Journal of the Nigerian Mathematical Society
Vol. 41, Issue 1, pp. 1-12, 2022

EXISTENCE OF FIXED POINTS OF MONOTONE ASYMPTOTIC POINTWISE LIPSCHITZIAN MAPPINGS IN HYPERBOLIC METRIC SPACES

S. C. EGBULEM[1], N. M. NWANKWOR[1] N. N. ARAKA[1] AND E. U. OFOEDU[2]

ABSTRACT. In this work, we establish fixed point theorems for the classes of monotone asymptotic pointwise Lipschitzian mappings and generalized monotone asymptotic pointwise nonexpansive mappings in the setting of complete hyperbolic metric spaces endowed with partial order. The Theorems obtained extend, generalize and unify some existing results.

Keywords and phrases: Hyperbolic metric space, Asymptotic pointwise Lipschitzian mapping, monotone mapping.
2010 Mathematical Subject Classification: 47H06,47H09, 47J05, 47J25

1. INTRODUCTION

Several existence results for different classes of nonexpansive-type mappings have been developed (see for example, [4, 5, 12, 13, 14, 16, 18, 19, 20]). The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [10] as an important generalization of the class of nonexpansive mappings. They proved fixed point theorems for this class of mappings under the assumption that the domain of the operator is a nonempty, bounded, closed and convex subset of a real uniformly convex Banach space. Generally, the space considered was a metric space which had a linear structure that allowed for easy modifications (see for example, [1, 2, 3, 7, 21]). However, the challenge was to modify some nonlinear operators in some nonlinear metric spaces and prove the existence of fixed points

Received by the editors August 16, 2021; Revised: September 29, 2021; Accepted: September 30, 2021

www.nigerianmathematicalsociety.org; Journal available online at https://ojs.ictp.it/jnms/

[1]Supported by Simons Foundation for Sub-Sahara Africa Nationals

[2]Corresponding author

of the mappings. One of the earliest attempt in providing an answers on existence of fixed points on these mappings was done by Khamsi and Khan [17]. Since then, several researches have been carried out in establishing existence of fixed points of various nonlinear operators in special metric spaces that need not be linear. In this work, we prove the fixed point result of monotone asymptotic pointwise Lipschitzian mapping which was earlier introduced by Dehaish and Khamsi [9] and obtain a more general result void of some conditions placed in their work.

Let (M, ρ) be a metric space. Suppose that for any two points $x, y \in M$, there is a unique metric segment denoted by $[x, y]$ which is an isometric image of the real line interval $[0, \rho(x, y)]$, then the unique point $z \in [x, y]$ is defined by

$$\rho(x, z) = \beta\rho(x, y), \quad \rho(z, y) = (1-\beta)\rho(x, y)$$

and will be denoted by $z = (1-\beta)x \oplus \beta y = \beta y \oplus (1-\beta)x$, where $\beta \in [0, 1]$. We call such metric space (M, ρ) convex metric spaces. A convex metric space (M, ρ) is called a hyperbolic space if for any $\beta \in [0, 1]$, $\forall\, w, x, y \in M$,

$$\rho\Big((1-\beta)w \oplus \beta x, (1-\beta)w \oplus \beta y\Big) \leq \beta\rho(x, y).$$

Thus, we obtain in particular that if (M, ρ) is a hyperbolic space, then $\forall\, w, x, y \in M$,

$$\rho\Big(\frac{1}{2}w \oplus \frac{1}{2}x, \frac{1}{2}w \oplus \frac{1}{2}y\Big) \leq \frac{1}{2}\rho(x, y).$$

Every normed linear space is clearly a linear hyperbolic space with the binary operation $\oplus \equiv +$, the vector addition on X. This is so, since for any given normed space $(X, \|\cdot\|)$, we obtain that $\forall\, x, y \in X$, the function $\rho : X \times X \to \mathbb{R}$ defined by $\rho(x, y) = ||x - y||$ is a metric on X; and for any $\beta \in [0, 1]$, $w, x, y \in X$,

$$\rho\Big((1-\beta)w \oplus \beta x, (1-\beta)w \oplus \beta y\Big) =$$
$$\|(1-\beta)w + \beta x - ((1-\beta)w + \beta y)\| = \beta\|x - y\| = \beta\rho(x, y).$$

