Journal of the Nigerian Mathematical Society
Vol. 41, Issue 2, pp. 163-191, 2022

GENERALIZED CASH-TYPE SECOND DERIVATIVE EXTENDED BACKWARD DIFFERENTIATION FORMULAS FOR STIFF SYSTEMS OF ODES

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ABSTRACT. In this paper, a generalized Cash-type second derivative extended backward differentiation formulas (GCE2BD) is developed as boundary value methods (BVMs) for the numerical solution of stiff systems of ordinary differential equations (ODEs). The proposed class of methods is $O_{v,\ (k+1)-v}-$stable and $A_{v,\ (k+1)-v}$ -stable with $(v,\ (k+1)-v\ )$-boundary conditions and order $k+3$ for all values of the step-length $k\ \geq\ 1$. The class of methods proposed is exceptional for the numerical solution of stiff systems whose Jacobians have some relatively large eigenvalues near the imaginary axis. The accuracy and efficiency of the constructed methods are examined in some details via the numerical experiments carried out on some well-known linear and non-linear stiff systems. The boundary value technique considered allows the solutions of a problem to be obtained simultaneously on the entire interval of integration. The new class of methods is found to compare favorably with existing standard methods in the literature.

Keywords and phrases: Linear multistep formulas, Boundary value methods, $O_{k_1,\ k_2}-$stable, $A_{k_1,\ k_2}-$stable, Super future points, Cash methods.
2010 Mathematical Subject Classification: 65L04, 65L05

1. INTRODUCTION

The problem of deriving efficient algorithms for the numerical integration of stiff systems of ODEs of the form:

$$y' = f\left(x, y\right), \qquad y\left(a\right) = y_0, \quad x \in [a, b] \tag{1}$$

Received by the editors October 11, 2021; Revised: July 05, 2022; Accepted: July 22, 2022
www.nigerianmathematicalsociety.org; Journal available online at https://ojs.ictp.it/jnms/

where $f{:}\mathbb{R}\times\mathbb{R}^m \longrightarrow \mathbb{R}^m$ satisfies a Lipschitz condition (see [25]) and $y, y_0 \in \mathbb{R}^m$ has been analyzed extensively over the years past and as a result a wide variety of approaches (see [41] for example) have been proposed. The system of ODEs (1) is stiff if the magnitude of the Jacobian is large. According to Dahlquist [18], a potentially good numerical method for the solution of stiff systems of ODEs must have good accuracy and some reasonably wide region of absolute stability. It was on this ground the so-called A-stability property was required. Accordingly, Cash [13] proposed that as far as stability is concerned, the property of A-stability is an excellent one for a code intended for the solution of stiff systems to possess. However, the A-stability requirement proposed by Dahlquist [18] came with the restriction that the most accurate A-stable method is the trapezoidal rule which is of order 2 and as a result "the problem confronting numerical analysts was to derive high-order methods which have the stability necessary for dealing with stiff differential systems" Cash [13]. For many years, the backward differentiation formulae (BDF) proposed by Gear [23] have been the most prominent and most widely used for the solution of stiff systems due to its A-stability property for step-lengths $k = 1$ and $k = 2$ and orders 1 and 2 respectively. Since then several authors carried out invaluable research to extend the A-stability property beyond $k = 2$ with higher order. In order to obtain methods with higher degree of accuracy Bickart and Rubin [7] stated that the conventional linear multistep method (LMM) should be modified to another class of methods to circumvent the Dahlquist's Barrier [18]. Afterwards Hairer and Wanner [24] stated that the search for higher order A-stable multistep methods is carried out in two main directions: use higher order derivatives of the solutions, throw in additional stages, off-step points, super-future points and the likes. This leads into the large field of general linear methods. Obrechkoff [40] introduced a general multi-derivative method for solving systems of ODEs. Special cases of the Obrechkoff multi-derivative method were later derived by Enright [20], Cash [14], Jia-Xiang and Jiao-Xun [29], Ehigie et al. [19], Ngwane and Jator [34], Longe and Adeniran [33], Nwachukwu and Okor [37, 38] and recently Ogunfeyitimi and Ikhile [42, 43]. Brugnano and Trigiante [9, 11] introduced BVMs which "are numerical methods based on Linear Multistep Formulae (LMF) and are renowned for high-order accuracy and unconditional stability" Chan et. al. [16]. These methods approximate a given continuous initial value problem (IVP) by means of a

discrete boundary value problem (BVP). The solution of the IVP is given simultaneously at all grid points. BVMs overcome the limitations of the well-known Dahlquist order and stability barrier for an A-stable LMM. More so, the reader is advice to see ([1, 5, 6, 8-11, 33, 36-39]) for more details on the stability properties of BVMs with respect to A-stability.
The second derivative linear multistep method (SDLMM) of Enright [20],

$$y_{n+k} = y_{n+k-1} + h\sum_{j=0}^{k}\beta_j f_{n+j} + h^2\gamma_k g_{n+k} \tag{2}$$

which is A-stable for $k = 1(1)2$ and $A(\alpha)-$stable for $k = 3(3)7$ with order $k+2$ is extended by Cash [14] via the super future point technique to obtain two new classes of methods called second derivative extended backward differentiation formulas (E2BD). These methods became superior to the Enright scheme in terms of order and accuracy. They are given in the form of formulas of Class 1 (A-stable up-to $k = 5$ with order $k+3$) and formulas of Class 2 (A-stable up-to $k = 3$ with order $k+3$)).
The aim of this paper is to develop Cash [14] E2BD-type BVMs with A-stable methods for all values of the step-length $k \geq 1$, larger regions of absolute stability and no barriers concerning the maximum order attainable. Also, the proposed methods will be implemented using the boundary value technique in the sense of [9, 11, 36-38] whereby all approximations $(y_1, y_2, y_3, \ldots, y_N)^T$ of the solution of (1) are simultaneously generated on the entire interval. The advantage of this implementation approach is that the global errors at the end of the interval are smaller than those produced by the step-by-step methods as in [31], see Ehigie et al. [19].
Next sections of the paper are organized as follows: In Section 2, the theoretical procedure on which BVMs are developed and analyzed is given. In Section 3, the generalized Cash-type second derivative extended backward differentiation formulas (GCE2BD) is derived and analyzed. Section 4 is devoted to the implementation approach considered. In Section 5 some numerical experiments to confirm the theoretical results in Section 3 are presented. Lastly, some concluding remarks are given in Section 6.

2. THEORETICAL PROCEDURES

The continuous IVP (1) is usually solved by means of a discrete IVP, that is, a set of k initial conditions $y_0,\ y_1, \ldots, y_{k-1}$ is associated with the LMF:

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j} \tag{3}$$

If k_1 and k_2 are two integers such that $k_1 + k_2 = k$ then one may impose the k conditions for the LMF (3) by fixing the first $k_1 \leq k$ values of the discrete solution $y_0,\ y_1, \ldots, y_{k_1-1}$ and the last $k_2 \equiv k - k_1$ values $y_{N-k_2+1}, \ldots, y_N$ so that the discrete problem becomes:

$$\sum_{j=-k_1}^{k_2} \alpha_{j+k_1} y_{n+j} = h \sum_{j=-k_1}^{k_2} \beta_{j+k_1} f_{n+j}, \qquad n = k_1, \ldots, N - k_2,$$

$$y_0,\ y_1, \ldots, y_{k_1-1}, \quad y_{N-k_2+1}, \ldots, y_N, fixed. \tag{4}$$

Thus, the given continuous IVP is approximated by means of a discrete BVP. The methods obtained in this way are called Boundary Value Methods (BVMs) with $(k_1,\ k_2)$-boundary conditions. Observe that, for $k_1 = k$ and therefore $k_2 = 0$, one has the initial value methods (IVMs). So, the class of IVMs is a subclass of BVMs for ODEs based on LMF (see [12]). Since the continuous problem (1) only provides the initial solution y_0, the remaining $y_1, \ldots, y_{k_1-1}$, initial values and $y_{N-k_{2+1}}, \ldots, y_N$ final values need to be found by introducing a set of $k-1$ additional equations which are derived by a set of $k_1 - 1$ additional initial methods.

$$\sum_{j=-k_1}^{k_2} \alpha_j^{(i)} y_j = h \sum_{j=-k_1}^{k_2} \beta_j^{(i)} f_j, \qquad i = 1, \ldots, k_1 - 1 \tag{5}$$

and k_2 final methods

$$\sum_{j=-k_1}^{k_2} \alpha_{K-j}^{(i)} y_{N-j} = h \sum_{j=-k_1}^{k_2} \beta_{k-j}^{(i)} f_{N-j}, \qquad i = N - k_2 + 1, \ldots, N \tag{6}$$

Practically equations (4), (5) and (6) form a composite scheme of the same order.

2.1 STABILITY CONCEPT FOR BVMs

The stability of the family of methods to be considered is characterized by two kinds of polynomials: $S_{k_1k_2}$ and $N_{k_1k_2}$.

Definition 1: *A* polynomial $p(z)$ of degree $k = k_1 + k_2$ *is an* $S_{k_1k_2}$-polynomial if its roots are such that

$$|z_1| \leq |z_2| \leq \cdots \leq |z_{k_1}| < 1 < |z_{k_1+1}| \leq \ldots \leq |z_k|$$

and it is an $N_{k_1k_2}$-polynomial if

$$|z_1| \leq |z_2| \leq \cdots \leq |z_{k_1}| \leq 1 < |z_{k_1+1}| \leq \ldots \leq |z_k|$$

with simple roots of unit modulus.
Observe that for $k_1 = k$ and $k_2 = 0$, an $N_{k_1k_2}$-polynomial reduces to a Von Neumann polynomial and an $S_{k_1k_2}$-polynomial reduces to a Schur polynomial. Let $\rho(z) = \sum_{j=0}^{k} \alpha_j z^j$ and $\sigma(z) = \sum_{j=0}^{k} \beta_j z^j$ denote the two characteristic polynomials associated with the LMF (3). Then $\pi(z, q) = \rho(z) - q\sigma(z)$, $q = h\lambda$ is the stability polynomial when (3) is applied on the test problem $y' = \lambda y, \quad Re(\lambda) < 0$. Thus, we have the following definitions for BVMs (see, [10, 12]):

Definition 2: A BVM with $(k_1,\ k_2)-$boundary conditions is $O_{k_1k_2}$-stable if ρ (z) is an $N_{k_1k_2}$-polynomial.
Observe that for $k_1 = k$ and $k_2 = 0$, $O_{k_1k_2}$-stablity reduces to the usual zero-stability for IVMs.

