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MORE RESULTS ON THE ALGEBRAIC PROPERTIES OF MIDDLE BOL LOOPS

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ABSTRACT. In this paper, more algebraic properties of middle Bol loop were unveiled. Efforts here paid tremendous attention to the parastrophes of middle Bol loop. In particular, it was shown that (12)−parastrophe of a middle Bol is also a middle Bol loop (MBL). It was further established that (13)− and (123)− parastrophes of middle of Bol loop satisfied respectively the left inverse property (LIP) and right inverse property (RIP) while (23)− parastrophe satisfied a flexible law and commutative square property if it has a middle symmetric property. It was shown that (123)− and (132)− parastrophes have respectively the left symmetric and right symmetric properties. The paper revealed that a commutative (13)− and (123)− parastrophes of (MBL) are Steiner loops.

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1. INTRODUCTION

Let Q be a non -empty set. Define a binary operation " $\cdot$ " on Q, such that, $x \cdot y \in Q$ for all $x, y \in Q$, then the pair $(Q, \cdot)$ is called a groupoid. If in addition to this, the equations: $a \cdot x = b$ and $y \cdot a = b$ have unique solutions $x, y \in Q$ for all $a, b \in Q$ then $(Q, \cdot)$ is called a quasigroup. Let $(Q, \cdot)$ be a quasigroup and there exist a unique element $e \in Q$ called the identity element such that for all $x \in Q, x \cdot e = e \cdot x = x$, then $(Q, \cdot)$ is called a loop. At times, we shall write xy instead of $x \cdot y$ and stipulate that $\cdot$ has lower priority than juxtaposition among factors to be multiplied. Let $(Q, \cdot)$ be a groupoid and a be a fixed element in Q, then the left and right translations L_a and R_a of a are respectively defined by $xL_a = a \cdot x$ and $xR_a = x \cdot a$ for all $x \in Q$. It can now be seen that a groupoid

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$(Q,\cdot)$ is a quasigroup if its left and right translation mappings are permutations. Since the left and right translation mappings of a quasigroup are bijective, then the inverse mappings L_x^{-1} and R_x^{-1} exist.

Let

$$a\backslash b = bL_a^{-1} = aP_b \qquad \text{and} \qquad a/b = aR_b^{-1} = bP_a^{-1}$$

and note that

$$a\backslash b = c \Longleftrightarrow a\cdot c = b \qquad \text{and} \qquad a/b = c \Longleftrightarrow c\cdot b = a.$$

Thus, for any quasigroup $(Q,\cdot)$, we have two new binary operations; right division $(/)$ and left division $(\backslash)$ and middle translation P_a for any fixed $a \in Q$. Consequently, $(Q,\backslash)$ and $(Q,/)$ are also quasigroups.

In a loop $(Q,\cdot)$ with identity element e, the *left inverse element* of $x \in Q$ is the element $xJ_\lambda = x^\lambda \in Q$ such that

$$x^\lambda \cdot x = e$$

while the *right inverse element* of $x \in G$ is the element $xJ_\rho = x^\rho \in G$ such that

$$x \cdot x^\rho = e.$$

It is well known that every quasigroup $(Q\cdot)$ belongs to a set of six quasigroups, called Adjugates by Fisher, Yates [1]; Parastrophes by Belousov [2] and Conjugates by Stein. A binary groupoid (Q,A) with a binary operation "A" such that in the equality $A(x_1,x_2) = x_3$, knowledge of any 2 elements of x_1, x_2, x_3 uniquely the third one, is called a binary quasigroup. It follows that any quasigroup (Q,A), associate $(3!-1)$ quasigroups called parastrophes of quasigroup (Q,A); $A(x_1,x_2) = x_3 \Longleftrightarrow$ $A^{(12)}(x_2,x_1) = x_3 \Longleftrightarrow A^{(13)}(x_3,x_2) = x_1 \Longleftrightarrow A^{(23)}(x_1,x_2) = x_2 \Longleftrightarrow A^{(123)}(x_2,x_3) = x_1 \Longleftrightarrow A^{(132)}(x_3,x_1) = x_2$.[see (Shcherbacov [3], 2008)]

Middle Bol loop were first studied in the work of V. D. Belousov [2], where he gave identity (0.3) characterizing loops that satisfy the universal anti-automorphic inverse property. After this beautiful characterization by Belousov and the laying of foundations for a classical study of this structure, Gwaramija in [4] proved that a loop $(Q,\circ)$ is middle Bol if there exist a right Bol loop $(Q,\cdot)$ such that $x\circ y = (y\cdot xy^{-1})y$ for all $x,y,\in Q$. If $(Q,\circ,//,\backslash\backslash)$ is a middle Bol loop and $(Q,\cdot,/,\backslash)$ is the corresponding right Bol loop, then

$$x\circ y = y^{-1}\backslash x \qquad \text{and} \qquad x\cdot y = y//x^{-1} \tag{0.1}$$

for every $x, y \in Q$. Also, if $(Q, \circ, //, \backslash\backslash)$ is a middle Bol loop and $(Q, \cdot, /, \backslash)$ is the corresponding left Bol loop, then

$$x \circ y = x/y^{-1} \qquad \text{and} \qquad x \cdot y = x//y^{-1} \tag{0.2}$$