Further examples of nonlinear hyperbolic metric spaces include CAT(0) spaces, Hadamard manifolds and the open unit ball of a Hilbert space equipped with the hyperbolic metric (see for example, [6, 8, 11, 15]). It is evident that most metric spaces in applications are hyperbolic metric spaces, hence their is need to consider some

analogous nonlinear operators and some properties of the hyperbolic metric spaces.
Observe that if (M,ρ) is a hyperbolic space, then for any $\beta \in [0,1]$, for any $x,y,w,z \in M$,

$$\rho\big((1-\beta)x \oplus \beta y, (1-\beta)z \oplus \beta w\big) \leq$$
$$\rho\big((1-\beta)x \oplus \beta y, (1-\beta)z \oplus \beta y\big) + \rho\big((1-\beta)z \oplus \beta y, (1-\beta)z \oplus \beta w\big)$$
$$\leq (1-\beta)\rho(x,z) + \beta\rho(y,w)$$

A subset C of a hyperbolic metric space (M,ρ) is called a convex set if for all $x,y \in C$, $[x,y] \subset C$.

The work embodied in this paper is motivated by the work of Dehaish and Khamsi [9] who obtained the following result:

Theorem 1.1 (Dehaish and Khamsi, 2017)**.** *Let $(M,d,\preceq)$ be a complete hyperbolic metric space endowed with a partial order $\preceq$ for which order intervals are closed and convex. Assume M satisfies the property (R). Let C be a nonempty, closed, convex, bounded subset of M not reduced to a point. Let $T: C \to C$ be a monotone asymptotically pointwise contraction. Then, T has a fixed point if and only if there exists $x_0 \in C$ such that x_0 and $T(x_0)$ are comparable.*

The aim of this paper therefore, is to improve on Theorem 1.1 by extending it from the class of monotone asymptotic pointwise contractions to the class of monotone asymptotic pointwise Lipschitzian mappings under the same setting. Moreover, a seemingly new class of generalized monotone asymptotic pointwise Lipschitzian mapping is introduced; and necessary and sufficient conditions for existence of fixed points of some mappings belonging to this class of operators is established.

2. PRELIMINARY

In this section, we consider some definitions of nonlinear operators, present some Lemmas and Theorems which shall play important roles in the sequel. We proceed as follows:
Let (M,ρ) be a hyperbolic metric space. Then, (M,ρ) is said to be uniformly convex if given $r > 0$, $\epsilon > 0$ and $x,y,w \in X$ with $\rho(x,w) \leq r$, $\rho(y,w) \leq r$, and $\rho(x,y) \geq r\epsilon$, there exists $\delta'(r,\epsilon) \geq 0$ such that

$$\frac{1}{r}\rho\Big(\frac{1}{2}x \oplus \frac{1}{2}y, w\Big) < 1 - \delta'(r,\epsilon).$$

The modulus of convexity of a hyperbolic metric space (X,ρ) is defined for all $r>0,\ \epsilon>0$ by

$$\delta(r,\epsilon) :=$$
$$\inf\Big\{1-\tfrac{1}{r}\rho\Big(\tfrac{1}{2}x\oplus\tfrac{1}{2}y,w\Big);\ \rho(x,w)\le r,\ \rho(y,w)\le r,\ \rho(x,y)\ge r\epsilon\Big\}.$$

The space (M,ρ) is said to be uniformly convex if and only if $\forall\ r>0$ and $\epsilon>0,\ \delta(r,\epsilon)>0$.
A hyperbolic metric space (M,ρ) is said to have the property (R) whenever any decreasing sequence of nonempty, closed, convex and bounded subsets of M has a nonempty intersection. That is, (M,ρ) is said to have property (R) if for any sequence $\{C_n\}_{n=0}^{\infty}$ of nonempty closed convex bounded subsets of M such that $\forall\ n\ge 0,\ C_{n+1}\subseteq C_n$, then $\bigcap\limits_{n=1}^{\infty} C_n\neq\emptyset$. The concept of property (R) in hyperbolic spaces is analogous to reflexivity of Banach space, and was introduced by Khamsi [16] for hyperbolic metric spaces.

