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A NEWLY FORMULATED ALGORITHM FOR THE NUMERICAL SOLUTION OF NONLINEAR KLEIN-GORDON EQUATION

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ABSTRACT. This paper proposes fast and reliable algorithm for the numerical solution of the nonlinear Klein-Gordon partial differential equation. A modified new iterative method was employed to formulate six steps algorithm. To demonstrate the efficiency of the proposed algorithm, we considered four examples from available literature, and the numerical solutions obtained confirmed that the proposed algorithm is efficient and reliable.

Keywords and phrases: Klein-Gordon partial differential equation, modified new iterative method (MNIM), six-step algorithm, four examples, size of computation.
2010 Mathematical Subject Classification: A80

1.0 INTRODUCTION

The Klein-Gordon equation was named after the physicists Oskar Klein and Walter Gordon (1926), who proposed an equation describing relativistic electrons; which Schrodinger considered a quantum wave equation. The equation plays a vital role in formulating mathematical models in quantum mechanics, which attracts much attention in studying the condensed matter and investigating the interaction of solutions in a collisionless plasma. They occur in various areas of physical sciences and engineering such as the propagation of fluxions in the Josephson junctions, the motion of rigid pendula attached to a stretched wire, solid-state physics, nonlinear optics, quantum field theory, fluid dynamics, mathematical biology, chemical kinematics, and dislocations in crystals. This equation is a relativistic version of the Schrodinger equation which describes scalar spineless particles [1, 2, 3]. In this paper, we consider a one-dimensional nonlinear Klein-Gordon equation of the form:

$$\frac{\partial^2}{\partial t^2}\phi(x,t) - \beta\frac{\partial^2}{\partial x^2}\phi(x,t) + F(x,t,\phi) = \psi(x,t): \ \beta > 0, \ a \le x \le b, \ t > t_0 \quad (1)$$

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subject to initial conditions

$$\phi(x, t_0) = h_1(x), \qquad \phi_t(x, t_0) = h_2(x) \tag{2}$$

and with boundary condition

$$\phi(x, t) = h(x, t), \tag{3}$$

where $\phi = \phi(x, t)$ represents the wave displacement at position x and time t, β is a known constant and $F(x, t, \phi)$ is the nonlinear force such that $\frac{\partial F}{\partial \phi} \geq 0, h_1(x), h_2(x), h(x, t)$, and $\phi(x, t)$ are known functions. Suppose $F(x, t, \phi)$ is of the form [4, 5]:

$$F(x, t, \phi) = \begin{cases} \sin(\phi) \\ \sin(\phi) + \sin(2\phi) \\ \sinh(\phi) + \sinh(2\phi) \\ e^{\phi} \end{cases} \tag{4}$$

$F(x, t, \phi)$ is characterized by any of the sine-Gordon, double sine Gordon, double sinh-Gordon and Liouville equations in (4) above. In recent years, several numerical and computational researchers have proposed and employed several techniques to solve the nonlinear Klein-Gordon equation such as [6] studied the adiabatic dynamics of topological solitons in presence of perturbation terms and the solitons due to sine-Gordon equation, double sine-Gordon equation and sineCosine Gordon equation, authors in [7] developed and applied numerical schemes to solve one-dimensional nonlinear KleinGordon equation with quadratic and cubic nonlinearity using collocation points and approximating the solution using thin plate splines (TPS) radial basis function (RBF), [8] developed a new approach to the nonlinear Klein-Gordon equations by using Taylor matrix method, [9] presented an algorithm to solve linear and nonlinear Klein-Gordon equation using pertubation iteration transform method, [10] presented numerical solutions for two-dimensional sine-Gordon equation using the radial basis functions, [11] investigated the numerical solutions of Klein-Gordon equation using Legendre wavelets, [12] presented numerical solutions of new traveling wave solutions to the boussinesq and the KleinGordon equations, [13] used Hes variational iteration method to obtain numerical solutions of the KleinGordon equation, [14] proposed and applied multiquadric Quasi-interpolation scheme for the numerical solutions of the nonlinear Klein-Gordon, [15] solving nonlinear Klien-Gordon equation with high accuracy multiquadric quasi-interpolation scheme was considered, [16] developed the spectral