Definition 3 (a) For a given q∈C, a BVM with $(k_1,\ k_2)-$boundary conditions is $(k_1,\ k_2)-$absolutely stable if π(z, q) is an $S_{k_1k_2}-$ polynomial. Again, $(k_1, k_2)-$ absolute stability reduces to the usual notion of absolute stability when $k_1 = k$ and $k_2 = 0$ for LMF (3).
Similarly, one defines
(b) the region of $(k_1,\ k_2)-$absolute stability of the method as $\mathcal{D}_{\mathrm{k_1k_2}} = \{\ \mathrm{q}\in\mathbb{C}\ :\ \pi(\mathrm{z,q})\ \ \textit{is an}\ \ S_{k_1k_2} -\ \textit{polynomial}\}$. Here $\pi(z, q)$ is a polynomial of type $(k_1, 0,\ k_2)$.
(c) A BVM with $(k_1,\ k_2)-$boundary conditions is said to be $A_{k_1k_2}-$stable if $\mathbb{C}^- \subseteq \mathcal{D}_{k_1k_2}$.
$\mathcal{D}_{k_1k_2} = \{\ q \in \mathbb{C} : \pi(\mathcal{Z},\ q)$ is of type $(k_1, 0,\ k_2)\}$.
Finally, a BVM with $(k_1,\ k_2)-$boundary conditions is said to be $A_{k_1k_2}-$stable if $\mathbb{C}^- \subseteq \mathcal{D}_{k_1k_2}$.

3. DERIVATION AND ANALYSIS OF THE METHOD

The general second derivative extended linear multistep formula for the numerical solution of (1) is given in the form:

$$\sum_{j=0}^{k}\alpha_j y_{n+j} = h\sum_{j=0}^{k+1}\beta_j^* f_{n+j} + h^2\sum_{j=0}^{k+1}\gamma_j^* g_{n+j} \tag{7}$$

where $y_{n+j} \approx y\left(x_n + jh\right), \quad f_{n+j} \equiv f(x_n + jh, \quad y\left(x_n + jh\right))$ and $g_{n+j} = \left.\frac{df(x, \quad y(x))}{dx}\right|_{y=y_{n+j}}^{x=x_{n+j}}$
while, $\alpha_j, \quad \beta_j^*$ and γ_j^* are parameters to be determined.
Following Brugnano and Trigiante [9-11] and Cash [14] the extended scheme associated with (2) can be written generally as

$$y_{n+i} - y_{n+i-1} = h\sum_{j=0}^{k+1}\beta_j^* f_{n+j} + h^2\gamma_i^* g_{n+i}\ ;\ i = 0\,(1)\,k \tag{8}$$

We note that for $i = k$ one gets the conventional E2BD of Cash [14]. However, for $i \neq k$ we can choose the values of i which provide methods with the best stability properties for all values of $k \geq 1$. Practically, we get the best stability properties for the choice $i = v$ such that

$$v = \begin{cases} \frac{k+1}{2} & for\ odd\ k \\ \\ \frac{k}{2} & for\ even\ k \end{cases} \tag{9}$$

Therefore (8) becomes

$$y_{n+v} - y_{n+v-1} = h\sum_{j=0}^{k+1}\beta_j^* f_{n+j} + h^2\gamma_v^* g_{n+v}, \tag{10}$$

The class of methods (10) having order $p = k+3$ is called the generalized Cash-type second derivative extended backward differentiation formulas (GCE2BD). It is $O_{v,(k+1)-v}-$stable and $A_{v,(k+1)-v}-$stable for all values of the step length $k \geq 1$ with $(v,\ (k+1)-v)$-boundary conditions (i.e. with v number of roots inside the unit circle and $(k+1)-v$ number of roots outside the unit circle). Also, the proposed class of methods have relatively small error constant as k increases (see Table 2, Fig. 1).
Rewriting (10) in the form:

$$y\left(x + vh\right) - y\left(x + (v-1)h\right) - h\sum_{j=0}^{k+1}\beta_j^* y'(x + jh)$$

$$- h^2\gamma_v^* y''(x+vh) = 0 \tag{11}$$

expanding in Taylor's series and applying the method of undetermined coefficient we obtained the coefficients of the methods (10) for $k = 1(1)10$ as shown in Table 1 and Table 2.

3.1 ORDER CONDITION OF THE METHOD

In the spirit of Fatunla [21] and Lambert [32] we define the local truncation error (LTE) associated with (10) as the linear difference operator $\mathcal{L}\left[y\left(x\right);h\right]$ such that;

$$\mathcal{L}\left[y\left(x\right);h\right] = y\left(x+vh\right) - y\left(x+\left(v-1\right)h\right) - h\sum_{j=0}^{k+1}\beta_j^* y'\left(x+jh\right)$$

$$- h^2\gamma_v^* y''\left(x+vh\right) \tag{12}$$

Assuming that $y(x)$ is sufficiently differentiable, we can find the Taylor series expansion of the terms in (12) about the point x

$$\mathcal{L}\left(y\left(x\right);h\right) = C_0 y\left(x\right) + C_1 h y'\left(x\right) + C_2 h^2 y''\left(x\right) + \cdots + C_r h^r y^r\left(x\right) + \ldots$$

Where,

$$C_0 = \sum_{j=0}^{k+1}\alpha_j$$

$$C_1 = 1 - \sum_{j=0}^{k+1}\beta_j$$

$$C_2 = \frac{1}{2!}\left(-(v-1)^2 + v^2\right) - \sum_{j=0}^{k+1} j\beta_j - \gamma_v$$

$$C_3 = \frac{1}{3!}\left(-(v-1)^3 + v^3\right) - \frac{1}{2!}\sum_{j=0}^{k+1} j^2\beta_j - v\gamma_v \tag{13}$$

$$\vdots$$

$$C_r = \frac{1}{r!}\left(-(v-1)^r + v^r\right) - \frac{1}{(r-1)!}\sum_{j=0}^{k+1} j^{r-1}\beta_j - \frac{v^{r-2}\gamma_v}{(r-2)!},$$

for $r = 0(1)p$. Thus, the class of methods (10) is of order p if

$$C_0 = C_1 = C_2 = \cdots = C_p = 0 \quad and \quad C_{p+1} \neq 0 \tag{14}$$

where C_{p+1} is the error constant (EC) of the methods (10) and $C_{p+1}h^{p+1}y^{p+1}(x)$ is the principal LTE at the point x (see [24]). The order and the error constants of the GCE2BD (10) are presented in Table 2.

Table 1. Coefficients, Error Constant (EC) and Order p of GCE2BD for $k = 1(1)10$

k	v	β_0^*	β_1^*	β_2^*	β_3^*	β_4^*	β_5^*	β_6^*	β_7^*
1	1	$\frac{7}{24}$	$\frac{2}{3}$	$\frac{1}{24}$					
2	1	$\frac{97}{360}$	$\frac{19}{30}$	$\frac{13}{120}$	$\frac{-1}{90}$				
3	2	$\frac{-7}{960}$	$\frac{59}{180}$	$\frac{19}{30}$	$\frac{1}{20}$	$\frac{-11}{2880}$			
4	2	$\frac{-107}{20160}$	$\frac{97}{315}$	$\frac{586}{945}$	$\frac{113}{1260}$	$\frac{-277}{20160}$	$\frac{1}{756}$		
5	3	$\frac{289}{362880}$	$\frac{-503}{40320}$	$\frac{13861}{40320}$	$\frac{586}{945}$	$\frac{2171}{40320}$	$\frac{-53}{8064}$	$\frac{191}{362880}$	
6	3	$\frac{409}{777600}$	$\frac{-17483}{1814400}$	$\frac{197611}{604800}$	$\frac{13903}{22680}$	$\frac{29843}{362880}$	$\frac{-9127}{604800}$	$\frac{13169}{5443200}$	$\frac{-23}{113400}$
7	4	$\frac{-3391}{29030400}$	$\frac{643}{362880}$	$\frac{-58703}{3628800}$	$\frac{640307}{1814400}$	$\frac{13903}{22680}$	$\frac{101741}{1814400}$	$\frac{-1241}{145152}$	$\frac{2129}{1814400}$
8	4	$\frac{-69823}{958003200}$	$\frac{4969}{3991680}$	$\frac{-519493}{39916800}$	$\frac{6748817}{19958400}$	$\frac{379571}{623700}$	$\frac{1560991}{19958400}$	$\frac{-42389}{2661120}$	$\frac{3119}{950400}$
9	5	$\frac{6289}{319334400}$	$\frac{-611321}{1916006400}$	$\frac{173839}{63866880}$	$\frac{-1510661}{79833600}$	$\frac{57292261}{159667200}$	$\frac{379571}{623700}$	$\frac{9186203}{159667200}$	$\frac{-7619}{760320}$
10	5	$\frac{25797689}{2179457280000}$	$\frac{-61321669}{290594304000}$	$\frac{523498609}{261534873600}$	$\frac{-113923639}{7264857600}$	$\frac{1005044731}{2905943040}$	$\frac{294593723}{486486000}$	$\frac{261825097}{3459456000}$	$\frac{-119892569}{7264857600}$