Lemma 2.1. [17] A complete hyperbolic metric space that is uniformly convex has the property (R).

Lemma 2.2. [17] Let (M,ρ) be a complete hyperbolic metric space and C is a nonempty, closed, convex and bounded subset of M. Let τ be a type function defined on C, that is, $\tau : C\to[0,\infty)$ is a function for which there exists a bounded sequence $\{x_n\}$ in C such that $\forall\ x\in C$,

$$\tau(x)=\limsup_{n\to\infty}\rho(x_n,x).$$

If M satisfies property (R), then τ has a minimum point $z\in C$. That is, if M satisfies property (R), then there exists $z\in C$ such that

$$\tau(z)=\inf\{\tau(x);\ x\in C\}:=\tau_0.$$

Moreover, if M is uniformly convex, then any minimizing sequence of τ is convergent.

Let $(M,\rho,\preceq)$ be a hyperbolic metric space with partial order $\preceq$. Given any $a,b\in M$, a,b are said to be comparable if either $a\preceq b$ or $b\preceq a$. An ordered interval of M is anyone of the following subsets; $[a,\to)=\{x\in M;\ a\preceq x\}$, $(\leftarrow,b]=\{x\in M;\ x\preceq b\}$ and $[a,b]=\{x\in M;\ a\preceq x\preceq b\}=[a,\to)\cap(\leftarrow,b]$. For this work, it will be assumed that ordered intervals are closed. For partially ordered set $(M,\preceq)$, a mapping $T:M\to M$ is said to be monotone if for any $x,y\in M$, we have that $x\preceq y$ implies that $T(x)\preceq T(y)$.

Definition 2.3. Let $(M, \rho, \preceq)$ be a metric space with partial order $\preceq$. Let $T : M \to M$ be a mapping.

(a.) We say that the mapping T is *monotone asymptotic pointwise Lipschitizian* if it is monotone and there exists a sequence $\{k_n\}_{n=1}^{\infty}$ of mappings $k_n : M \to [0, \infty)$ such that for any $x \in M$, we have that $\rho(T^n(x), T^n(y)) \leq k_n(x)\rho(x, y)$ for any $y \in M$ comparable to x.

  (i.) If $\{k_n\}$ converges pointwise to $k : M \to [0, 1)$, then the mapping T is called a *monotone asymptotic pointwise contraction*,

  (ii.) If $\limsup_{n\to\infty} k_n(x) \leq 1$, then the mapping T is called a *monotone asymptotic pointwise nonexpansive mapping.*

(b.) The mapping T is called *generalized monotone asymptotic pointwise Lipschitizian* if there exists sequences $\{k_n\}_{n=1}^{\infty}$ and $\{l_n\}_{n=1}^{\infty}$ of mappings $k_n : M \to [0, 1), l_n : M \to [0, \infty)$ such that for any $x \in M$, we have that $\rho(T^n(x), T^n(y)) \leq k_n(x)\rho(x, y) + l_n(x)$ for any y comparable to x.

(i.) If for any $x \in M$, $\limsup_{n\to\infty} k_n(x) = k(x)$, where $k : M \to [0, 1)$ and $\limsup_{n\to\infty} l_n(x) = 0$, then T is called a *generalized monotone asymptotic pointwise contraction*,

(ii.) If for any $x \in M$, $\limsup_{n\to\infty} k_n(x) \leq 1$ and $\lim_{n\to\infty} l_n(x) = 0$, then T is called a *generalized monotone asymptotic pointwise nonexpansive mapping.*

Remark 2.4. The above definition of generalized monotone asymptotic pointwise contraction and generalized monotone asymptotic pointwise nonexpansive mappings were motivated by the work of Zegeye and Shazad [22] who introduced the class of generalized asymptotic nonexpansive mappings in normed linear spaces.