for every $x, y \in Q$. Grecu [5] showed that right multiplication group of a middle Bol loop coincides with the left multiplication group of the corresponding right Bol loop. After then, middle Bol loops resurfaced in literature in 1994 and 1996 when Syrbu [6, 7] considered them in-relation to the universality of the elasticity law. In 2003, Kuznetsov [8], while studying gyrogroups (a special class of Bol loops) established some algebraic properties of middle Bol loop and designed a method of constructing a middle Bol loop from a gyrogroup. In 2010, Syrbu [9] studied the connections between structure and properties of middle Bol loops and of the corresponding left Bol loops. It was noted that two middle Bol loops are isomorphic if and only if the corresponding left (right) Bol loops are isomorphic, and a general form of the autotopisms of middle Bol loops was deduced. Relations between different sets of elements, such as nucleus, left (right,middle) nuclei, the set of Moufang elements, the center, e.t.c. of a middle Bol loop and left Bol loops were established. In 2012, Grecu and Syrbu [10] proved that two middle Bol loops are isotopic if and only if the corresponding right (left) Bol loops are isotopic. In 2012, Drapal and Shcherbacov [11] rediscovered the middle Bol identities in a new way. In 2013, Syrbu and Grecu [12] established a necessary and sufficient condition for the quotient loop of a middle Bol loop and of its corresponding right Bol loop to be isomorphic. In 2014, Grecu and Syrbu [13] established that the commutant (centrum) of a middle Bol loop is an AIP-subloop and gave a necessary and sufficient condition when the commutant is an invariant under the existing isostrophy between middle Bol loop and the corresponding right Bol loop. Osoba et al. in [14] and [15] investigate further the multiplication group of middle Bol loop in relation to left Bol loop and the relationship of multiplication groups and isostrophic quasigroups respectively while Jaiyéọlá [16, 17] studied second Smarandache Bol loops. The Smarandache nuclei of second Smarandache Bol loops was further studied by Osoba [18]. For more on quasigroups and loops, see Jaiyéọlá [25] and Pflugfelder [19].

In [20], Jaiyéọlá et al. studied the holomorphic structure of Middle Bol loops and showed that the holomorph of a commutative loop is a commutative middle Bol loop if and only if the loop is a middle

Bol loop and its automorphism group is abelian. Adeniran et al. [21, 22], Jaiyéọlá and Popoola [23] studied generalised Bol loops.

Jaiyéọlá et al. in [24], in furtherance to their exploit obtained new algebraic identities of middle Bol loop, where necessary and sufficient conditions for a bi-variate mapping of a middle Bol loop to have RIP, LIP, RAIP, LAIP and flexible property were presented. Additional algebraic properties of middle Bol loop were announced in Jaiyéọlá et al. [26]. The new algebraic connections between right and middle Bol loops and their cores was presented by Osoba and Jaiyéọlá in [27].

2. PRELIMINARY

Using the operations $(\backslash)$ and $(/)$ earlier described in the introduction above, the definition of a loop can be restated as follows.

Definition 1: A *loop* $(Q, \cdot, /, \backslash, e)$ is a set G together with three binary operations $(\cdot)$, $(/)$, $(\backslash)$ and one nullary operation e such that

- **(i):** $a \cdot (a\backslash b) = b$, $(b/a) \cdot a = b$ for all $a, b \in Q$,
- **(ii):** $a\backslash(a \cdot b) = b$, $(b \cdot a)/a = b$ for all $a, b \in Q$ and
- **(iii):** $a\backslash a = b/b$ or $e \cdot a = a$ for all $a, b \in Q$.

We also stipulate that $(/)$ and $(\backslash)$ have higher priority than $(\cdot)$ among factors to be multiplied. For instance, $a \cdot b/c$ and $a \cdot b\backslash c$ stand for $a(b/c)$ and $a(b\backslash c)$ respectively.

Definition 2: A groupoid (quasigroup) $(Q, \cdot)$ is said to have the

1. left inverse property (LIP) if there exists a mapping $J_\lambda : x \mapsto x^\lambda$ such that $x^\lambda \cdot xy = y$ for all $x, y \in Q$.
2. right inverse property (RIP) if there exists a mapping $J_\rho : x \mapsto x^\rho$ such that $yx \cdot x^\rho = y$ for all $x, y \in Q$.
3. inverse property (IP) if it has both the LIP and RIP. for all $x, y \in Q$.
4. flexibility or elasticity if $xy \cdot x = x \cdot yx$ holds for all $x, y \in Q$.
5. cross inverse property (CIP) if there exist mapping $J_\lambda : x \mapsto x^\lambda$ or $J_\rho : x \mapsto x^\rho$ such that $xy \cdot x^\rho = y$ or $x \cdot yx^\rho = y$ or $x^\lambda \cdot yx = y$ or $x^\lambda y \cdot x = y$ for all $x, y \in Q$.

Definition 3: A loop $(Q, \cdot)$ is said to be

1. commutative loop if $R_x = L_x$ and a commutative square loop if $R_x^2 = L_x^2$ for all $x, y \in Q$
2. an automorphic inverse property loop (AIPL) if $(xy)^{-1} = x^{-1}y^{-1}$ for all $x, y \in Q$
3. an anti- automorphic inverse property loop (AAIPL) if $(xy)^{-1} = y^{-1}x^{-1}$ for all $x, y \in Q$.