Table 2. Table 1 continued

k	v	β_8^*	β_9^*	β_{10}^*	β_{11}^*	γ_v^*	C_{p+1}	p
1	1					$-\frac{1}{4}$	$-\frac{1}{180}$	4
2	1					$-\frac{19}{60}$	$\frac{7}{2400}$	5
3	2					$-\frac{11}{48}$	$\frac{1}{1512}$	6
4	2					$-\frac{271}{1008}$	$-\frac{289}{846720}$	7
5	3					$-\frac{191}{864}$	$-\frac{23}{226800}$	8
6	3					$-\frac{3233}{12960}$	$\frac{3391}{65318400}$	9
7	4	$\frac{-2497}{29030400}$				$-\frac{2497}{11520}$	$\frac{263}{14968800}$	10
8	4	$\frac{-153707}{319334400}$	$\frac{263}{7484400}$			$-\frac{90817}{380160}$	$-\frac{6289}{702535680}$	11
9	5	$\frac{192103}{106444800}$	$\frac{-50063}{212889600}$	$\frac{14797}{958003200}$		$-\frac{14797}{69120}$	$-\frac{133787}{40864824000}$	12
10	5	$\frac{345704453}{87178291200}$	$\frac{-27037529}{34871316480}$	$\frac{74011757}{726485760000}$	$\frac{-133787}{20432412000}$	$-\frac{109551893}{471744000}$	$\frac{4522787}{2719962685440}$	13

We observed from the literature that, although some generalizations of the Enright and BDF family of methods exist, which include:

the GBDF by Brugnano and Trigiante [11] with order $p = k$, the SDGBDF ($p = k+1$) and TDGBDF ($p = k+2$) of Nwachukwu and Okor [37, 38], the GSDLMME ($p = k+2$) and SDGEBDFs ($p = 2k$) of Ogunfeyitimi and Ikhile [42, 43], the GCE2BD ($p = k + 3$) promises better approximate solutions to stiff systems of ODE per step than the other generalizations in the literature (see, Brugnano and Trigiante [11]). This is because the GCE2BD possesses a significantly smaller error constant per step than the other methods mentioned. Hence, in Fig. 1, we show their comparisons for $k = 1\,(1)\,5$.

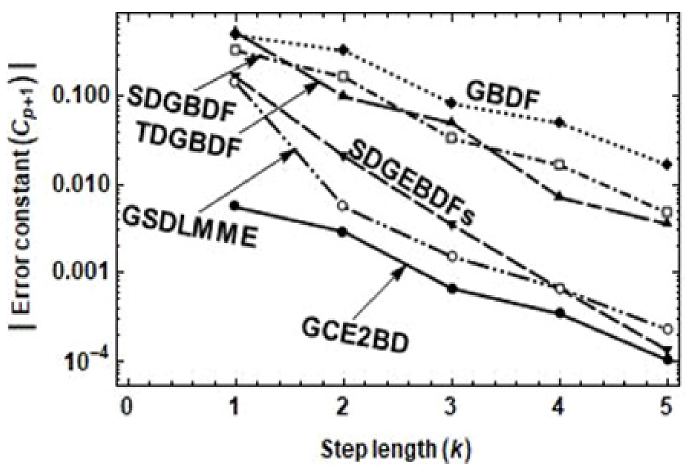

Fig. 1. Semi-Log plot of the Absolute value of the error constant against step length (k) of the GBDF, SDGBDF, TDGBDF, SDGEBDFs, GSDLMME and the GCE2BD

3.2 STABILITY PROPERTIES OF GCE2BD

To analyze the stability of the proposed class of methods (10) (see [24]), we apply it on the test problem

$$y' = \lambda y, \qquad y'' = \lambda^2 y \tag{15}$$

to obtain the characteristic equation

$$z^v\left(1 - q^2\gamma_v^*\right) - z^{v-1} - \sum_{j=0}^{k+1} q\beta_j^* z^j = 0, \qquad q = \lambda h, \qquad q \in \mathbb{C} \tag{16}$$

Inserting $z = e^{i\theta}$, $i = 0\,(1)\,k$, $\theta\ \epsilon\ [0, 2\pi]$ in (16) yields a polynomial of degree two in q. The two roots of q are functions of θ describing the stability domain of the GCE2BD (10) given in Figs. 1 and 2 for odd and even values of k respectively. Note that the stability plots of the proposed class of methods (10) for the first 30 values of k presented in Figs. 1 and 2 show distinctively that methods associated with the new class of methods are all $A_{v,\ (k+1)-v} - stable$(since its region of $\mathcal{D}_{v,(k+1)-v}$ contains the entire left half of the complex q plane $\left(\mathbb{C}^- \subseteq \mathcal{D}_{v,(k+1)-v}\right)$, see Definition 3) and the sizes of the region of absolute stability (the exterior of the closed curves) is considerably large.

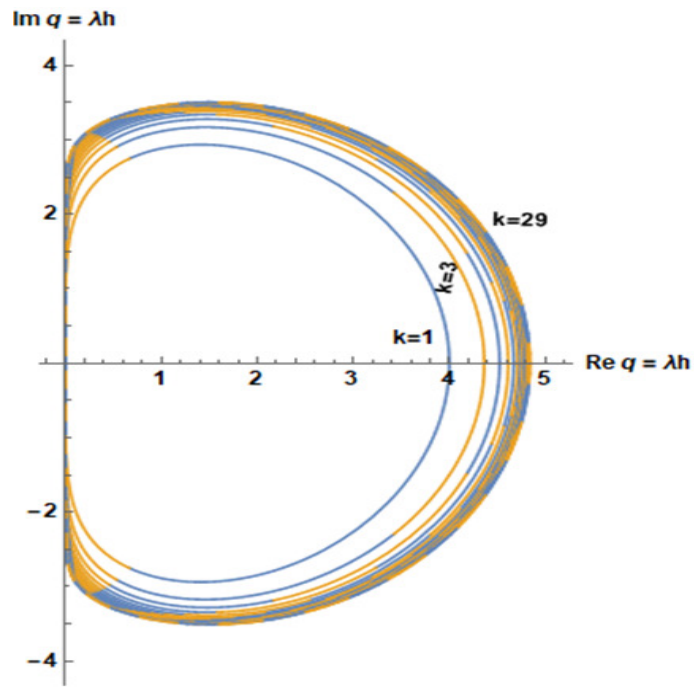

Fig. 2. Stability plot of GCE2BD for $k = 1(2)29$

If the values of the coefficients (given in Tables 1 and 2) for a specified method of the class of methods (10) are substituted in its characteristic equation (16), the root distribution can be obtained. In fact, the coefficients in row 2 of Table 1 and 2 which defines the GCE2BD of order 4 when substituted in (16), taking $q = -100$ (since the region $\mathcal{D}_{v,(k+1)-v}$ contains the entire left half of the complex q plain) yields,

$$-1 + z = -\frac{175}{6} - \frac{7700z}{3} - \frac{25z^2}{6}$$

and solving for the roots $(z_i\ ; i = 1(1)k + 1)$ we obtain the root distribution for the GCE2BD of order 4 as,

$$\{\{z_1 = -0.01096994732530235\}\,,\ \{z_2 = -616.2290300526746\}\}\,.$$

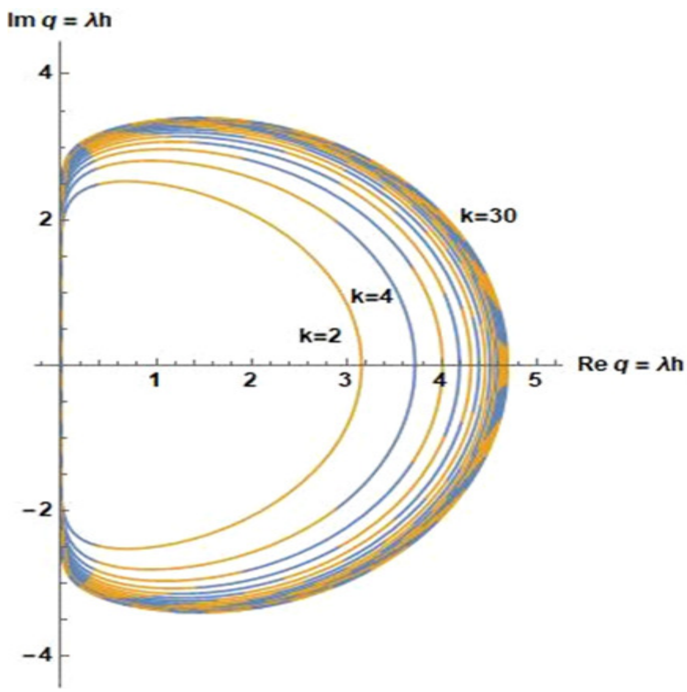

Fig. 3. Stability plot of GCE2BD for $k = 2(2)30$

Obviously, the root $|z_1|$ is strictly inside the unit circle, no root(s) with unit modulus and the root $|z_2|$ is strictly outside the unit circle. Thus, we have a root distribution equivalent to type $(1,\ 0,\ 1)$.
Similarly,
The roots of the GCE2BD of order 11 is obtained as,
$\{\{z_1 = -11.6263\}, \{z_2 = -0.0560\}, \{z_3 = 0.0340\}, \{z_4 = -1.8627$ $-14.2981\}, \{z_5 = -1.8627 + 14.2981\}, \{z_6 = 0.0043 - 0.0393\},$ $\{z_7 = 0.0043 + 0.0393\}, \{z_8 = 14.5314 - 8.7955\}, \{z_9 = 14.5314$ $+8.7955\}\}$, thus has a root distribution of type $(4,\ 0,\ 5)$.

The roots of the GCE2BD of order 12 is obtained as,
$\{z_1 = -13.5354\}, \{z_2 = -0.0846\}, \{z_3 = -2.2628 - 16.4919\}, \{z_4 = -2.2628 + 16.4919\},$ $\{z_5 = -0.0095 - 0.0621\}, \{z_6 = -0.0095 + 0.0621\}, \{z_7 = 0.0439 - 0.0274\},$ $\{z_8 = 0.0439 + 0.0274\}, \{z_9 = 16.6509 - 10.1458\}, \{z_{10} = 16.6509 + 10.1458\}$
,
thus has a root distribution of type $(5,\ 0,\ 5)$.