3. THE HEART OF THE MATTER

We now turn to the heart of the matter, which is presentation of the main contributions of this paper. Let us proceed as follows:

3.1. **Fixed point theorem for asymptotic Lipschitzian mappings.**

Theorem 3.1. *Let $(M, \rho, \preceq)$ be a complete hyperbolic metric space endowed with partial order $\preceq$ for which order intervals are closed and convex, and such that M satisfies property (R). Let C be a* *nonempty, closed, bounded and convex subset of M and $T : C \to C$ be a monotone asymptotic pointwise Lipschitizian mapping with a sequence of functions $k_n : C \to [0, \infty)$, such that $k_{n_0}(x) \in (0, 1)$ for all $x \in C$, for some $n_0 \in \mathbb{N}$. Then, T has a fixed point in C if and only if there exists $x_0 \in C$ such that x_0 and $T(x_0)$ are comparable.*

Proof. If $C = \{z\}$ for some $z \in M$, then the result holds trivially. Hence, suppose C is not a singleton, then it is clear that if there is $z^* \in C$ such that $z^* = Tz^*$, then z^* and Tz^* are comparable. Thus, there exists $x_0 = z^* \in C$ such that $x_0 \preceq Tx_0$.

Suppose there exists $x_0 \in C$ such that $x_0 \preceq Tx_0$, then since T is monotone, we obtain that for all $n \in \mathbb{N}$, $T^n(x_0) \preceq T^{n+1}(x_0)$. So, the sequence $\{T^n(x_0)\}_{n=0}^{\infty}$ is monotone increasing. It is very clear that for any $n \in \mathbb{N}$, the set $C_n = \{x \in C \ : \ T^n(x_0) \preceq x\}$ is nonempty, closed, bounded and convex subset of C, and that the collection $\{C_n\}_{n=1}^{\infty}$ is a decreasing sequence of sets in M. Since M satisfies property (R), we obtain that

$$C_\infty = \bigcap_{n=0}^{\infty} C_n = \bigcap_{n=0}^{\infty} \{x \in C \ : \ T^n(x_0) \preceq x\} \neq \emptyset.$$

Now, define the type function $\tau : C_\infty \to [0, \infty)$ for any $x \in C_\infty$ by

$$\tau(x) = \limsup_{n\to\infty} \rho(T^n(x_0), x).$$

It is easy to see that C_∞ is invariant under T. That is, $T(C_\infty) \subseteq (C_\infty)$. To see this, observe that for any $x \in C_\infty$, for any $n \in \mathbb{N}$, $T^n(x_0) \preceq x$ and $x_0 \preceq Tx_0$. Thus, by monotonicity of T, we obtain that $T^n(x_0) \preceq T^{n+1}(x_0) = T(T^n(x_0)) \preceq T(x)$. Hence, for all $x \in C_\infty$, we have that $T(x) \in C_\infty$. Therefore, $T(C_\infty) \subseteq C_\infty$.
So, by Lemma 2.2, there exists $z^* \in C_\infty$ such that

$$\tau(z^*) = \inf\{\tau(x) \ : \ x \in C_\infty\} = \tau_0.$$

By our hypothesis, there exists $n_0 \in \mathbb{N}$ such that for all $x \in C$, $k_{n_0}(x) \in (0, 1)$.
Since $z^* \in C_\infty$, we obtain that $T^{n_0}(z^*) \in C_\infty$. So, for all $n \geq n_0$,

$$\rho(T^n(x_0), T^{n_0}(z^*)) \leq k_{n_0}(x_0)\rho(T^{n-n_0}(x_0), z^*).$$

This implies that,

$$\begin{aligned}\tau(T^{n_0}(z^*)) &= \limsup_{n\to\infty} \rho(T^n(x_0), T^{n_0}(z^*)) \\ &\leq k_{n_0}(x_0) \limsup_{n\to\infty} \rho(T^{n-n_0}(x_0), z^*)\end{aligned}$$