Definition 4: A groupoid (quasigroup) $(Q, \cdot)$ is

(1) right symmetric if $yx \cdot x = y$ for all $x, y \in Q$
(2) left symmetric if $x \cdot xy = y$ for all $x, y \in Q$
(3) middle symmetric if $x \cdot yx = y$ or $xy \cdot x = y$ for all $x, y \in Q$
(4) idempotent if $x \cdot x = x$ for all $x \in Q$

Definition 5: [19] If a totally symmetric quasigroup $(Q, \cdot)$ is a loop, then it is called Steiner loop.
Theorem 1: [19] A quasigroup $(Q, \cdot)$ is totally symmetric if and only if it is commutative ($xy = yx$ for all $x, y \in Q$) and is right or left symmetric.
Theorem 2: [19] A loop $(Q, \cdot)$ is totally symmetric if and only if $(Q, \cdot)$ is an IP loop of exponent 2.
Corollary 1: [19] Every T.S. quasigroup is a commutative I.P. quasigroup.
A loop $(Q, \cdot)$ is called a middle Bol loop if is satisfies the identity

$$(x/y)(z\backslash x) = x(zy\backslash x) \text{ or } (x/y)(z\backslash x) = (x/(zy))x \tag{0.3}$$

Furtherance to earlier studies, this paper present the algebraic properties of middle Bol loop using its parastrophes. We shown that $(12)-$parastrophe of a middle Bol is also a middle Bol loop and the other four parastrophes are not. Since the other four parastrophes of Q are not a middle Bol loop, we further investigate the algebraic properties of the four parastrophes to obtain some of the related properties and identities they shares with the underline structure. Interestingly, it was found that $(13)-$ and $(123)-$ parastrophes of middle of Bol loop satisfies left and right inverse properties respectively.

3. MAIN RESULTS

Theorem 3.1: Let $(Q, \cdot, /, \backslash)$ be a middle Bol loop. Then, $(12)-$parastrophe of Q is a middle Bol loop.
Proof: Let operation " $\circ$ " denotes (12)-parastrophe of Q. Let

$$a \cdot b = x(zy\backslash x) \tag{0.4}$$

in equation (0.3) where $a = x/y \Rightarrow x = ay \Rightarrow y \circ_{(12)} a = x \Rightarrow a = y\backslash^{(12)} x$.
And $b = z\backslash x \Rightarrow zb = x \Rightarrow b \circ_{(12)} z = x \Rightarrow b = x/^{(12)} z$. Put a and b into equation (0.4), we have

$$(y\backslash^{(12)} x) \cdot (x/^{(12)} z) = x(zy\backslash x) \tag{0.5}$$

Doing (12)−permutation on equation (0.5), gives

$$(x/^{(12)}z)\circ_{(12)}(y\backslash^{(12)}x)=\big((y\circ_{(12)}z)\backslash x\big)\circ_{(12)}x \tag{0.6}$$

Let

$$(y\circ_{(12)}z)\backslash x=c\Rightarrow (y\circ_{(12)}z)\cdot c=x\Rightarrow c\circ_{(12)}(z\circ_{(12)}y)=x\Rightarrow$$
$$c=x/^{(12)}(y\circ_{(12)}z)$$

Put c into equation (0.6), gives

$$(x/^{(12)}z)\circ_{(12)}(y\backslash^{(12)}x)=\big(x/^{(12)}(y\circ_{(12)}z)\big)\circ_{(12)}x$$

Thus, $(x/^{(12)}z)\circ_{(12)}(y\backslash^{(12)}x)=\big(x/^{(12)}(y\circ_{(12)}z)\big)\circ_{(12)}x$ is an identity of middle Bol loop.

Theorem 2: In middle Bol loop $(Q,\cdot,/,\backslash)$, the following hold in (13)−parastrophe of Q

(1) $(L_x,L_x^{-1},L_x^{-1}P_x^{-1})\in AATP(Q,/^{(13)})$
(2) $t^\lambda\cdot_{(13)}(t\cdot_{(13)}y)=y$ that is left inverse property for all $t\in Q$
(3) $(x\cdot_{(13)}y)/^{(13)}x^\rho=x/^{(13)}(x\backslash^{(13)}y^\lambda)$
(4) $L_xP_x=L_x^{-1}P_x^{-1}$
(5) $x\cdot_{(13)}x=x/^{(13)}x^\rho$ that is right self inverse property for all $x\in Q$
(6) $y=(y^\lambda)^\lambda$ for all $y\in Q$
(7) $L_xR_{x^\lambda}^{-1}=\lambda L_{x^\lambda}P_x^{-1}$
(8) $(y\cdot_{(13)}x)\cdot_{(13)}y=x$ that is middle symmetric property if $|Q|=2$

Proof: Let

$$a\cdot b=x(zy\searrow x) \tag{0.7}$$

in equation (0.3) where $a=x/y\Rightarrow x=ay\Rightarrow$

$$a=x\cdot_{(13)}y \tag{0.8}$$

and $b=z\backslash x\Rightarrow zb=x\Rightarrow$

$$z=x\cdot_{(13)}b\Rightarrow x\backslash^{(13)}z=b \tag{0.9}$$

Let $c=zy$ in identity (0.3), this implies that $z=c\cdot_{(13)}y\Rightarrow c=y/^{(13)}z$.

Also, let $d=c\backslash x\Rightarrow c\cdot d=x\Rightarrow x\cdot_{(13)}d=c\Rightarrow d=x\backslash^{(13)}c$. Then, putting c into d give

$$d=x\backslash^{(13)}(y/^{(13)}z) \tag{0.10}$$

Let $t=x\cdot d\Rightarrow x=t\cdot_{(13)}d\Rightarrow t=x/^{(13)}d\Rightarrow$

$$t=x/^{(13)}\big[x\backslash^{(13)}(z/^{(13)}y)\big] \tag{0.11}$$

Now, $a \cdot b = t$ according to identity (0.3) which implies that $t \cdot_{(13)} b = a \Rightarrow a/^{(13)} b = t$. Substituting (0.8), (0.9) and (0.11) for a, b and t respectively, we have

$$(x \cdot_{(13)} y)/^{(13)}(x\backslash^{(13)} z) = x/^{(13)}\big[x\backslash^{(13)}(z/^{(13)}y)\big] \qquad (0.12)$$