method for solving sine-Gordon equation by new orthogonal polynomial, [17] presented the finite difference method for ϕ^4 nonlinear Klein Gordon equation, and [18] applied variational method and finite element techniques for the numerical solution of damped nonlinear KleinGordon equations.
In this paper, we employed modified new iterative method (MNIM) discussed in [19] to formulate MAPLE18 software codes for the development of six steps algorithm to solve the nonlinear Klein-Gordon equation. The procedures are more accurate in comparison with many numerical techniques available in literature.
This paper is organized as follows; In section 1, a brief introduction on the Klein-Gordon equation is discussed, and in section 2, formulation of modified new iterative method and modified new iterative algorithm are discussed while in section 3, we present four computational experiments to illustrate the proposed algorithm. 3Dplots for the numerical solutions for four Klein-Gordon equation examples are presented in section 4. Finally, the discussion and conclusion are presented in section 5.

2.0 METHOD OF SOLUTION

2.1 Modified New Iterative Method (MNIM)

New iterative method (NIM) was proposed by [20] and simple easy to implement on computer using symbolic computation packages such as Maple, Mathematica, Matlab e.t.c. This method is better than some numerical techniques as it is free from rounding off errors and does not require large computer power [21, 22].
Consider new iterative method (NIM) as a numerical technique for solving a functional equation of the form

$$\phi(\bar{x}) = f(\bar{x}) + N(\phi(\bar{x})) \tag{5}$$

where N a nonlinear operator from a Banach space $B \to B$ and $f(\bar{x})$ is a known function, $\bar{x} = (x_1, x_2, x_3, \ldots, x_n)$. We need to obtain the solution $\phi(\bar{x})$ of equation (5) having the series of the form;

$$\phi(\bar{x}) = \sum_{i=0}^{\infty} \phi_i(\bar{x}) \tag{6}$$

The nonlinear operator which is on the right-hand side of equation (5) can be decomposed as follow:

$$N\left(\sum_{i=0}^{\infty} \phi_i(\bar{x})\right) = N(\phi_0) + \sum_{i=1}^{\infty}\left(N\left(\sum_{j=0}^{i} \phi_j\right) - N\left(\sum_{j=0}^{i-1} \phi_j\right)\right) \tag{7}$$

substitute equations equation (6) and (7) into the equation (5) and (5) becomes:

$$\sum_{i=0}^{\infty}\phi_i(\bar{x}) = f(\bar{x})+N(\phi_0)+\sum_{i=1}^{\infty}\left(N\left(\sum_{j=0}^{i}\phi_j\right)-N\left(\sum_{j=0}^{i-1}\phi_j\right)\right) \tag{8}$$

The recurrence relation is given by

$$\begin{cases}\phi_0 = f,\\ \phi_1 = N(\phi_0),\\ \vdots \\ \phi_{m+1} = N(\phi_0+\phi_1+\cdots+\phi_m) - N(\phi_0+\phi_1+\cdots+\phi_{m-1})\\ m = 1,2,3,\ldots\end{cases} \tag{9}$$

Then

$$(\phi_1+\phi_2+\cdots+\phi_{m+1}) = N(\phi_0+\phi_1+\cdots+\phi_m), \qquad m = 1,2,3,\ldots \tag{10}$$

and

$$\sum_{i=0}^{\infty}\phi_i = f + N\left(\sum_{i=0}^{\infty}\phi_i\right) \tag{11}$$

The $q-$ term approximate solution of equation (5) is given by;

$$\phi = \phi_0+\phi_1+\phi_2+\cdots+\phi_{q-1} \tag{12}$$

In order to improve the convergence rate of NIM discussed in Section 2.1 for the numerical solution of the nonlinear Klein-Gordon equation where source term was difficult to evaluate using NIM, authors [19] discussed the introduction of source terms into the integral representing $N(\phi)$ and the modification was stated as follows:

(1) if the source term is a function of the independent variable x only, it is included in $N(\phi)$;
(2) if the source term is a function of the independent variables x and t, it is included in $N(\phi)$;
(3) if the source term is a function containing terms with functions of x, t, and both x and t then, we include in $N(\phi)$ the terms involving t and both x and t;
(4) and if the source term is $\sin(x)\sin(t)$, then MNIM can be applied to obtain the close exact solution.