So, $\tau_0 \leq \tau(T^{n_0}(z^*)) \leq k_{n_0}(x_0)\tau_0$, which implies that $\tau_0 \leq k_{n_0}(x_0)\tau_0$ or equivalently $(1 - k_{n_0})\tau_0 \leq 0$. Since, $k_{n_0}(x_0) \in (0, 1)$, we obtain that $\tau_0 = 0$. Thus,

$$\tau(T^{n_0}(z^*)) = 0, \text{ or equivalently, } \limsup_{n\to\infty} \rho(T^n(x_0), T^{n_0}(z^*)) = 0.$$

This means that $\lim_{n\to\infty} T^n(x) = T^{n_0}(z^*)$. But for all $n \in \mathbb{N}$,

$$\begin{aligned} 0 \leq \rho(z^*, T^{n_0}(z^*)) &\leq \rho(z^*, T^n(x_0)) + \rho(T^n(x_0), T^{n_0}(z^*)) \\ &= \rho(T^n(x_0), z^*) + \rho(T^n(x_0), T^{n_0}(z^*)) \end{aligned}$$

Thus,

$$\begin{aligned} 0 \leq \rho(z^*, T^{n_0}(z^*)) &\leq \limsup_{n\to\infty} \rho(T^n(x_0), z^*) \\ &\quad + \limsup_{n\to\infty} \rho(T^n(x_0), T^{n_0}(z^*)) \\ &= \tau_0 + 0 = 0 \text{ (since } \tau_0 = 0) \end{aligned}$$

So, $\rho(z^*, T^{n_0}(z^*)) = 0$, which implies that $T^{n_0}(z^*) = z^*$.
Observe therefore that,

$$\begin{aligned} \rho(z^*, T(z^*)) &= \rho(T^{n_0}(z^*), T^{n_0+1}(z^*)) \\ &= \rho(T^{n_0}(z^*), T^{n_0}(Tz^*)) \leq k_{n_0}(z^*)\rho(z^*, T(z^*)). \end{aligned}$$

Since $k_{n_0}(z^*) \in (0, 1)$, we obtain that $\rho(z^*, Tz^*) = 0$. Thus, $z^* = Tz^*$. This completes the proof. □

Remark 3.2. If we consider $T : C \to C$ to be a monotone asymptotic pointwise contraction, then we have the following corollary which is an improved result of Theorem 1.1 of Dehaish and Khamsi [[9]]. This is so, since the condition of C not reduced to one point is dispensed with.

Corollary 3.3. Let $(M, \rho, \preceq)$ be a complete hyperbolic metric spacce endowed with partial order $\preceq$ for which order intervals are convex and closed. Suppose M satisfies the property (R). Let C be a nonempty, bounded, closed, convex subset of M. Let $T : C \to C$ be a monotone asymptotic pointwise contraction. Then, T has a fixed point if and only if there exists $x_0 \in C$ such that x_0 and Tx_0 are comparable.

Proof. Observe that since T is a monotone asymptotic pointwise contraction, there exists a sequence of functions $k_n : C \to [0, \infty)$ such that $\forall x \in C$, $\lim_{n\to\infty} k_n(x) = k(x) \in (0, 1)$. Thus, for all $\epsilon > 0$, there exists $n_\epsilon \in \mathbb{N}$ such that for all $n \geq n_\epsilon$, $k_n(x) < k(x) + \epsilon$. In particular, for $\epsilon_x = 1 - k(x)$, there exists $n_0(x) \in \mathbb{N}$ such that for all $n \geq n_0(x)$, $k_n(x) < k_n(x) + (1 - k(x)) = 1$. In particular, $k_{n_0}(x) < 1$. The result thus follows as in the proof of Theorem 3.1. This completes the proof □