The roots of the GCE2BD of order 13 is obtained as,
$\{z_1 = -9.4267\}, \{z_2 = -0.0748\}, \{z_3 = 14.8481\}, \{z_4 = -3.5932 - 10.9978\},$ $\{z_5 = -3.5932 + 10.9978\}, \{z_6 = -0.0087 - 0.0552\}, \{z_7 = -0.0087 + 0.0552\},$

$\{z_8 = 0.0391 - 0.0245\}, \{z_9 = 0.0391 + 0.0245\}, \{z_{10} = 8.6689 - 10.9156\}$, $\{z_{11} = 8.6689 + 10.9156\}$ and thus it has a root distribution of type $(5,\ 0,\ 6)$.

So, it is easy to see that the characteristic polynomial $\pi(z,q) = z^v(1 - q^2\gamma_v^*) - z^{v-1} - \sum_{j=0}^{k+1} q\beta_j^* z^j$ (where "z" is the shift operator which also denotes the characteristic roots of the GCE2BD (10) and $q \in \mathbb{C}$) of the GCE2BD is of type $(v, 0, (k+1)-v)\forall k \geq 1$. Hence it is an $S_{v,(k+1)-v} - polinomial$. In analogous with $(k_1, k_2) - stability$ illustrated for BVMs in Section 2, here, $k_1 = v,\ k_2 = (k+1) - v$ and $k_1 + k_2 = k+1$. Thus, the distribution, $|z_1| \leq |z_2| \leq \cdots \leq |z_{k_1}| < 1 < |z_{k_1+1}| \leq \ldots \leq |z_{k+1}|$. The knowledge of the root distribution of the scheme ensures the correct use of the methods as would be seen in Section 4.

4. USE OF METHODS

Here, the implementation procedure for the GCE2BD (10) as BVMs in the sense of [9, 11, 36-38] is given. The proposed methods (10) are conveniently used with the following set of $v-1$ additional initial methods

$$y_i - y_{i-1} = h\sum_{j=0}^{k+1} \beta_j^{*(i)} f_j + h^2\gamma_v^{*(i)} g_v\ ; i = 1,\ 2,\ \ldots,\ v-1;\ (n=0) \tag{17}$$

and $(k+1) - v$ final methods

$$y_{N+i} - y_{N+i-1} = h\sum_{j=0}^{k+1} \beta_{N+j}^{*(i)} f_{N+j} + h^2\gamma_{N+v}^{*(i)} g_{N+v}\ ;\ \ i = v+1,\ \ldots,\ N;\ (n=N) \tag{18}$$

Since the continuous problem (1) provides only the initial solution y_0. As a result, the GCE2BD of order 6 (GCE2BD3) requires 1 initial method and 2 final methods, the GCE2BD of order 7 (GCE2BD4) requires 1 initial method and 3 final methods, the GCE2BD of order 8 (GCE2BD5) requires 2 initial methods and 3 final methods, GCE2BD of order 9 (GCE2BD6) requires 2 initial methods and 4 final method, GCE2BD of order 10 (GCE2BD7) requires 3 initial methods and 4 final methods and GCE2BD of order 11 (GCE2BD8) requires 3 initial methods and 5 final methods.

Example 1: The, GCE2BD3 given as:

$$y_{n+2} - y_{n+1} = h\left(-\frac{7}{960}f_n + \frac{59}{180}f_{n+1} + \frac{19}{30}f_{n+2} + \frac{1}{20}f_{n+3} - \frac{11}{2880}f_{n+4}\right) - \frac{11}{48}h^2 g_{n+2}$$

is used with 1 initial method given as:

$y_1 - y_0 = h\left(\frac{869}{2880}f_0 + \frac{229}{180}f_1 - \frac{11}{30}f_2 - \frac{41}{180}f_3 + \frac{59}{2880}f_4\right) + \frac{9}{16}h^2 g_2$

and 2 final methods given respectively as:

$y_{N+3} - y_{N+2} = h\left(-\frac{11}{2880}f_N + \frac{1}{20}f_{N+1} + \frac{19}{30}f_{N+2} + \frac{59}{180}f_{N+3} - \frac{7}{960}f_{N+4}\right) + \frac{11}{48}h^2 g_{N+2},$

$y_{N+4} - y_{N+3} = h\left(\frac{59}{2880}f_N - \frac{41}{180}f_{N+1} - \frac{11}{30}f_{N+2} + \frac{229}{180}f_{N+3} + \frac{869}{2880}f_{N+4}\right) - \frac{9}{16}h^2 g_{N+2}.$

Example 2: The GCE2BD4 given as:

$$y_{n+2} - y_{n+1} = h\left(-\frac{107}{20160}f_n + \frac{97}{315}f_{n+1} + \frac{586}{945}f_{n+2} + \frac{113}{1260}f_{n+3} - \frac{277}{20160}f_{n+4} + \frac{1}{756}f_{n+5}\right) - \frac{271}{1008}h^2 g_{n+2}$$

is used with 1initial method given as:

$y_1 - y_0 = h\left(\frac{643}{2240}f_0 + \frac{149}{105}f_1 - \frac{254}{945}f_2 - \frac{73}{140}f_3 + \frac{631}{6720}f_4 - \frac{37}{3780}f_5\right) + \frac{863}{1008}h^2 g_2$

and 3 final methods given respectively as:

$y_{N+3} - y_{N+2} = h\left(-\frac{37}{20160}f_N + \frac{19}{630}f_{N+1} + \frac{586}{945}f_{N+2} + \frac{463}{1260}f_{N+3} - \frac{347}{20160}f_{N+4} + \frac{1}{756}f_{N+5}\right) + \frac{191}{1008}h^2 g_{N+2},$

$y_{N+4} - y_{N+3} = h\left(\frac{13}{2240}f_N - \frac{17}{210}f_{N+1} - \frac{254}{945}f_{N+2} + \frac{137}{140}f_{N+3} + \frac{2521}{6720}f_{N+4} - \frac{37}{3780}f_{N+5}\right) - \frac{271}{1008}h^2 g_{N+2},$

$y_{N+5} - y_{N+4} = h\left(-\frac{97}{4032}f_N + \frac{97}{315}f_{N+1} + \frac{586}{945}f_{N+2} - \frac{1777}{1260}f_{N+3} + \frac{24293}{20160}f_{N+4} + \frac{1139}{3780}f_{N+5}\right) + \frac{863}{1008}h^2 g_{N+2}.$

Example 3: The GCE2BD5 given as:

$$y_{n+3}-y_{n+2} = h\left(\frac{289}{362880}f_n - \frac{503}{40320}f_{n+1} + \frac{13861}{40320}f_{n+2} + \frac{586}{945}f_{n+3} + \frac{2171}{40320}f_{n+4} - \frac{53}{8064}f_{n+5} + \frac{191}{362880}f_{n+6}\right) - \frac{191}{864}h^2 g_{n+3}$$

is used with 2 initial methods given respectively as:

$$y_1 - y_0 = h\left(\frac{104897}{362880}f_0 + \frac{53033}{40320}f_1 - \frac{79099}{40320}f_2 + \frac{586}{945}f_3 + \frac{34651}{40320}f_4 - \frac{5417}{40320}f_5 + \frac{4447}{362880}f_6\right) - \frac{1375}{864}h^2 g_{3,}$$

$$y_2 - y_1 = h\left(-\frac{907}{120960}f_0 + \frac{953}{2688}f_1 + \frac{14537}{13440}f_2 - \frac{254}{945}f_3 - \frac{2473}{13440}f_4 + \frac{71}{2688}f_5 - \frac{277}{120960}f_6\right) + \frac{13}{32}h^2 g_3$$

and 3 final methods given respectively as:

$$y_{N+4} - y_{N+3} = h\left(\frac{191}{362880}f_N - \frac{53}{8064}f_{N+1} + \frac{2171}{40320}f_{N+2} + \frac{586}{945}f_{N+3} + \frac{13861}{40320}f_{N+4} - \frac{503}{40320}f_{N+5} + \frac{289}{362880}f_{N+6}\right) + \frac{191}{864}h^2 g_{N+3},$$

$$y_{N+5} - y_{N+4} = h\left(-\frac{277}{120960}f_N + \frac{71}{2688}f_{N+1} - \frac{2473}{13440}f_{N+2} - \frac{254}{945}f_{N+3} + \frac{14537}{13440}f_{N+4} + \frac{953}{2688}f_{N+5} - \frac{907}{120960}f_{N+6}\right) - \frac{13}{32}h^2 g_{N+3,}$$

$$y_{N+6} - y_{N+5} = h\left(\frac{4447}{362880}f_N - \frac{5417}{40320}f_{N+1} + \frac{34651}{40320}f_{N+2} + \frac{586}{945}f_{N+3} - \frac{79099}{40320}f_{N+4} + \frac{53033}{40320}f_{N+5} + \frac{104897}{362880}f_{N+6}\right) + \frac{1375}{864}h^2 g_{N+3.}$$