(1) From equation (0.12), we have $yL_x/^{(13)}zL_x^{-1} = (z/^{(13)}y)L_x^{-1}P_x^{-1} \Rightarrow$

$(L_x, L_x^{-1}, L_x^{-1}P_x^{-1}) \in AATP(Q, /^{(13)})$

(2) Put $x = e$ in (0.12) the identity element in Q, we have

$$ey/^{(13)}z = e/^{(13)}(z/^{(13)}y) \Rightarrow y/^{(13)}z = (z/^{(13)}y)^{\lambda} \Rightarrow y = (z/^{(13)}y)^{\lambda} \cdot_{(13)} z \qquad (0.13)$$

Let $t = z/^{(13)}y \Rightarrow z = t \cdot_{(13)} y$, put z and t in (0.13), give $y = t^{\lambda} \cdot_{(13)} (t \cdot_{(13)} y)$ for all $t \in Q$

(3) Set $z = e$ in (0.12), we have $(x \cdot_{(13)} y)/x^{\rho} = x/^{(13)}(x\backslash^{(13)}y^{\lambda})$

(4) $z = x$ in (0.12), we have $x \cdot_{(13)} y = x/^{(13)}\big(x\backslash^{(13)}(x/^{(13)}y) \Rightarrow yL_x = yP_x^{-1}L_x^{-1}P_x^{-1} \Rightarrow L_xP_x = L_x^{-1}P_x^{-1}$

(5) Put z=y in (0.12), give $x/^{(13)}(x\backslash^{(13)}x) = x \cdot_{(13)} y/^{(13)}(x\backslash^{(13)}y) \Rightarrow x/^{(13)}x^{\rho} = (x \cdot_{(13)} y)/^{(13)}(x\backslash^{(13)}y)$. Let $y = x$ then, we have $x \cdot_{(13)} x = x/^{(13)}x^{\rho}$

(6) $z = x = e$ in (0.12) give $(y^{\lambda})^{\lambda} = y$

(7) Apply the LIP to equality (0.12), give $x \cdot_{(13)} y/^{(13)}(x^{\lambda} \cdot_{(13)} z) = x/^{(13)}(x^{\lambda} \cdot_{(13)} (z/^{(13)}y))$, set $z = e$ get $x \cdot_{(13)} y/^{(13)}x^{\lambda} = x/^{(13)}(x^{\lambda} \cdot_{(13)} y^{\lambda}) \Rightarrow yL_xR_{x^{\lambda}}^{-1} = y\lambda L_{x^{\lambda}}P_x^{-1} \Rightarrow L_xR_{x^{\lambda}}^{-1} = \lambda L_{x^{\lambda}}P_x^{-1}$

(8) Recall equation 7, $L_xR_{x^{\lambda}}^{-1} = \lambda L_x^{-1}P_x^{-1}$, then use 4 and LIP to get $L_xR_{x^{\lambda}}^{-1} = \lambda L_x^{-1}P_x^{-1} = \lambda L_xP_x \Rightarrow L_xR_{x^{\lambda}}^{-1} = \lambda L_xP_x \Rightarrow tL_xR_{x^{\lambda}}^{-1} = t\lambda L_xP_x \Rightarrow (x \cdot_{(13)} t)/^{(13)}x^{\lambda} = (x \cdot_{(13)} t^{\lambda})\backslash^{(13)}x$. Since $|Q| = 2$, we have $(x \cdot_{(13)} t)/^{(13)}x = (x \cdot_{(13)} t)\backslash^{(13)}x$. Replacing $(x \cdot_{(13)} t)$ with s for all $s \in Q$, give $s/^{(13)}x = s\backslash^{(13)}x \Rightarrow s \cdot_{(13)} (s/^{(13)}x) = x$. Let $s/^{(13)}x = y \Rightarrow s = y \cdot_{(13)} x$ for all $y \in Q$ give $(y \cdot_{(13)} x) \cdot_{(13)} y = x$.

Corollary 1: A commutative $(13)-$parastrophe of a middle Bol loop $(Q, \cdot, /, \backslash)$ satisfies an inverse property.
Proof: Apply the commutative property to Theorem 2(2), it give an inverse property

Corollary 2: A commutative $(13)-$parastrophe of a middle Bol loop $(Q, \cdot, /, \backslash)$ is a totally symmetric.
Proof: Apply the commutative property to Theorem 2(8).

Corollary 3: A commutative $(13)-$parastrophe of a middle Bol loop $(Q,\cdot,/,\backslash)$ is a Steiner loop.
Proof: Result follows from Corollary 2.

Theorem 3: In middle Bol loop $(Q,\cdot,/,\backslash)$, the following hold in $(23)-$parastrophe of Q

(1) $(L_x^{-1},R_x,R_xL_x^{-1})\in AATP(Q,\backslash^{(23)})$ for all $x\in Q$
(2) $(z\cdot_{(23)}t)\cdot_{(23)}t=z$ for all $z,t\in Q$
(3) if $Q^{(23)}$ is middle symmetric then, $x\cdot_{(23)}(z\cdot_{(23)}x)=(x\cdot_{(23)}z)\cdot_{(23)}x$ that is, flexible law
(4) $|x|=2$ for all $x\in Q$
(5) $R_x^{-1}P_x=R_xL_x^{-1}$
(6) $R_xL_x^{-1}=\lambda R_xL_{x^\lambda}^{-1}$