3.2. **Fixed point theory for generalized asymptotic pointwise mappings.**

Theorem 3.4. *Let $(M, \rho, \preceq)$ be a complete hyperbolic metric spacce endowed with partial order $\preceq$ for which order intervals are convex and closed. Suppose M satisfies the property (R). Let C be a nonempty, bounded, closed, convex subset of M. Let $T : C \to C$ be a generalized monotone continuous asymptotic pointwise nonexpansive mapping. Then, T has a fixed point if and only if there exists $x_0 \in C$ such that x_0 and Tx_0 are comparable.*

Proof. If $C = \{z\}$ for some $z \in M$, the result holds trivially. Hence, suppose C is not a singleton. Then, it is clear that if there is $z^* \in C$ such that $z^* = Tz^*$, then z^* and Tz^* are comparable. Thus, there exists $x_0 = z^* \in C$ such that $x_0 \preceq Tx_0$.
Suppose there exists $x_0 \in C$ such that $x_0 \preceq Tx_0$, then since T is monotone, we obtain (as in the proof of Theorem 3.1) that for all $n \in \mathbb{N}$, $T^n(x_0) \preceq T^{n+1}(x_0)$. So, the sequence $\{T^n(x_0)\}_{n=1}^{\infty}$ is monotone increasing. Clearly for any $n \in \mathbb{N}$, the set $C_n = \{x \in C \ : \ T^n(x_0) \preceq x\}$ is nonempty closed bounded and convex subset of C, and that the collection $\{C_n\}_{n=1}^{\infty}$ is a decreasing sequence of sets in M. Since M satisfies property (R), we obtain that

$$C_\infty = \bigcap_{n=0}^{\infty} C_n = \bigcap_{n=0}^{\infty} \{x \in C \ : \ T^n(x_0) \preceq x\} \neq \emptyset.$$

Again, define the type function $\tau : C_\infty \to [0, \infty)$ by

$$\tau(x) = \limsup_{n\to\infty} \rho(T^n(x_0), x).$$

It is easy to see that C_∞ is invariant under T. That is, $T(C_\infty) \subseteq (C_\infty)$. To see this, observe that for any $x \in C_\infty$, for any $n \in \mathbb{N}$, $T^n(x_0) \preceq x$ and $x_0 \preceq Tx_0$. Thus, by monotonicity of T, $T^n(x_0) \preceq T^{n+1}(x_0) = T(T^n(x_0)) \preceq T(x)$. Hence, for all $x \in C_\infty$, we have that $Tx \in C_\infty$. Therefore, $T(C_\infty) \subseteq C_\infty$.
So, by Lemma 2.2, there exists $z^* \in C_\infty$ such that

$$\tau(z^*) = \inf\{\tau(x) \ : \ x \in C_\infty\} = \tau_0.$$

Since $z^* \in C_\infty$, then $T^b(z^*) \in C_\infty$ for every $b \in \mathbb{N}$ and

$$\begin{aligned}\tau(T^b(z^*)) &= \limsup_{n\to\infty} \rho(T^n(x_0), T^b(z^*)) \\ &\leq k_b(x_0) \limsup_{n\to\infty} \rho(T^{n-b}(x_0), z*) + l_b(x_0) \\ &= k_b(x_0)\tau_0 + l_b(x_0).\end{aligned}$$

So, $\limsup_{b\to\infty} \tau(T^b(z*)) \leq \limsup_{b\to\infty} k_b(x_0)\tau_0 + \limsup_{b\to\infty} l_b(x_0)$.
But $\limsup_{b\to\infty} k_b(x_0) \leq 1$ and $\lim_{b\to\infty} l_b(x_0) = 0$ (see Definition 2.3).
Hence, $\tau_0 \leq \limsup_{b\to\infty} \tau(T^b(z^*)) \leq \tau_0$.
It is now easy to show that

$$\tau_0 = \lim_{b\to\infty} \tau(T^b(z^*)).$$

Therefore, $\{T^b(x)\}$ is a minimizing sequence of τ. Applying Theorem 3.1, we get that $\{T^b(z^*)\}$ converges to z^*. By continuity of T, we have that

$$T(z^*) = T(\lim_{n\to\infty} T^n(z^*)) = \lim_{n\to\infty} T^{n+1}(z^*) = z^*.$$