Example 4: The GCE2BD6 given as:

$$y_{3+n} - y_{2+n} = h\left(\frac{409}{777600}f_n - \frac{17483}{1814400}f_{n+1} + \frac{197611}{604800}f_{n+2} + \frac{13903}{22680}f_{n+3} + \frac{29843}{362880}f_{n+4} - \frac{9127}{604800}f_{n+5} + \frac{13169}{5443200}f_{n+6} - \frac{23}{113400}f_{n+7}\right) - \frac{3233}{12960}h^2 g_{n+3},$$

is used with 2 initial methods given respectively as:

$$y_1 - y_0 = h\left(\frac{1520143}{5443200}f_0 + \frac{2573077}{1814400}f_1 - \frac{1559669}{604800}f_2 + \frac{8233}{22680}f_3 + \frac{685043}{362880}f_4 - \frac{267847}{604800}f_5 + \frac{439889}{5443200}f_6 - \frac{119}{16200}f_7\right) - \frac{33953}{12960}h^2 g_{3,}$$

$$y_2 - y_1 = h\left(-\frac{32687}{5443200}f_0 + \frac{614827}{1814400}f_1 + \frac{711061}{604800}f_2 - \frac{5207}{22680}f_3 - \frac{123667}{362880}f_4 + \frac{44423}{604800}f_5 - \frac{69361}{5443200}f_6 + \frac{127}{113400}f_7\right) + \frac{7297}{12960}h^2 g_3$$

and 4 final methods given respectively as:

$$y_{N+4} - y_{N+3} = h\left(\frac{199}{777600}f_N - \frac{6773}{1814400}f_{N+1} + \frac{22261}{604800}f_{N+2} + \frac{13903}{22680}f_{N+3}\right.$$

$$+\frac{135053}{362880}f_{N+4} - \frac{12697}{604800}f_{N+5} + \frac{14639}{5443200}f_{N+6} - \frac{23}{113400}f_{N+7}\Big) + \frac{2497}{12960}h^2 g_{N+3},$$

$$y_{N+5} - y_{N+4} = h\Big(-\frac{4337}{5443200}f_N + \frac{19477}{1814400}f_{N+1} - \frac{54389}{604800}f_{N+2} - \frac{5207}{22680}f_{N+3} + \frac{335603}{362880}f_{N+4} + \frac{242873}{604800}f_{N+5} - \frac{97711}{5443200}f_{N+6} + \frac{127}{113400}f_{N+7}\Big) - \frac{3233}{12960}h^2 g_{N+3},$$

$$y_{N+6} - y_{N+5} = h\Big(\frac{13393}{5443200}f_N - \frac{57173}{1814400}f_{N+1} + \frac{146581}{604800}f_{N+2} + \frac{8233}{22680}f_{N+3} - \frac{338707}{362880}f_{N+4} + \frac{608903}{604800}f_{N+5} + \frac{1946639}{5443200}f_{N+6} - \frac{119}{16200}f_{N+7}\Big) + \frac{7297}{12960}h^2 g_{N+3},$$

$$y_{N+7} - y_{N+6} = h\Big(-\frac{73937}{5443200}f_N + \frac{305077}{1814400}f_{N+1} - \frac{743189}{604800}f_{N+2} - \frac{31457}{22680}f_{N+3} + \frac{1320083}{362880}f_{N+4} - \frac{1084327}{604800}f_{N+5} + \frac{7243889}{5443200}f_{N+6} + \frac{32377}{113400}f_{N+7}\Big) - \frac{33953}{12960}h^2 g_{N+3}.$$

Example 5: The GCE2BD7 given as:

$$y_{4+n} - y_{3+n} = h\Big(-\frac{3391}{29030400}f_n + \frac{643}{362880}f_{n+1} - \frac{58703}{3628800}f_{n+2} + \frac{640307}{1814400}f_{n+3} + \frac{13903}{22680}f_{n+4} + \frac{101741}{1814400}f_{n+5}\frac{1241}{145152}f_{n+6} + \frac{2129}{1814400}f_{n+7} - \frac{2497}{29030400}f_{n+8}\Big) - \frac{2497}{11520}h^2 g_{n+4}$$

is used with 3 initial methods obtained by evaluating,

$$y_i - y_{i-1} = h\sum_{j=0}^{8}\beta_j^{*(i)}f_j + h^2\gamma_4^{*(i)}g_4 \;; \qquad i = 1, \ldots, 3$$

and 4 final methods obtained by evaluating,

$$y_{N+i} - y_{N+i-1} = h\sum_{j=0}^{8}\beta_{N+j}^{*(i)}f_{N+j} + h^2\gamma_{N+4}^{*(i)}g_{N+4} \;; \qquad i = 5, \ldots, 8$$

Following the same procedure as in Section 3, the coefficients $\{\beta^{*(i)}, \gamma^{*(i)}\}$ are obtained for the additional initial and final methods of the GCE2BD7. They are readily available in Table 13 for brevity.

Example 6: The GCE2BD8 given as:

$$y_{n+4} - y_{n+3} = h\Big(-\frac{69823}{958003200}f_n + \frac{4969}{3991680}f_{n+1} - \frac{519493}{39916800}f_{n+2} + \frac{6748817}{19958400}f_{n+3} + \frac{379571}{623700}f_{n+4} + \frac{1560991}{19958400}f_{n+5} - \frac{42389}{2661120}f_{n+6} + \frac{3119}{950400}f_{n+7} - \frac{153707}{319334400}f_{n+8} + \frac{263}{7484400}f_{n+9}\Big) - \frac{90817}{380160}h^2 g_{n+4}$$

is used with 3 initial methods obtained by evaluating,

$$y_i - y_{i-1} = h\sum_{j=0}^{9}\beta_j^{*(i)}f_j + h^2\gamma_4^{*(i)}g_4 \;; \qquad i = 1, \ldots, 3$$

and 5 final methods obtained by evaluating,

$$y_{N+i} - y_{N+i-1} = h\sum_{j=0}^{9}\beta_{N+j}^{*(i)}f_{N+j} + h^2\gamma_{N+4}^{*(i)}g_{N+4} \;; \qquad i = 5, \ldots, 9$$

The coefficients $\{\beta^{*(i)}, \gamma^{*(i)}\}$ of the additional methods associated with GCE2BD8 are readily available in Table 14.
The main methods (derived from (10)) are implemented as BVMs efficiently by combining them with their respective additional initial and final methods (derived from (17) and (18)). They are combined as simultaneous numerical integrators for the solution of the specified problem. Practically, the main methods and the additional methods are combined as BVMs to give a single matrix of finite difference equations which simultaneously provides the values of the solution. Effectively, a modified Newton-Raphson method is used (see Lambert [31]). The accumulation of error is not significant on the numerical results so obtained since the solutions are obtained simultaneously (see [19, 27, 36-38]).

5. NUMERICAL EXAMPLES

In this section we tested extensively the GCE2BD on some standard stiff problems to illustrate the accuracy and efficiency of the scheme. All computations were carried out using our written code in MATLAB R2015a software package.

Problem 1: Consider the stiff test given by Cash [14]

$$y_1' = -\alpha y_1 - \beta y_2 + (\alpha + \beta - 1)\, e^{-t} y_1(0) = 1$$
$$y_2' = \beta y_1 - \alpha y_2 + (\alpha - \beta - 1)\, e^{-t} y_2(0) = 1$$

In other to make this system homogeneous, an additional variable y_3 is introduced such that:

$$y_3' = 1 \qquad y_3(0) = 0$$

The eigenvalues of the Jacobian associated with the resulting system are $-\alpha \pm i\beta$, 0 and the required solution is

$$y_1(t) = y_2(t) = e^{-t}\ ,\ y_3(t) = t.$$

This problem is solved using step length $k = 5$, step size $h = 0.09$, $\alpha = 1$ and $\beta = 30$ and the absolute errors are computed. The numerical results displayed in Table 3 show that the GCE2BD5 is more accurate than the conventional E2BD of Cash [14]. Further comparison in Table 4 shows that even with a large step size our scheme performs better than the conventional E2BD.

Table 3. Absolute error for Problem 1,
$k = 5, \quad h = 0.09, \quad \alpha = 1, \ \beta = 30$, and Error $y_i = |y_i - y(t_i)|\,,\ i = 1, 2$

t	y_i	Error in E2BD-Class1	Error in E2BD-Class2	Error in GCE2BD5
4.5	y_1	0.1×10^{-10}	0.1×10^{-10}	0.6×10^{-14}
	y_2	0.1×10^{-10}	0.1×10^{-10}	0.8×10^{-15}
9.0	y_1	0.1×10^{-12}	0.1×10^{-12}	0.3×10^{-16}
	y_2	0.1×10^{-12}	0.1×10^{-12}	0.1×10^{-16}
13.5	y_1	0.1×10^{-15}	0.8×10^{-11}	0.8×10^{-18}
	y_2	0.1×10^{-15}	0.6×10^{-11}	0.5×10^{-18}
18.0	y_1	0.1×10^{-17}	0.1×10^{-11}	0.1×10^{-19}
	y_2	0.1×10^{-17}	0.1×10^{-11}	0.2×10^{-20}

Table 4. Absolute error for Problem 1, $h = 0.15$, $\alpha = 1$, $\beta = 30$ and Error $y_i = |y_i - y(t_i)|$, $i = 1, 2$

t	y_i	Error in GCE2BD5	Error in GCE2BD6	Error in GCE2BD7	Error in GCE2BD8
4.5	y_1	0.2×10^{-11}	0.1×10^{-12}	0.9×10^{-13}	0.4×10^{-14}
	y_2	0.4×10^{-11}	0.2×10^{-12}	0.2×10^{-12}	0.3×10^{-14}
9.0	y_1	0.9×10^{-14}	0.2×10^{-14}	0.2×10^{-14}	0.3×10^{-16}
	y_2	0.3×10^{-13}	0.3×10^{-14}	0.2×10^{-15}	0.1×10^{-15}
13.5	y_1	0.8×10^{-16}	0.1×10^{-16}	0.6×10^{-18}	0.3×10^{-18}
	y_2	0.2×10^{-15}	0.1×10^{-16}	0.1×10^{-16}	0.1×10^{-17}
18.0	y_1	0.7×10^{-18}	0.2×10^{-18}	0.2×10^{-18}	0.5×10^{-19}
	y_2	0.2×10^{-17}	0.2×10^{-18}	0.2×10^{-18}	0.1×10^{-19}

Problem 2: Consider the stiffly nonlinear singularly perturbed problem which was proposed by Kaps [30] in the range of $0 \leq t \leq 10$

$$y_1' = -\left(\epsilon^{-1} + 2\right) y_1 + \epsilon^{-1} y_2^2, \qquad y_1(0) = 1$$

$$y_2' = y_1 - y_2 - y_2^2, \qquad y_2(0) = 1$$

The smaller ϵ is, the more serious the stiffness of the system (see [3, 4]). Its exact solution is given by

$$y_1 = y_2^2, \qquad y_2 = e^{-t}$$

The GCE2BD4 is applied to this problem using $\epsilon = 10^{-3}$ and the step sizes $h = 0.01,\ 0.05$ and the absolute errors $|y_i - y(t_i)|,\ i = 1, 2$ in the interval $0 \leq t \leq 10$ are compared with the $BBDF_8$

of Akinfenwa et al. [3] and the method of Wu and Xia [45]. It is observed that the new method even though it is of order 7 compares favorably with the $BBDF_8$ of Akinfenwa et. al. [3] of order p = 8 and performs better than the method of Wu and Xia [45] of order p = 8 with smaller step sizes $h = 0.001,\ 0.002$. $N = \left(\frac{T-t_0}{h}\right)$ is number of integration point. The details of the numerical results are displayed in Table 5. In Table 6, it is noticed that the GCE2BD4 is more accurate than the SDGBDF of Nwachukwu and Okor [37] and the SDGAM of Nwachukwu and Mokwunyei [35] using $\epsilon = 10^{-4}$, $k = 4$ and $h = 0.01$.