Proof: Let

$$a\cdot b=x(zy\backslash x) \qquad (0.14)$$

in an identity (0.3), where

$$a=x/y\Rightarrow x=a\cdot y\Rightarrow y=a\cdot_{(23)}x\Rightarrow a=y/^{(23)}x \qquad (0.15)$$

and

$$b=z\backslash x\Rightarrow z\backslash b=x\Rightarrow z\cdot_{(23)}x=b \qquad (0.16)$$

Let $c=zy$ in identity (0.3), then $z\cdot_{(23)}c=y\Rightarrow c=z\backslash^{(23)}y$. Let $d=c\backslash x\Rightarrow c\cdot_{(23)}d=x\Rightarrow c\cdot_{(23)}x=d$, putting c into d gives

$$d=(z\backslash^{(23)}y)\cdot_{(23)}x. \qquad (0.17)$$

Also, let $t=x\cdot d\Rightarrow x\cdot_{(23)}t=d\Rightarrow t=x\backslash^{(23)}d$, put d into t

$$t=x\backslash^{(23)}\left[(z\backslash^{(23)}y)\cdot_{(23)}x\right] \qquad (0.18)$$

Now, going by the identity (0.3), we have $a\cdot b=t\Rightarrow a\cdot_{(23)}t=b\Rightarrow a\backslash^{(23)}b=t$. Then, substituting for a,b and t in $a\backslash^{(23)}b=t$, we have

$$(y/^{(23)}x)\backslash^{(23)}(z\cdot_{(23)}x)=x\backslash^{(23)}\left[(z\backslash^{(23)}y)\cdot_{(23)}x\right] \qquad (0.19)$$

(1) this follows from equation (0.19),

$$yR_x^{-1}\backslash^{(23)}zR_x=(z\backslash^{(23)}y)R_xL_x^{-1}\Rightarrow(R_x^{-1},R_x,R_xL_x^{-1})\in AATP(Q,\backslash)$$

(2) put $x=e$ the identity element in (0.19), give $y\backslash^{(23)}z=z\backslash^{(23)}y\Rightarrow y\cdot_{(23)}(z\backslash^{(23)}y)=z$. Let $t=z\backslash^{(23)}y\Rightarrow z\cdot_{(23)}t=y$. Put y into the last equality to get $(z\cdot_{(23)}t)\cdot_{(23)}t=z$ for any $t\in Q$

(3) put $y=x$ in (0.19), we have $z\cdot_{(23)}x=x\backslash^{(23)}[(z\backslash^{(23)}x)\cdot_{(23)}x]\Rightarrow x\cdot_{(23)}(z\cdot_{(23)}x)=(z\backslash^{(23)}x)\cdot_{(23)}x\Rightarrow zR_xL_x=zP_xR_x$. Use middle symmetric as $L_x=P_x$. Then, we have $zR_xL_x=zL_xR_x$ i.e $x\cdot_{(23)}(z\cdot_{(23)}x)=(x\cdot_{(23)}z)\cdot_{(23)}x$

(4) put $z = y$ and $y = x$ in (0.19), we have $x \cdot_{(23)} x = e$ that is $|x| = 2$ for all $x \in Q$
(5) put $z = e$, the identity element in (0.19), we have $(y/^{(23)}x)\backslash^{(23)}x = x\backslash^{(23)}(y \cdot^{(23)} x) \Rightarrow yR_x^{-1}P_x = yR_xL_x^{-1} \Rightarrow R_x^{-1}P_x = R_xL_x^{-1}$
(6) $y = e$ in (0.19), give $x^\lambda\backslash^{(23)}(z \cdot_{(23)} x) = x\backslash^{(23)}(z^\rho \cdot_{(23)} x)$, set $z = z^\lambda$, then $x^\lambda\backslash^{(23)}(z^\lambda \cdot_{(23)} x) = x\backslash^{(23)}(z \cdot_{(23)} x)$ or $zR_xL_x^{-1} = z\lambda R_xL_{x^\lambda}^{-1} \Rightarrow R_xL_x^{-1} = \lambda R_xL_{x^\lambda}^{-1}$

Corollary 4: A commutative $(23)-$parastrophe of a middle Bol loop $(Q, \cdot, /, \backslash)$ is totally symmetric.
Proof: Using Theorem 3, with the commutative property give the desire result.

Corollary 5: In middle Bol loop $(Q, \cdot, /, \backslash)$, if $(23)-$parastrophe of Q is commutative loop, then it is a Steiner loop.
Proof: Using Theorem 3 with Corollary 4.

Corollary 6: If $(23)-$parastrophe of a middle Bol loop $(Q, \cdot, /, \backslash)$ has middle symmetric, then $L_x^2 = R_x^2$ for all $x \in Q$.
Proof: Use 5 and 6 in Theorem 3, we have $R_xL_x^{-1} = \lambda R_xL_{x^\lambda}^{-1} = R_x^{-1}P_x$, use the middle symmetric property as $L_x = P_x$, give $\lambda R_xL_{x^\lambda}^{-1} = R_x^{-1}L_x$. Then, for all $s \in Q$, we have

$$s\lambda R_xL_{x^\lambda}^{-1} = sR_x^{-1}L_x \Rightarrow x^\lambda\backslash^{(23)}(s^\lambda \cdot_{(23)} x) =$$
$$x \cdot_{(23)} (s/^{(23)}x) \Rightarrow x^\lambda \cdot_{(23)} (x \cdot_{(23)} (s/^{(23)}x)) = s^\lambda \cdot_{(23)} x$$