Hence, z^* is a fixed point of T. This completes the proof. □

Corollary 3.5. Let $(M, \rho, \preceq)$ be a complete hyperbolic metric spacce endowed with partial order $\preceq$ for which order intervals are convex and closed. Suppose M satisfies the property (R). Let C be a nonempty, bounded, closed, convex subset of M. Let $T : C \to C$ be a generalized monotone continuous asymptotic pointwise contraction. Then, T has a fixed point if and only if there exists $x_0 \in C$ such that x_0 and Tx_0 are comparable.

Proof: Observe that since T is a generalized monotone asymptotic pointwise contraction, there exists a sequence of functions $k_n : C \to [0, \infty)$ with $\lim_{n\to\infty} k_n(x_0) = k(x_0) \in (0, 1)$ and $l_n : C \to [0, \infty)$ with $\lim_{n\to\infty} l_n(x_0) = 0$. The result easily follows as in Theorem 3.4. Thus, this completes the proof □

Remark 3.6. If we consider that $T : C \to C$ is monotone asymptotic pointwise nonexpansive, then we have the following corollary which is a better version of Theorem 3.3 of Dehaish and Khamsi [9] as it does not require the condition that C is not to be reduced to one point.

Corollary 3.7. Let $(M, \rho, \preceq)$ be a complete hyperbolic metric space endowed with partial order $\preceq$ for which order intervals are convex and closed. Suppose M satisfies the property (R). Let C be a nonempty, bounded, closed, convex subset of M. Let $T : C \to C$ be a monotone asymptotic continuous pointwise nonexpansive. Then, T has a fixed point if and only if there exists $x_0 \in C$ such that x_0 and Tx_0 are comparable.

Proof. The result easily follows from Theorem 3.4 with $\{l_n(x)\} = 0$ for all $n \in \mathbb{N}$. □

4. CONCLUDING REMARKS

The theorems obtained in this paper extend, generalize, improve and unify some existing results; in particular, the work of Dehaish and Khamsi [9] is a special case of the results documented in this paper.

ACKNOWLEDGEMENTS

The authors will like to thank the Simons Foundation and the coordinators of Simons Foundation for Sub-Sahara Africa Nationals with base at Department of Mathematics and Statistical Sciences, Botswana International University of Science and Technology, Botswana, for providing financial support that helped in carrying out this research. Constructive criticisms by the anonymous reviewers that helped to improve the quality of this paper are equally acknowledged.

NOMENCLATURE

All terminologies, notations and symbols used in this work are clearly explained in the body of this paper.

References

[1] M. R. Alfuriaidan and M. A. Khamsi (2015). Fixed points of monotone nonexpansive mappings on hyperbolic metric space with a graph. Fixed point theory and applications, *Fixed Point Theory and Applications* (2015) 2015:44, DOI 10.1186/s13663-015-0294-5. .

[2] M. R. Alfuriaidan and M. A. Khamsi. A fixed point theorem for monotone asymptotically nonexpansive mappings, *Proc. Amer. Math. Soc.*, 146, (2018) number 6, 2451-2456.

[3] I. Beg. Inequalities in metric spaces with applications, *Topol. Methods Nonlinear Anal.* 17 (2001) 183-190.

[4] X. Belong and M. A. Noor. Fixed point iterations for asymptotically nonexpansive mappings in Banach spaces. *J. Math. Anal. Appl.* 267 (2002) 444-453.

[5] B. A. Bin Dehaish and M. A. Khamsi. Browder and Ghode fixed point theorem for monotone nonexpansive mappings. *Fixed point theory and Applications,* (2016), vol. 9, no. 20.

