{-10}	1.2×10^{-13}
0.9	1.4×10^{-09}	1.0×10^{-13}
1.0	8.0×10^{-10}	9.3×10^{-14}

Problem 4:

Furthermore, consider the initial value problem given by

$$y' = -y^2,\ y(0) = 1, \tag{23}$$

whose exact solution is $y(x) = \frac{1}{x+1}$.

Table 5a. Result for problem 4, with $h = 0.01$

X	YEX	YN	ER
0.0	1.000000000000000	1.000000000000000	0.00
0.1	0.909090909090909	0.909090909061729	2.91799×10^{-11}
0.2	0.833333333333333	0.833333333296176	3.71577×10^{-11}
0.3	0.769230769230769	0.769230769191403	3.93663×10^{-11}
0.4	0.714285714285714	0.714285714251721	3.39936×10^{-11}
0.5	0.666666666666667	0.666666666637174	2.94922×10^{-11}
0.6	0.625000000000000	0.624999999973872	2.61278×10^{-11}
0.7	0.588235294117647	0.588235294094498	2.31487×10^{-11}
0.8	0.555555555555556	0.555562362598403	6.80704×10^{-06}
0.9	0.526315789473684	0.526324106923278	8.31745×10^{-06}
1.0	0.500000000000000	0.500007506492327	7.50649×10^{-06}

Table 5b. Comparison of Errors for problem 4

X	Areo et al(2008)	Proposed scheme
0.1	2.4×10^{-04}	2.9×10^{-11}
0.2	5.6×10^{-04}	3.7×10^{-11}
0.3	7.1×10^{-04}	3.9×10^{-11}
0.4	8.4×10^{-04}	3.4×10^{-11}
0.5	9.6×10^{-04}	2.9×10^{-11}
0.6	1.1×10^{-04}	2.6×10^{-11}
0.7	1.1×10^{-03}	2.3×10^{-11}
0.8	1.3×10^{-03}	6.8×10^{-06}
0.9	1.5×10^{-03}	8.3×10^{-06}
1.0	1.6×10^{-02}	7.5×10^{-06}

Problem 5:
Lastly, we consider the initial value problem given by

$$y' = 8(x - y) + 1, \; y(0) = 2, \tag{24}$$

whose exact solution is $y(x) = x + 2e^{-8x}$.

Table 6a. Result for problem 5, with $h = 0.01$

X	YEX	YN	ER
0.0000	2.000000000000000	2.000000000000000	0.00
0.1000	0.998657928234443	0.998657928234362	8.07132×10^{-14}
0.2000	0.603793035989311	0.603793035989697	3.86469×10^{-13}
0.3000	0.481435906578825	0.481435906578969	1.44384×10^{-13}
0.4000	0.481524407956732	0.481524407956806	7.30527×10^{-14}
0.5000	0.536631277777468	0.536631277777507	3.86358×10^{-14}
0.6000	0.616459494098040	0.616459494098048	7.54952×10^{-15}
0.7000	0.707395727432966	0.707395727432942	2.34257×10^{-14}
0.8000	0.803323114546348	0.803323114546321	2.70894×10^{-14}
0.9000	0.901493171616753	0.901493171616708	4.57412×10^{-14}
1.0000	1.000670925255805	1.000670925255766	3.95239×10^{-14}

Table 6b. Comparison of Errors for problem 5

X	Areo et al(2008)	Proposed scheme
0.1	1.7×10^{-05}	8.1×10^{-14}
0.2	1.6×10^{-05}	3.9×10^{-13}
0.3	9.3×10^{-06}	1.4×10^{-13}
0.4	4.6×10^{-06}	7.3×10^{-14}
0.5	1.8×10^{-06}	3.9×10^{-14}
0.6	4.2×10^{-07}	7.5×10^{-15}
0.7	1.8×10^{-06}	2.3×10^{-14}
0.8	2.3×10^{-06}	2.7×10^{-14}
0.9	3.8×10^{-07}	4.6×10^{-14}
1.0	3.2×10^{-07}	4.0×10^{-14}

5. CONCLUSION

The numerical schemes generated in this paper are consistent and convergent and can compete favorably with existing schemes.

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