Let $s/^{(23)}x = t \Rightarrow s = t \cdot_{(23)} x$ Then, $x^\lambda \cdot_{(23)} (x \cdot_{(23)} t) = (t \cdot_{(23)} x)^\lambda \cdot_{(23)} x$. Since in Theorem 3(4), $|x| = 2$ for all $x \in Q$, we have $x \cdot_{(23)} (x \cdot_{(23)} t) = (t \cdot_{(23)} x) \cdot_{(23)} x \Rightarrow tL_xL_x = tR_xR_x \Rightarrow L_x^2 = R_x^2$ for all $x \in Q$

Theorem 4: In middle Bol loop $(Q, \cdot, /, \backslash)$, the following hold in $(123)-$parastrophe of Q
(1) $(L_x^{-1}, R_x, R_x^{-1}P_x) \in AATP(Q, \backslash^{(123)})$
(2) $(y \cdot_{(123)} t) \cdot_{(123)} t^\rho = y$, i.e right inverse property
(3) $(z/^{(123)}x)[x^\lambda\backslash^{(123)}x] = z \cdot_{(123)} x$
(4) $L_{x^\lambda}^{-1} = \rho L_x^{-1}P_x$
(5) $P_x^{-1}R_x = L_x^{-1}P_x$
(6) $(x \cdot_{(123)} t) \cdot_{(123)} x = (x\backslash^{(123)}t)\backslash^{(123)}x$ for all $x, t \in Q$

Proof: Let $a \cdot b = x \cdot (zy\backslash x)$ in equation 0.3 where

$$a = x/y \Rightarrow a \cdot y = x \Rightarrow y \cdot_{(123)} x = a \qquad (0.20)$$

$$b = z\backslash x \Rightarrow z \cdot b = x \Rightarrow b \cdot_{(123)} x = z \Rightarrow b = z/^{(123)}x \qquad (0.21)$$

Let $c = z \cdot y$ in equation 0.3, then, we have $y \cdot_{(123)} c = z \Rightarrow c = y\backslash^{(123)} z$. Also, let $d = c\backslash x \Rightarrow c \cdot d = x \Rightarrow d \cdot_{(123)} x = c \Rightarrow d = c/^{(123)} x$. Put c into d to get

$$d = (y\backslash^{(123)} z)/^{(123)} x \tag{0.22}$$

Next, let $t = x \cdot d \Rightarrow d \cdot_{(123)} t = x \Rightarrow t = d\backslash^{(123)} x$. Put d into t give

$$t = \big[(y\backslash^{(123)} z)/^{(123)} x\big]\backslash^{(123)} x \tag{0.23}$$

It is follows from the identity (0.3) $a \cdot b = t \Rightarrow b \cdot_{(123)} t = a \Rightarrow b\backslash^{(123)} a = t$. Substituting for a, b and t in $b\backslash^{(123)} a = t$, give the equality

$$(z/^{(123)} x)\backslash^{(123)}(y \cdot_{(123)} x) = \big[(y\backslash^{(123)} z)/^{(123)} x\big]\backslash^{(123)} x \tag{0.24}$$

(1) From equation (0.24) , we have

$$zR_x^{-1}\backslash^{(123)} yR_x = (y\backslash^{(123)} z)R_x^{-1}P_x \Rightarrow$$
$$(Rx^{-1}, R_x, R_x^{-1}P_x) \in AAPT(Q, \backslash^{(123)})$$

(2) Set $x = e$ the identity element in equation (0.24), we have $((z/^{(123)} e)\backslash^{(123)}(y\cdot_{(123)} e) = \big((y\backslash^{(123)} z)/^{(123)} e\big)\backslash^{(123)} e \Rightarrow z\backslash^{(123)} y =$

$(y\backslash^{(123)} z)^{\rho} \Rightarrow z \cdot_{(123)} (y\backslash^{(123)} z)^{\rho} = y$. Let $t = y\backslash^{(123)} z \Rightarrow y\cdot_{(123)} t = z$ for any $t \in Q$, this implies that $(y\cdot_{(123)} t)\cdot_{(123)} t^{\rho} = y$.

(3) Set $y = z$ in equation (0.24), we have $(z/^{(123)} x)\backslash^{(123)}(z \cdot_{(123)} x) = x^{\lambda}\backslash^{(123)} x \Rightarrow z/^{(123)} x[x^{\lambda}\backslash(123) x = z \cdot_{(123)} x]$

(4) put $z = e$ in equation (0.24), to get $x^{\lambda}\backslash^{(123)} y \cdot_{(123)} x) = (y^{\rho}/^{(123)} x)\backslash^{(123)} x$ or $yR_x L_{x^{\lambda}}^{-1} = y\rho R_x^{-1}P_x \Rightarrow R_x L_{x^{\lambda}}^{-1} = \rho R_x^{-1} P_x$

(5) put $z = x$ in equation (0.24), give $y\cdot_{(123)} x = \big((y\backslash^{(123)} x)/^{(123)} x\big)\backslash^{(123)} x \Rightarrow yR_x = P_x R_x^{-1} P_x \Rightarrow P_x^{-1} R_x = R_x^{-1} P_x$

Corollary 7: A commutative $(123)-$parastrophe of a middle Bol loop $(Q, \cdot, /, \backslash)$ satisfies the inverse property.
Proof: Apply the commutative property to Theorem 4, it give an inverse property

Corollary 8: A commutative $(123)-$parastrophe of a middle Bol loop $(Q, \cdot, /, \backslash)$ is totally symmetric if $|Q^{(123)}| = 2$
Proof: Using 3, give right symmetric property in Theorem 4, with the commutative property, gives the desire result.

Corollary 9: In middle Bol loop $(Q, \cdot, /, \backslash)$, if $(123)-$parastrophe of Q is commutative loop, then it is a Steiner loop.