[6] M. Bridson, A. Haefliger. Metric spaces of non-positive curvature, *Springer-Verlag, Berlin, Heidelberg, New York*, (1999).

[7] F. Bruhat and J. Tits. Groupes reductifs sur un corps local. I. Donnees radicielles valuees, *Publ. Math. Inst. Hautes Etudes Sci.* 41 (1972) 5251.

[8] H. Busemann. Spaces with non-positive curvature, *Acta. Math.* 80 (1948) 259-310.

[9] B. A. B. Dehaish and M. A. Khamsi. Monotone Asymptotic Pointwise Contractions, *Faculty of Sc. and Math.* (2017) 3291-3294.

[10] K. Goebel and W. A. Kirk. A fixed point theorem for asymptotically nonexpansive mappings, *Proc. Amer. Math. Soc.* (1972), 171-174.

[11] K. Goebel and S. Reich. Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings, *Series of Monographs and Textbooks in Pure and Applied Mathematics* vol. 83, Dekker, New York, (1984).

[12] N. Hussain and M. A. Khamsi. On asymptotic pointwise contractions in Metric spaces. *Nonlinear Anal.* (2009) doi-10.1016/j.na.2009.02.126.

[13] W. A. Kirk. Fixed points of asymptotic contractions, *J. Math. Anal. Appl.*, 277, (2003) 645-650.

[14] W. A. Kirk and H. K. Xu. Asymptotic pointwise contractions,*Nonlinear Anal.* 69 (2008), 4706-4712.

[15] W. A. Kirk. Fixed point theory for nonexpansive mappings, I and II, Lecture notes in Mathematics, *Springer Berlin* 886(1981).

[16] M. A. Khamsi. On metric spaces with uniform structure, *Proc. Amer. Math. Soc.* 106 (1989), 723-726.

[17] M. A. Khamsi. A. R. Khan, Inequalities in metric spaces with applications, *Nonlinear Anal. TMA* 74 (2011) no. 12, 4036-4045.

[18] U. Kohlenbach and L. Leustean. Asymptotically nonexpansive mappings in uniformly convex hyperbolic spaces, *Euro. Math. Soc.*, 12, (2010) 71-92.

[19] L. Leustean. Nonexpansive iterations in uniformly convex W-hyperbolic space Nonlinear Analysis and optimisation. Nonlinear Analysis in Contemporary Maths., vol. 513 (2010) 193-210. American Mathematical Society, Providence, RI, USA.

[20] A. A. Mebawondu, L. O. Jolaos and H. A. Abass. On some fixed points properties and convergence theorems for a Banach operator in hyperbolic spaces. Int. J. Nonlinear Anal. and Appl. 8 (2017) no. 2, 293-306.

[21] H.K. Xu. Inequalities in Banach spaces with applications, *Nonlinear Anal.* 16 (1991) 1127-1138.

[22] H. Zegeye and N. Shahzad. Convergence of Mann's type iteration method for Generalized asymptotically nonexpansive mappings, *Comp. Math. Appl.*, vol. 62 (2011), no. II, pp. 4007-4014.

DEPARTMENT OF MATHEMATICS, NNAMDI AZIKIWE UNIVERSITY, AWKA, NIGERIA

E-mail address: chisarah2013@yahoo.com

DEPARTMENT OF MATHEMATICS, NNAMDI AZIKIWE UNIVERSITY, AWKA, NIGERIA
E-mail address: nchedonwankwor@gmail.com

DEPARTMENT OF MATHEMATICS, FEDERAL UNIVERSITY OF TECHNOLOGY, OWERRI, NIGERIA
E-mail address: nnamdi.araka@futo.edu.ng

DEPARTMENT OF MATHEMATICS, NNAMDI AZIKIWE UNIVERSITY, AWKA, NIGERIA
E-mail addresses: eu.ofoedu@unizik.edu.ng or euofoedu@yahoo.com