2.2 Modified New Iterative Algorithm (MNIA)

In this section, we develop six steps algorithm using the MINM discussed in Section 2.1 in order to reduce computational time taken in simplifying and evaluating the derivatives involve in the modified new iterative method while the convergence rate is faster. Modified new iterative algorithm formulation goes thus:

restart
Step 1:
Digits := 8;
N := $\mathbb{R}^+$;

$$\beta := \mathbb{R}^+$$

;
$\phi(x, t_0) := h_1(x)$;
$\phi_t(x, t_0) := h_2(x)$;
$\phi[0] := \phi(x, t_0) + t * \phi_t(x, t_0)$;
Step 2:
KGPDE := value($\beta * diff(\phi[0], x, x) - F(x, t, \phi[0]) + \psi(x, t)$);
$\phi[1]$:= value(int(KGPDE, $t = 0..t, t = 0..t$));
Step 3:
for m from 1 to N do
$\phi[m + 1]$ = value((int((β * Diff(sum($\phi[n]$,n=0..m, x,x)-F(x,t,sum($\phi[n]$, n=0..m))
+ $\psi(x, t)$))))) - (int((β * Diff(sum($\phi[n]$,n=0..m-1, x,x)-F(x,t,sum($\phi[n]$, n=0..m-1)) + $\psi(x, t)$))))))); end do;
Step 4:
$\phi*$:= sum($\phi[k]$, k=0..N+1);
$\phi[sol]$:= evalf($\phi*$);
Step 5:
for i from -1 by 0.2 to 1 do
ϕ := evalf(eval($\phi[sol]$, $x = i, t = i$))
end do
Step 6:
$\phi[3Dplot]$:= plot3d(ϕ [sol], x=-10..10, t=-60..60, grid=[100,100], color);
$\phi[3Dplot]$:= plot3d($\phi[example1, example2]$, x=-10..10, t=-60..60, grid = [100,100], color:red,blue);
$\phi[3Dplot]$:= plot3d(ϕ [example3, example4], x=-10..10, t=-60..60, grid = [100,100], color:green,purple);
Output: See Table 1, Table 2, Table 3,Table 4 and Figures (1,2,3,..,6) where N is the computational length and is positive constant.

3.0 NUMERICAL EXAMPLES

Example 1: We consider the nonlinear KleinGordon equation (1) with $\beta = \frac{5}{2}, F(x,t,\phi) = \frac{3}{2}\phi^3$ and $\psi(x,t) = 0$. [7, 8]

$$\frac{\partial^2\phi}{\partial t^2} - \frac{5}{2}\frac{\partial^2\phi}{\partial x^2} + \phi + \frac{3}{2}\phi^3 = 0 \qquad -1 \leq x \leq 1 \tag{13}$$

with the initial conditions

$$\phi(x,0) = B\tan(Kx), \quad \phi_t(x,0) = BcK\sec^2(kx) \tag{14}$$

Here $B = \sqrt{\frac{2}{3}}$, $K = \sqrt{\frac{-1}{2(-2.5+c^2)}}$ and $c = 0.05$.
The exact solution is given as;

$$\phi(x,t) = B\tan(K(x+ct)) \tag{15}$$

Apply algorithm proposed when $h_1(x) = B\tan(Kx), h_2(x) = BcK$ $\sec^2(Kx)$ and taking computational length $N = 2$. We obtain numerical solutions given in Table 1.

Table 1: Numerical solutions for $\phi(x,t)$ wave displacement in Example 1.