Table 5. A comparison of methods for Problem 2, using $\epsilon = 10^{-3}$, Absolute error, Error $y_i = |y_i - y(t_i)|,\ i = 1, 2$

Methods	t	h	N	Error y_1	Error y_2
GCE2BD4	1	0.05	20	7.8530×10^{-13}	1.4732×10^{-13}
	10	0.01	1000	2.3988×10^{-23}	2.7105×10^{-19}
$BBDF_8$	1	0.05	20	4.5602×10^{-13}	6.2638×10^{-13}
	10	0.01	1000	6.6466×10^{-20}	2.3988×10^{-17}
Wu and Xia[45]	1	0.002	500	2.5606×10^{-7}	8.0150×10^{-8}
	10	0.001	10000	5.5468×10^{-16}	6.0936×10^{-12}

Table 6. A comparison of methods for Problem 2 using $\epsilon = 10^{-4}$ and $h = 0.01$, Absolute error, Error $y_i = |y_i - y(t_i)|,\ i = 1, 2$

t	y_i	Error in SDGBDF5 $k=4$	Error in SDGAM $k=4$	Error in GCE2BD4 $k=4$
2	y_1	5.156×10^{-12}	2.038×10^{-13}	2.082×10^{-17}
	y_2	1.911×10^{-11}	7.087×10^{-13}	8.327×10^{-17}
4	y_1	9.571×10^{-14}	3.518×10^{-15}	1.572×10^{-18}
	y_2	2.613×10^{-13}	9.594×10^{-14}	4.510×10^{-17}
6	y_1	1.771×10^{-15}	6.428×10^{-17}	1.143×10^{-19}
	y_2	3.573×10^{-13}	1.297×10^{-14}	2.472×10^{-17}
8	y_1	3.278×10^{-17}	1.176×10^{-18}	4.764×10^{-21}
	y_2	4.886×10^{-14}	1.752×10^{-15}	7.373×10^{-18}
10	y_1	6.065×10^{-19}	2.150×10^{-20}	2.399×10^{-23}
	y_2	6.680×10^{-15}	2.368×10^{-16}	2.711×10^{-19}

Problem 3: Consider the stiff system giving in Fatunla [22] which has been solved by Akinfenwa and Jator [2], Akinfenwa et al. [4], Ismail and Ibrahim [26]

$$y_1' = -2000y_1 + 1000y_2 + 1, \qquad y_1(0) = 0$$
$$y_2' = y_1 - y_2, \qquad y_2(0) = 0$$

The eigenvalues of the Jacobian are -2000.5 and -0.5. Thus, the stiffness ratio is 4001. The theoretical solution is

$$y_1(t) = -4.97\times10^{-4}e^{-2000.5t} - 5.034\times10^{-4}e^{-0.5t} + 0.001$$
$$y_2(t) = -2.5\times10^{-7}e^{-2000.5t} - 1.007\times10^{-3}e^{-0.5t} + 0.001$$

Tables 7 and 8 contain the absolute errors, Error $y_i = |y_i - y(t_i)|$, $i = 1, 2$ at the end points $t = 5$ and $t = 10$ using the GCE2BD5. In Tables 7, for the purpose of comparison, the system is integrated with $h = 0.0001$. It can be seen in Tables 7 that our method is superior in terms of accuracy to the method of Ismail and Ibrahim [26] and compares favorably with the CBBDF5 of Akinfenwa et al. [4] and the ECBBDF5 of Akinfenwa and Jator [2] for the same number of steps. In Tables 8, for a larger step size $h = 0.1$, our method performs excellently compared with the method of Akinfenwa and Jator [2].

Table 7. Comparison of methods at the end points $t = 5$ and $t = 10$, $h = 0.0001$ and $k = 5$ for Problem 3, Error $y_i = |y_i - y(t_i)|$ $\quad i = 1, 2$

t	Ismail- Ibrahim $k=5$ Error y_1 Error y_2	CBBDF5 $k=5$ Error y_1 Error y_2	ECBBDF5 $k=5$ Error y_1 Error y_2	GCE2BD5 $k=5$ Error y_1 Error y_2
5	3.64920×10^{-7}	2.328953×10^{-7}	2.328953×10^{-7}	2.328953×10^{-7}
	7.670023×10^{-7}	5.027468×10^{-7}	5.027468×10^{-7}	5.027468×10^{-7}
10	2.454035×10^{-7}	1.700858×10^{-8}	1.700858×10^{-8}	1.699965×10^{-8}
	4.942995×10^{-7}	3.705176×10^{-8}	3.705176×10^{-8}	3.703239×10^{-8}

Table 8. Comparison of methods at the end points $t = 5$ and $t = 10$, $h = 0.1$ and $k = 5$ for Problem 3 Error $y_i = |y_i - y(t_i)| \quad i = 1, 2$

h	t	ECBBDF5 $k = 5$ Error y_1 Error y_2	GCE2BD5 $k = 5$ Error y_1 Error y_2
0.1	5	3.163426×10^{-4}	2.328953×10^{-7}
		6.610743×10^{-7}	5.027469×10^{-7}
0.1	10	2.005234×10^{-4}	1.700858×10^{-8}
		1.373470×10^{-7}	3.705176×10^{-8}

Problem 4: Consider the classical stiff test which was solved by D'Ambrosio et al. [17]

$$y' = Ay, \qquad t \in [0,\ 50]$$

where $A = \begin{bmatrix} -a & -b \\ b & -a \end{bmatrix}$
with eigenvalues of A given as $\{-a + bi,\ -a - bi\}$.
We integrated Problem 4 using the GCE2BD with step lengths $k = 6(1)8$ and the numerical results are presented in Table 9. In Table 9, our results are compared with those of the following methods: the MEBDF proposed by Cash [15] as a better version of the EBDF family of methods and the Perturbed MEBDF (PMEBDF) and the fully Perturbed MEBDF (FPMEBDF) proposed by D'Ambrosio et. al. [17] as improved versions of the MEBDF. From Table 9, it is obvious that our class of methods performs excellently compared with the methods of Cash [15] and D'Ambrosio et al. [17]. Further examinations using the GCE2BD were made considering several cases of {a, b} to investigate the performance of our scheme on stiff problem whose Jacobian has some large eigenvalues close to the imaginary axis. The results are given in Figs. 4, 5, 6 and 7. From our findings (Figs. 4, 5, 6 and 7), it is clear that our scheme is well suited for stiff system whose Jacobian has some large eigenvalues near the imaginary axis.

Table 9. Maximum Absolute error for methods, MEBDF, PMEBDF, FPMEBDF, GCE2BD applied to Problem 4

h	a	b	k	MEBDF	PMEBDF	FPMEBDF	GCE2BD
0.1	5	25	6	$9.1458e{+}67$	$1.0827e{-}10$	$6.4619e{-}10$	$1.2796e{-}45$
0.05	5	25	6	$9.8280e{-}46$	$4.2093e{-}42$	$3.1724e{-}51$	$5.7578e{-}108$
0.1	10	25	7	$3.7745e{+}60$	$2.8380e{-}08$	$1.8857e{-}10$	$7.9167e{-}74$
0.05	10	25	7	$4.2158e{-}24$	$8.6327e{-}43$	$1.0682e{-}41$	$9.1520e{-}218$
0.1	10	15	8	$3.2440e{+}19$	$2.2573e{-}10$	$4.7513e{-}13$	$5.3678e{-}121$
0.05	10	15	8	$2.1582e{-}21$	$5.9876e{-}31$	$6.2765e{-}38$	$2.1623e{-}218$

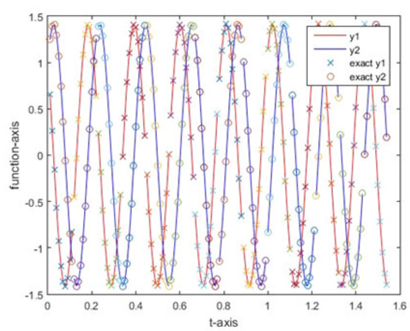

Fig. 4. Solution of Problem 4 for $\{a = 0, b = 30\}$ using GCE2BD8 with $h = 0.01$

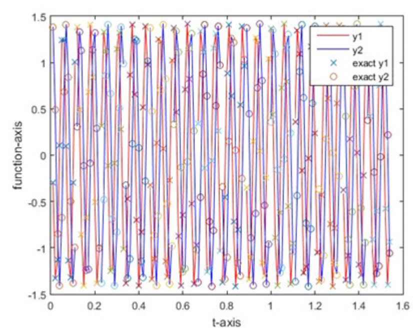

Fig. 5. Solution of Problem 4 for $\{a = 0, b = 100\}$ using GCE2BD8 with $h = 0.01$

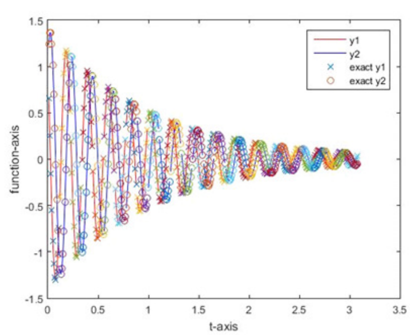

Fig. 6. Solution of Problem 4 for $\{a = 1, b = 30\}$ using GCE2BD8 with $h = 0.01$

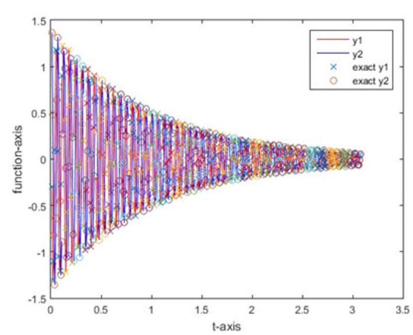

Fig. 7. Solution of Problem 4 for $\{a = 1, b = 100\}$ using GCE2BD8 with $h = 0.01$