Proof: Using Theorem 4 with Corollary 8.

Corollary 10: Let $(Q, \cdot, /, \backslash)$ middle Bol loop. If the $(123)-$parastrophe of Q is a commutative loop of exponent 2, then Q is a Moufang loop.
Proof: Using the Corollary 7 on equation (0.24), give $(z \cdot_{(123)} x^{-1})^{-1} \cdot_{(123)} (y \cdot_{(123)} x) = y^{-1} \cdot_{(123)} z \cdot_{(123)} x^{-1})^{-1} \cdot_{(123)} x$. Since $|Q^{(123)}| = 2$, then we have $(z \cdot_{(123)} x) \cdot_{(123)} (y \cdot_{(123)} x) = ((y \cdot_{(123)} z) \cdot_{(123)} x) \cdot_{(123)} x \Rightarrow$. Since $Q^{(123)}$ is commutative, then we have $(x \cdot_{(123)} z) \cdot_{(123)} (y \cdot_{(123)} x) = x \cdot_{(123)} ((z \cdot_{(123)} y) \cdot_{(123)} x)$

Corollary 11: Let $(Q, \cdot, /, \backslash)$ middle Bol loop. If the $(123)-$parastrophe of Q is an automorphic inverse property loop, then Q is a commutative loop.
Proof: By Corollary 7, $Q^{(123)}$ has an inverse property. Then, re-written (123)-parastrophe in equation (0.24), give $(z \cdot_{(123)} x^{-1})^{-1} \cdot_{(123)} (y \cdot_{(123)} x) = ((y^{-1}_{(123)} z) \cdot_{(123)} x)^{-1} \cdot_{(123)} x$, set $x = e$ give

$$z^{-1} \cdot_{(123)} y = (y^{-1} \cdot_{(123)} z)^{-1} \tag{0.25}$$

Set $y^{-1} = y$ in equation (0.25), we have $z^{-1} \cdot_{(123)} y^{-1} = (y \cdot_{(123)} z)^{-1}$

Theorem 5: In middle Bol loop $(Q, \cdot, /, \backslash)$, the following hold in $(132)-$parastrophe of Q

(1) $(L_x, L_x^{-1}, L_x R_x^{-1}) \in AATP(Q, /^{(132)})$ for all $x \in Q$
(2) $z = t \cdot_{123} (t \cdot_{123} z)$ i.e left symmetric property
(3) $(x \cdot_{(132)} z) \cdot_{(132)} x = x \cdot_{(132)} (x/^{(132)} z)$ or $P_x^{-1} L_x = L_x R_x$
(4) $(x \cdot_{(132)} z)/^{(132)} x^{\rho} = (x \cdot_{(132)} z^{\lambda})/^{(132)} x$ or $LxR_{x^{\rho}}^{-1} = \lambda R_x^{-1} L_x$
(5) $R_x^{-1} = P_x^{-1}$

Proof: Let $a \cdot b = x \cdot (zy \backslash x)$ in equation 0.3 where

$$x/y = a \Rightarrow x = a \cdot y \Rightarrow x \cdot_{(132)} a = y \Rightarrow a = x \backslash^{(132)} y \tag{0.26}$$

and

$$z \backslash x = b \Rightarrow z \cdot b = x \Rightarrow x \cdot_{(132)} z = b \tag{0.27}$$

Let $c = z \cdot y \Rightarrow c \cdot_{(132)} z = y \Rightarrow c = y/^{(132)} z$. Also, let $d = c \backslash x \Rightarrow c \cdot d = x \Rightarrow x \cdot_{(132)} c = d$. Thus, $d = x \cdot_{(132)} (y/^{(132)} z)$ Let $t = x \cdot d \Rightarrow t \cdot_{(132)} x = d \Rightarrow t = d/^{(132)} x$. Hence, putting d into t, we have

$$t = \left[x \cdot_{(132)} (y/^{(132)} z)\right]/^{(132)} x \tag{0.28}$$

Also, let $a \cdot b = t \Rightarrow t \cdot_{(132)} a = b \Rightarrow b/^{(132)} a = t$ Thus,

$$(x \cdot_{(132)} z)/(x \backslash^{(132)} y) = \left[x \cdot_{(132)} (y/^{(132)} z)\right]/^{(132)} x \tag{0.29}$$

(1) From (0.29), we have $zL_x/^{(132)}yL_x^{-1} = (y/^{(132)}z)L_xR_x^{-1} \Rightarrow (L_x, L_x^{-1}, L_xR_x^{-1}) \in AATP(Q, /^{(132)})$ for all $x \in Q$
(2) put $x = e$ in (0.29), give $z/^{(132)}y = y/^{(132)}z = t\cdot_{(132)}(t\cdot_{(132)}z)$ by setting $t = y/^{(132)}z \Rightarrow y = (z \cdot_{(132)} t)$
(3) put $y = x$ in (0.29), to get $(x \cdot_{(132)} z) \cdot_{(132)} x = x \cdot_{(132)} (x/^{(132)}z) \Rightarrow zP_x^{-1}L_x = zL_xR_x \Rightarrow P_x^{-1}L_x = L_xR_x$ for all $x \in Q$
(4) set $y = e$ in (0.29), we have $(x \cdot_{(132)} z)/^{(132)}x^\rho = (x \cdot_{(132)} z^\lambda)/^{(132)}x$ or $zL_xR_{x^\rho}^{-1} = z\lambda L_xR_x^{-1} \Rightarrow L_xR_{x^\rho}^{-1} = \lambda L_xR_x^{-1}$
(5) set $z = e$, we have $x/^{(132)}(x\backslash^{(132)}y) = (x \cdot_{(132)} y)/^{(132)}x \Rightarrow yL_xP_x^{-1} = yL_xR_x^{-1} \Rightarrow L_x^{-1}P_x^{-1} = L_xR_x^{-1}$ using the left symmetric property implies that $L_x = L_x^{-1}$. So, $R_x^{-1} = P_x^{-1}$ for all $x \in Q$

Corollary 12: Let $(Q, \cdot, /, \backslash)$ be a middle Bol loop. Then, (132)−parastrophe of Q is totally symmetric if it has a commutative property
Proof Using Theorem 5.