$\phi(x,t)$	Exact solution	MNIA	[7]	[8]
(0.1, 0.01)	0.03674054	0.00×10^0	0.00×10^0	5.20×10^{-7}
(0.2, 0.01)	0.07344602	1.10×10^{-9}	1.78×10^{-6}	9.42×10^{-6}
(0.3, 0.01)	0.11044836	0.00×10^0	1.77×10^{-7}	3.68×10^{-6}
(0.4, 0.01)	0.14790165	1.10×10^{-8}	1.54×10^{-6}	2.62×10^{-6}
(0.5, 0.01)	0.18596741	1.10×10^{-8}	1.53×10^{-7}	2.19×10^{-6}
(0.6, 0.01)	0.22481755	0.00×10^0	1.72×10^{-6}	3.79×10^{-6}
(0.7, 0.01)	0.26463756	1.10×10^{-8}	1.73×10^{-7}	2.96×10^{-6}
(0.8, 0.01)	0.30563029	0.00×10^0	2.00×10^{-6}	1.12×10^{-5}
(0.9, 0.01)	0.34802048	0.00×10^0	1.99×10^{-7}	3.63×10^{-6}
(1.0, 0.01)	0.39206013	0.00×10^0	0.00×10^0	2.68×10^{-6}

Example 2: We consider nonlinear KleinGordon equation (1) with $\beta = 1, F(x,t,\phi) = \phi^2$ and $\psi(x,t) = -x\cos(t) + x^2\cos^2(t)$. [8, 11]

$$\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2} + \phi^2 = -x\cos(t) + x^2\cos^2(t) \qquad -1 \leq x \leq 1 \tag{16}$$

with the initial conditions

$$\phi(x,0) = x, \quad \phi_t(x,0) = 0 \tag{17}$$

The exact solution is given as;

$$\phi(x,t) = x\cos(t) \tag{18}$$

Apply algorithm proposed when $h_1(x) = x, h_2(x) = 0$ and taking computational length $N = 3$. We obtain numerical solutions given in Table 2.

Table 2: Numerical solutions for $\phi(x,t)$ wave displacement in Example 2.

$\phi(x,t)$	Exact solution	MNIA	[8]	[11]
(0.0, 0.0)	0.00000000	0.00×10^{0}	0.00×10^{0}	0.00×10^{0}
(0.1, 0.1)	0.09950042	1.83×10^{-7}	2.89×10^{-6}	**
(0.2, 0.2)	0.19601332	1.68×10^{-6}	1.65×10^{-6}	1.62×10^{-7}
(0.3, 0.3)	0.28660095	3.05×10^{-6}	3.59×10^{-6}	**
(0.4, 0.4)	0.36842440	4.40×10^{-6}	4.77×10^{-6}	7.78×10^{-7}
(0.5, 0.5)	0.43879128	3.28×10^{-6}	4.42×10^{-6}	**
(0.6, 0.6)	0.43879128	7.63×10^{-6}	4.44×10^{-6}	2.81×10^{-6}
(0.7, 0.7)	0.53538953	8.04×10^{-6}	1.18×10^{-5}	**
(0.8, 0.8)	0.55736537	9.96×10^{-5}	4.20×10^{-5}	1.65×10^{-5}
(0.9, 0.9)	0.55944897	3.28×10^{-5}	1.22×10^{-4}	**
(1.0, 1.0)	0.99500417	6.28×10^{-5}	2.95×10^{-4}	1.39×10^{-5}

** - Not available

Example 3: We consider nonlinear KleinGordon equation (1) with $\beta = 1, F(x,t,\phi) = \phi^2$ and $\psi(x,t) = 6tx^3 - 6xt^3 + (xt)^6$. [8, 9]

$$\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2} + \phi^2 = 6tx^3 - 6xt^3 + (xt)^6 \qquad -1 \leq x \leq 1 \tag{19}$$

with the initial conditions

$$\phi(x,0) = 0, \quad \phi_t(x,0) = 0 \tag{20}$$

The exact solution is given as;

$$\phi(x,t) = x^3t^3 \tag{21}$$

Apply algorithm proposed when $h_1(x) = 0, h_2(x) = 0$ and taking computational length $N = 4$. We obtain numerical solutions given in Table 3.

Table 3: Numerical solutions for $\phi(x,t)$ wave displacement in Example 3.