Problem 5: Consider the linear stiff test solved by [11, 42, 44]

$$y_1' = -21y_1 + 19y_2 - 20y_3\ ; \qquad y_1(0) = 1$$

$$y_2' = 19y_1 - 21y_2 + 20y_3\ ; \qquad y_2(0) = 0$$

$$y_3' = 40y_1 - 40y_2 - 40y_3 \ ; \qquad\qquad y_3(0) = -1$$

The theoretical solution is given by:

$$y_1(t) = \frac{1}{2}(e^{-2t} + e^{-40t}(\cos(40t) \ + \sin(40t) \ ))$$

$$y_2(t) = \frac{1}{2}(e^{-2t} - e^{-40t}(\cos(40t) \ + \sin(40t) \ ))$$

$$y_3(t) = \frac{1}{2}(2e^{-40t}(\sin(40t) \ - \cos(40t) \ ))$$

This problem was solved using the GCE2BD of order 6, 9 and 10 (GCE2BD3, GCE2BD6 and GCE2BD7 respectively), the rate of convergence and number of computational steps $\left(N = \left(\frac{T-t_0}{h}\right)\right)$ were also computed. The results are reproduced in Table 10 and Table 11. From Table 10, our comparison with the results produced by SDGEBDFs given in Ogunfeyitimi and Ikhile [42], ETR_2s and TOMs given in [11] shows that our scheme performs much better than those of [11] and [42] for the same order (order 6). In Table 11, our scheme displays superiority to the ETR_2s and TOMs for same order (order 10). Thus, for this test, our scheme is more efficient and accurate than the other schemes considered. Although only the maximum absolute values are presented in Tables 10 and 11, Fig. 8 shows the complete solution for all the values of $y_i; i = 1(1)3$ with $N = 180$points.

Table 10. A comparison of methods for Problem 5, Error $= Max\,|y - y(t)|$ $Rate = log_2\left(\frac{e^{2h}}{e^h}\right)$ where e^h is the maximum absolute error for h, $0 \le t \le 1$

h	N	GCE2BD3 (Rate)	h	N	SDGEBDFs3 (Rate)	TOM (Rate)	ETR_{2s} (Rate)
2e-1	5	8.18×10^{-07} (—)	2e-2	50	3.22×10^{-7} (—)	1.55×10^{-3} (—)	3.51×10^{-3} (—)
1e-1	10	6.09×10^{-10} (10.4)	1e-2	100	3.79×10^{-9} (6.40)	9.77×10^{-6} (7.31)	8.62×10^{-5} (5.35)
5e-2	20	9.57×10^{-12} (5.99)	5e-3	200	5.39×10^{-11} (6.14)	1.20×10^{-7} (6.35)	7.23×10^{-7} (6.90)
2.5e-2	40	8.19×10^{-14} (6.87)	2.5e-3	400	8.89×10^{-13} (5.92)	1.85×10^{-9} (6.01)	8.86×10^{-9} (6.39)

Table 11. A comparison of methods for Problem 5, Error $= Max\,|y(t) - y|$ $Rate = log_2\left(\frac{e^{2h}}{e^h}\right)$ where e^h is the maximum absolute error for h, $0 \le t \le 1$

h	N	GCE2BD7 (Rate)	h	N	TOMs (Rate)	ETR_{2s} (Rate)
1.25e-1	8	$7.719e-04$ (—)	2e-2	50	$1.523e-04$ (—)	$2.866e-04$ (—)
6.25e-2	16	$5.242e-08$ (13.85)	1e-2	100	$2.504e-07$ (9.25)	$2.013e-06$ (7.15)
3.125e-2	32	$8.825e-14$ (19.18)	5e-3	200	$7.490e-11$ (11.70)	$2.208e-09$ (9.83)
1.5625e-2	64	$2.082e-17$ (12.05)	2.5e-3	400	$3.009e-14$ (11.28)	$1.004e-12$ (11.10)

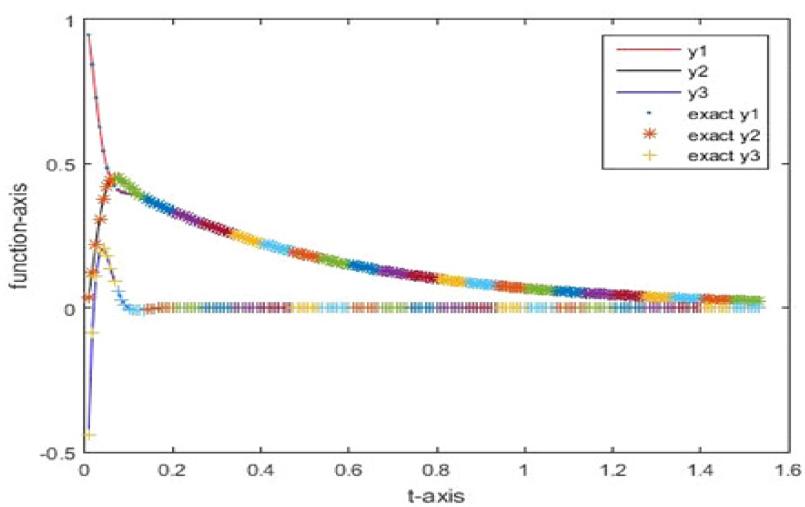

Fig. 8. Solution of Problem 5 using GCE2BD6 with $N = 180$

Problem 6: Consider the classical linear stiff system (Fatunla [22], Yakubu and Markus [46])

$$y' = \begin{bmatrix} -10 & -100 & 0 & 0 & 0 & 0 \\ -100 & -10 & 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.1 \end{bmatrix} y,$$

$$y_i(0) = 1, i = 1\,(1)\,6$$

The eigenvalues of the Jacobian are $\lambda_{1,2} = -10 \pm 100i$, $\lambda_3 = -4$, $\lambda_4 = -1$, $\lambda_5 = -0.5$ and $\lambda_6 = -0.1$. This problem is particularly troublesome for most stiff systems due to the eigenvalues near the imaginary axis (see, [22]). We integrated this problem using GCE2BD8 and compared our results with Method(3.4) given in [46]. The comparison is presented in Table 12. Although only the first four components of the computed solutions are considered in Table 12, the graphical plots are displayed in Fig. 9 for all the values of $y_i; i = 1\,(1)\,6$ with number of integration points $(N) = 100$. From Table 12, it is obvious that our scheme performs better and it is much accurate than the method in [46].

Table 12. Absolute Error for Problem 6 using $h\ = 0.01$

t	y_i	Method (3.4) in [46] *Order* $p = 14$	GCE2BD8 *Order* $p = 11$
5	y_1	$2.22044604925031 \times 10^{-16}$	$1.608442239844966 \times 10^{-23}$
	y_2	$1.74166236988071 \times 10^{-15}$	$1.605539145513942 \times 10^{-23}$
	y_3	$3.33066907387547 \times 10^{-16}$	$9.305781891221561 \times 10^{-24}$
	y_4	$2.22044604925031 \times 10^{-16}$	$6.071532165918825 \times 10^{-17}$
50	y_1	$1.66533453693773 \times 10^{-15}$	$4.505288687121358 \times 10^{-218}$
	y_2	$8.24340595784179 \times 10^{-15}$	$2.067926191149865 \times 10^{-218}$
	y_3	$2.55351295663786 \times 10^{-15}$	$9.172920294489976 \times 10^{-100}$
	y_4	$3.77475828372553 \times 10^{-15}$	$3.749826979123097 \times 10^{-35}$
250	y_1	$5.86336534880161 \times 10^{-16}$	$4.940656458412465 \times 10^{-324}$
	y_2	$4.18068357710411 \times 10^{-16}$	$6.916919041777452 \times 10^{-323}$
	y_3	$3.63598040564739 \times 10^{-15}$	$4.940656458412465 \times 10^{-323}$
	y_4	$5.10702591327572 \times 10^{-15}$	$3.760035257753905 \times 10^{-120}$
500	y_1	$8.73121562029733 \times 10^{-18}$	$458459520888726 \times 10^{-323}$
	y_2	$4.43167638003450 \times 10^{-18}$	$9.881312916824931 \times 10^{-324}$
	y_3	$8.15320033709099 \times 10^{-16}$	$4.940656458412465 \times 10^{-323}$
	y_4	$1.66533453693773 \times 10^{-16}$	$3.054171332054006 \times 10^{-228}$

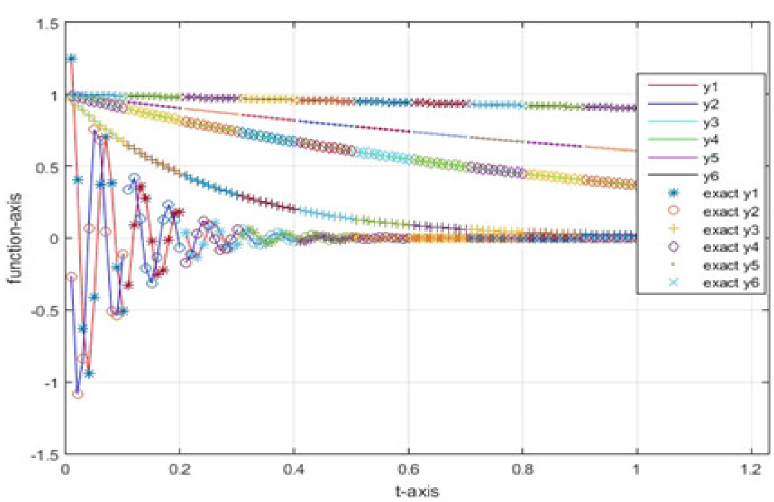

Fig. 9. Solution of Problem 6 using GCE2BD6 with $N = 100$

Problem 7: Chemistry stiff test suggested by Gear [23] which was also solved by Cash [14]

$$y_1' = -0.013y_1 - 1000y_1y_3\ ; \qquad y_1(0) = 1$$
$$y_2' = -2500y_2y_3\ ; \qquad y_2(0) = 1$$
$$y_3' = -0.013y_1 - 1000y_1y_3 - 2500y_2y_3\ ; \qquad y_3(0) = 0$$

This problem was integrated using the newly constructed methods within the range [0, 10] and the efficiency curves obtained are compared with the ODE15s code of MATLAB in Fig. 10 where nfe = number of function evaluation and Tol = tolerance. It can be observed from the figure (Fig. 10) that for a large tolerance level $\left(Tol\ = 10^{-3}\right)$ the scheme produced solutions as accurate as those of ODE15s code of MATLAB.