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References

[1] R. O. Fisher and F. Yates, *The 6x6 latin squares*, Proc. Soc 30, 429-507,1934.
[2] V. D. Belousov *Foundations of the theory of quasigroups and loops*, (Russian) Izdat. "Nauka", Moscow 223pp, 1967
[3] V. A. Shcherbacov, *A-nuclei and A-centers of quasigroup*, Institute of mathematics and computer Science Academiy of Science of Moldova Academiei str. 5, Chisinau, MD -2028, Moldova ,2011.
[4] A. Gvaramiya, *On a class of loops* (Russian), Uch. Zapiski MAPL. 375, 25-34, 1971.
[5] I. Grecu, *On multiplication groups of isostrophic quasigroups*, Proceedings of the Third Conference of Mathematical Society of Moldova IMCS-50, August 19-23, 2014, Chisinau, Republic of Moldova, 78-81, 2014.
[6] P. Syrbu, *Loops with universal elasticity*, Quasigroups Related Systems (1), 57–65, 1994.
[7] P. Syrbu, *On loops with universal elasticity*, Quasigroups Related Systems (3), 41–54, 1996.

[8] Kuznetsov, E, *Gyrogroups and left gyrogroups as transversals of a special kind*, Algebraic and discrete Mathematics 3, 54–81, 2003.

[9] P. Syrbu, *On middle Bol loops*, ROMAI J. 6 (2), 229–236, 2010.

[10] I. Grecu and P. Syrbu, *On Some Isostrophy Invariants of Bol Loops*, Bulletin of the Transilvania University of Brasov, Series III: Mathematics, Informatics, Physics, 5(54), 145–154, 2012.

[11] A. Drapal and V. A. Shcherbacov, *Identities and the group of isostrophisms*, Comment. Math. Univ. Carolin. 53(3), 347–374, 2012.

[12] P. Syrbu and I. Grecu, *On some groups related to middle Bol loops*, Revistă Ştiinţifică a Universităţii de Stat din Moldova, 7(67), 10–18, 2013.

[13] I. Grecu and P. Syrbu, *Commutants of middle Bol loops*, Quasigroups and Related Systems, 22, 81–88, 2014.

[14] B. Osoba and Y. T. Oyebo, *On Multiplication Groups of Middle Bol Loop Related to Left Bol Loop*, Int. J. Math. And Appl., 6(4), 149–155, 2018.

[15] B. Osoba and Y.T. Oyebo, *On Relationship of Multiplication Groups and Isostrophic quasogroups*, International Journal of Mathematics Trends and Technology (IJMTT), 58(2), 80–84,2018. doi:10.14445/22315373/IJMTT-V58P511

[16] T. G. Jaiyéọlá, *Basic Properties of Second Smarandache Bol Loops*, International Journal of Mathematical Combinatorics, 2, 11–20, 2009. http://doi.org/10.5281/zenodo.32303.

[17] T. G. Jaiyéọlá, *Smarandache Isotopy Of Second Smarandache Bol Loops*, Scientia Magna Journal, 7, (1), 82–93,2011. http://doi.org/10.5281/zenodo.234114.

[18] B. Osoba, *Smarandache Nuclei of Second Smarandache Bol Loops*, Scientia Magna Journal, (7) 16, 2022. to be appeared

[19] H. O. Pflugfelder,*Quasigroups and loops: introduction. Sigma Series in Pure Mathematics*, 7. Heldermann Verlag, Berlin. viii+147 1990

[20] T. G. Jaiyéọlá, S. P. David, E. Ilojide, Y. T. Oyebo, *Holomorphic structure of middle Bol loops*. Khayyam J. Math. 3 (2), 172–184 2017.

[21] J. O. Adeniran, T. G. Jaiyéọlá and K. A. Idowu, *Holomorph of generalized Bol loops*, Novi Sad Journal of Mathematics, 44, (1), 37–51, 2014.

[22] J. O. Adeniran, T.G. Jaiyéọlá and K. A. Idowu, *On some characterizations of generalized Bol loops*, Proyecciones Journal of Mathematics, accepted for publication.

[23] T. G. Jaiyéọlá and B. A. Popoola, *Holomorph of generalized Bol loops II*, Discussiones Mathematicae-General Algebra and Applications, 35, no. 1, 59-–78. doi:10.7151/dmgaa.1234, 2015.

[24] T. G. Jaiyéọlá, S. P. David and Y. T. Oyebo, *New algebraic properties of middle Bol loops* ROMAI J. 11, (2), 161–183, 2015.

[25] T. G. Jaiyéọlá, *A study of new concepts in smarandache quasigroups and loop*, ProQuest Information and Learning(ILQ), Ann Arbor, 2009.

[26] T. G. Jaiyéọlá, S. P. David and O. O Oyebola, *New algebraic properties of middle Bol loops II*, Proyecciones 40, (1), 85–106, 2021.

[27] B. Osoba and T. G. Jaiyéọlá, *Algebraic Connections Between Right and Middle Bol loops and their Cores, Quasigroups and Related Systems*, 30, 149-160, 2022.

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