$\phi(x,t)$	Exact solution	MNIA	[8]	[9]
(0.0, 0.0)	0.00000000	0.00×10^0	0.00×10^0	0.00×10^0
(0.1, 0.1)	0.00000100	0.00×10^0	0.00×10^0	0.00×10^0
(0.2, 0.2)	0.00006400	0.00×10^0	0.00×10^0	0.00×10^0
(0.3, 0.3)	0.00072900	0.00×10^0	0.00×10^0	0.00×10^0
(0.4, 0.4)	0.00409600	0.00×10^0	0.00×10^0	0.00×10^0
(0.5, 0.5)	0.01562500	0.00×10^0	0.00×10^0	0.00×10^0
(0.6, 0.6)	0.04665600	0.00×10^0	0.00×10^0	0.00×10^0
(0.7, 0.7)	0.11764900	0.00×10^0	0.00×10^0	0.00×10^0
(0.8, 0.8)	0.26214400	0.00×10^0	0.00×10^0	0.00×10^0
(0.9, 0.9)	0.53144100	0.00×10^0	0.00×10^0	0.00×10^0
(1.0, 1.0)	0.00100000	0.00×10^0	0.00×10^0	0.00×10^0

Example 4: We consider nonlinear KleinGordon equation (1) with $\beta = 1, F(x,t,\phi) = \phi^2$ and $\psi(x,t) = x^2t^2$. [9, 19]

$$\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2} + \phi^2 = x^2t^2 \qquad -1 \le x \le 1 \tag{22}$$

with the initial conditions

$$\phi(x,0) = 0, \quad \phi_t(x,0) = x \tag{23}$$

The exact solution is given as;

$$\phi(x,t) = xt \tag{24}$$

Apply algorithm proposed when $h_1(x) = 0, h_2(x) = x$ and taking computational length $N = 2$. We obtain numerical solutions given in Table 4.

Table 4: Numerical solutions for $\phi(x,t)$ wave displacement in Example 4.

$\phi(x,t)$	Exact solution	MNIA	[9]	[19]
(0.0, 0.0)	0.00000000	0.00×10^0	0.00×10^0	0.00×10^0
(0.1, 0.1)	0.01000000	0.00×10^0	0.00×10^0	0.00×10^0
(0.2, 0.2)	0.04000000	0.00×10^0	0.00×10^0	0.00×10^0
(0.3, 0.3)	0.09000000	0.00×10^0	0.00×10^0	0.00×10^0
(0.4, 0.4)	0.16000000	0.00×10^0	0.00×10^0	0.00×10^0
(0.5, 0.5)	0.25000000	0.00×10^0	0.00×10^0	0.00×10^0
(0.6, 0.6)	0.36000000	0.00×10^0	0.00×10^0	0.00×10^0
(0.7, 0.7)	0.49000000	0.00×10^0	0.00×10^0	0.00×10^0
(0.8, 0.8)	0.64000000	0.00×10^0	0.00×10^0	0.00×10^0
(0.9, 0.9)	0.81000000	0.00×10^0	0.00×10^0	0.00×10^0
(1.0, 1.0)	0.10000000	0.00×10^0	0.00×10^0	0.00×10^0

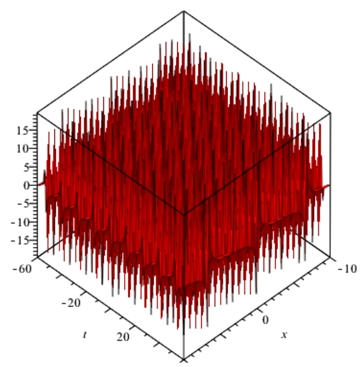

Fig.1.0 $\phi(x,t)$ wave displacement of position x and time t on 3D plot for nonlinear Klein-Gordon equation: Example 1

4.0 CONCLUSION

In this paper, we developed fast and accurate six steps algorithm using modified new iterative method to obtain numeric-analytic solutions of nonlinear Klein-Gordon equation. Four examples are considered from available literature to demonstrate the feasibility of the proposed algorithm which shows a good agreement with exact solutions. Figures 1, 2, 3, 4, 5 and 6 depict $\phi(x,t)$ wave displacement at position x and time t which are plotted on 3Dplots. From a computational point of view, MNIA takes few computational lengths to achieve a close form solution. Therefore, we recommend the newly introduced algorithm for similar problems in computational physics and engineering sciences.