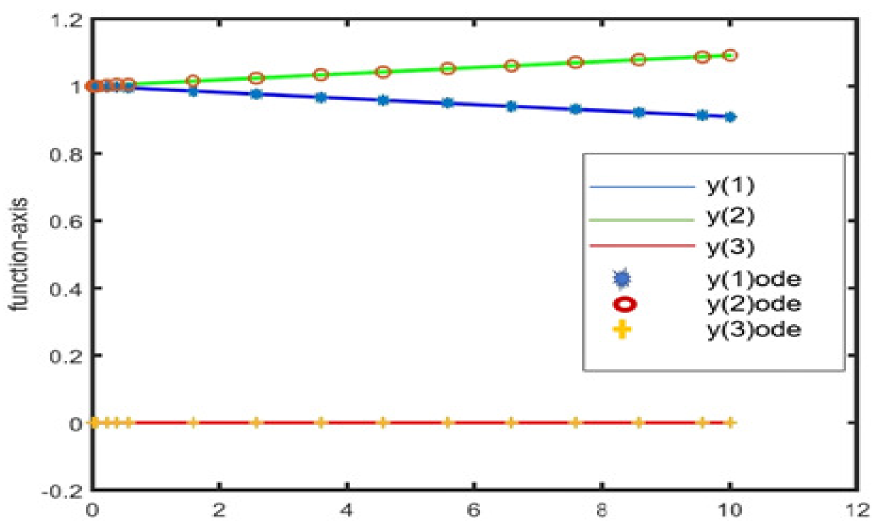

Fig. 10. Solution of Problem 7 using GCE2BD8 with $Tol = 10^{-3}, nfe = 33$

Table 13. Coefficients and order (p) of the additional initial and final methods for GCE2BD of order 10 (GCE2BD7)

i	β_0	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	γ_4	p
					Initial additional methods						
1	$\frac{8044607}{29030400}$	$\frac{2577233}{1814400}$	$\frac{-8213297}{3628800}$	$\frac{2003017}{362880}$	$\frac{-31457}{22680}$	$\frac{-5644237}{1814400}$	$\frac{2317489}{3628800}$	$\frac{-187249}{1814400}$	$\frac{48781}{5806080}$	$\frac{57281}{11520}$	10
2	$\frac{-175999}{29030400}$	$\frac{624047}{1814400}$	$\frac{3914161}{3628800}$	$\frac{-2215021}{1814400}$	$\frac{8233}{22680}$	$\frac{961229}{1814400}$	$\frac{-375089}{3628800}$	$\frac{29297}{1814400}$	$\frac{-37249}{29030400}$	$\frac{-2125}{2304}$	10
3	$\frac{4531}{5806080}$	$\frac{-25999}{1814400}$	$\frac{1388239}{3628800}$	$\frac{1816013}{1814400}$	$\frac{-5207}{22680}$	$\frac{-60233}{362880}$	$\frac{107953}{3628800}$	$\frac{-8017}{1814400}$	$\frac{9857}{29030400}$	$\frac{441}{1280}$	10
					Final additional methods						
5	$\frac{-2497}{29030400}$	$\frac{2129}{1814400}$	$\frac{-1241}{145152}$	$\frac{101741}{1814400}$	$\frac{13903}{22680}$	$\frac{640307}{1814400}$	$\frac{-58703}{3628800}$	$\frac{643}{362880}$	$\frac{-3391}{29030400}$	$\frac{2497}{11520}$	10
6	$\frac{9857}{29030400}$	$\frac{-8017}{1814400}$	$\frac{107953}{3628800}$	$\frac{-60233}{362880}$	$\frac{-5207}{22680}$	$\frac{1816013}{1814400}$	$\frac{1388239}{3628800}$	$\frac{-25999}{1814400}$	$\frac{4531}{5806080}$	$\frac{-441}{1280}$	10
7	$\frac{-37249}{29030400}$	$\frac{29297}{1814400}$	$\frac{-375089}{3628800}$	$\frac{961229}{1814400}$	$\frac{8233}{22680}$	$\frac{-2215021}{1814400}$	$\frac{3914161}{3628800}$	$\frac{624047}{1814400}$	$\frac{-175999}{29030400}$	$\frac{2125}{2304}$	10
8	$\frac{48781}{5806080}$	$\frac{-187249}{1814400}$	$\frac{2317489}{3628800}$	$\frac{-5644237}{1814400}$	$\frac{-31457}{22680}$	$\frac{2003017}{362880}$	$\frac{-8213297}{3628800}$	$\frac{2577233}{1814400}$	$\frac{8044607}{29030400}$	$\frac{-57281}{11520}$	10

Table 14. Coefficients and order (p) of the additional initial and final methods for GCE2BD of order 11 (GCE2BD8)

i	β_0	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	β_9	γ_5	p
						Initial additional methods						
1	$\frac{36953033}{136857600}$	$\frac{30049763}{19958400}$	$\frac{-110748667}{39916800}$	$\frac{31554307}{3991680}$	$\frac{-83753}{124740}$	$\frac{-133495007}{19958400}$	$\frac{24365993}{13305600}$	$\frac{-2953513}{6652800}$	$\frac{4617071}{63866880}$	$\frac{-8501}{1496880}$	$\frac{3250433}{380160}$	11
2	$\frac{-4910527}{958003200}$	$\frac{6640157}{19958400}$	$\frac{45748091}{39916800}$	$\frac{-30647311}{19958400}$	$\frac{167513}{623700}$	$\frac{19996639}{19958400}$	$\frac{-1156451}{4435200}$	$\frac{135523}{2217600}$	$\frac{-3102059}{319334400}$	$\frac{5609}{7484400}$	$\frac{-530113}{380160}$	11
3	$\frac{109363}{191600640}$	$\frac{-235789}{19958400}$	$\frac{14668229}{39916800}$	$\frac{21381743}{19958400}$	$\frac{-26003}{124740}$	$\frac{-1084243}{3991680}$	$\frac{864361}{13305600}$	$\frac{-96329}{6652800}$	$\frac{710827}{319334400}$	$\frac{-251}{1496880}$	$\frac{171137}{380160}$	11
						Final additional methods						
5	$\frac{-40321}{958003200}$	$\frac{12899}{19958400}$	$\frac{-43007}{7983360}$	$\frac{824591}{19958400}$	$\frac{379571}{623700}$	$\frac{7485217}{19958400}$	$\frac{-104477}{4435200}$	$\frac{1721}{443520}$	$\frac{-23363}{45619200}$	$\frac{263}{7484400}$	$\frac{14797}{76032}$	11
6	$\frac{17783}{136857600}$	$\frac{-37987}{19958400}$	$\frac{585083}{39916800}$	$\frac{-381443}{3991680}$	$\frac{-26003}{124740}$	$\frac{17867743}{19958400}$	$\frac{5558743}{13305600}$	$\frac{-162263}{6652800}$	$\frac{170321}{63866880}$	$\frac{-251}{1496880}$	$\frac{-90817}{380160}$	11
7	$\frac{-331777}{958003200}$	$\frac{97907}{19958400}$	$\frac{-1433659}{39916800}$	$\frac{4291439}{19958400}$	$\frac{167513}{623700}$	$\frac{-14942111}{19958400}$	$\frac{12257897}{13305600}$	$\frac{369617}{950400}$	$\frac{-661187}{45619200}$	$\frac{5609}{7484400}$	$\frac{171137}{380160}$	11
8	$\frac{35659}{27371520}$	$\frac{-359539}{19958400}$	$\frac{5089979}{39916800}$	$\frac{-14481007}{19958400}$	$\frac{-83753}{124740}$	$\frac{7751507}{3991680}$	$\frac{-4748963}{4435200}$	$\frac{2394307}{2217600}$	$\frac{108893077}{319334400}$	$\frac{-8501}{1496880}$	$\frac{-530113}{380160}$	11
9	$\frac{-8691073}{958003200}$	$\frac{491719}{3991680}$	$\frac{-33973243}{39916800}$	$\frac{93431567}{19958400}$	$\frac{2227571}{623700}$	$\frac{-218177759}{19958400}$	$\frac{13451861}{2661120}$	$\frac{-15749417}{6652800}$	$\frac{464544043}{319334400}$	$\frac{2046263}{7484400}$	$\frac{3250433}{380160}$	11

6. CONCLUDING REMARKS

The generalized Cash-type second derivative extended backward differentiation formulas (GCE2BD) has been developed and implemented as self-starting BVMs which requires only the initial value in (1) for the numerical solution of ordinary differential equations. The proposed class of methods is not only accurate (see Tables 3-12, Figs. 1, 4-10) but also reduces computational cost with fewer numbers of step length (k), function evaluations (nfe) and computational steps (N). Also, because of the single block computation, the accumulation error is not significant on the numerical results obtained. The exceptional stability properties (see Fig. 2 and Fig. 3) of our new class of methods makes it suitable for dealing with standard stiff problems. The GCE2BD displayed better stability properties ($O_{v,\ (k+1)-v}-$stable and $A_{v,\ (k+1)-v}-$stable for all values of $k \geq 1$) than the conventional E2BD which are only $A-$stable upto $k = 5$. The numerical results (see Tables 3-12 and Figs. 1, 4-10) demonstrate its efficiency and good accuracy over other standard methods in the literature for the integration of stiff systems. Considering the accuracy of this scheme and the fact that BVMs are natural candidate for the direct solution of Boundary value Problems (BVPs) (see, Brugnano and Trigiante [11]), we wish to propose a direct solution using the GCE2BD for singularly perturbed and/or stiff BVPs in the future.

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