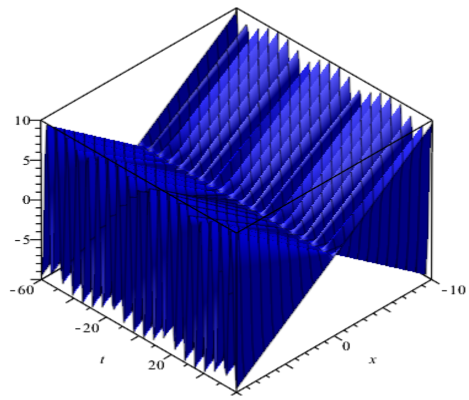

Fig.2.0 $\phi(x,t)$ wave displacement of position x and time t on 3D plot for nonlinear Klein-Gordon equation: Example 2

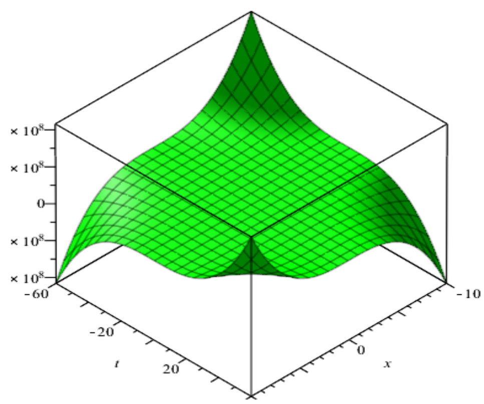

Fig.3.0 $\phi(x,t)$ wave displacement of position x and time t on 3D plot for nonlinear Klein-Gordon equation: Example 3

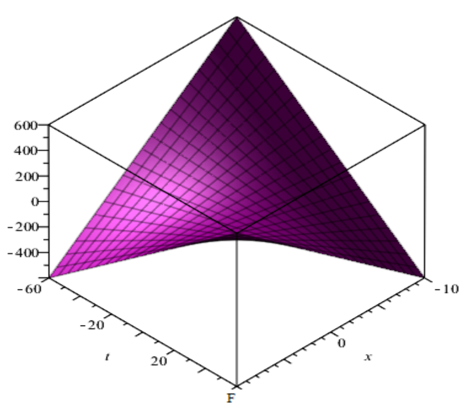

Fig.4.0 $\phi(x,t)$ wave displacement of position x and time t on 3D plot for nonlinear Klein-Gordon equation: Example 4

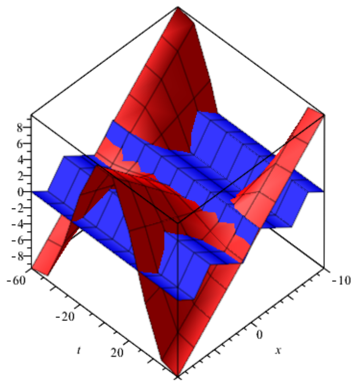

Fig.5.0 $\phi(x,t)$ wave displacement of position x and time t on 3D plot for nonlinear Klein-Gordon equation: Example 1 and Example 2

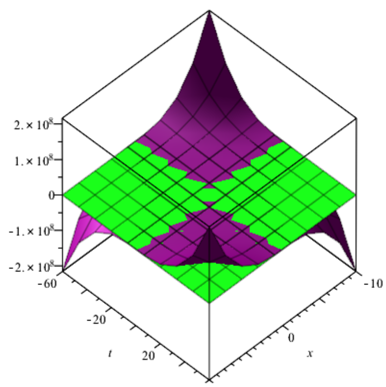

Fig.6.0 $\phi(x,t)$ wave displacement of position x and time t on 3D plot for nonlinear Klein-Gordon equation: Example 3 and Example 4

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