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A REVIEW OF THE RESEARCH PUBLICATIONS OF PROFESSOR ANTHONY UYI AFUWAPE

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1. Introduction

AFUWAPE: THE MAN

When I received a call from Professor S. S. Okoya, the Editor-in-Chief of the Journal of the Nigerian Mathematical Society, to do a review of Professor A. U. Afuwape's research publications, I was not comfortable. This was not because I am not qualified to do it, but because there are other more qualified senior colleagues who in my opinion would do a better job. I knew it would be a very difficult task for me to make a concise portrait of an academic giant as Afuwape but I took the risk by accepting the offer.

In 1977, at the University of Nigeria, Nsukka, Professor H. O. Tejumola delivered a paper entitled "A Review of Ezeilo's Research Publications" during a symposium organized in honour of Professor J. O. C. Ezeilo. Twenty years after, in 1997, Professor A. U. Afuwape was invited to give "A Review of the Research Publications of Professor H. O. Tejumola" at a Conference in honour of Professor Tejumola on his retirement from the services of the University of Ibadan, Ibadan. Again, twenty years after, this year 2017, I have been asked to do "A Review of the Research Publications of Professor A. U. Afuwape." It must be pointed out that the 1977 and 1997 events were quite different from this 2017 event. One common factor to all the mentioned reviewers is that they were the first Professors of Mathematics produced by their supervisors: Professor Tejumola was produced by Professor Ezeilo; Professor Afuwape was produced by Professor Tejumola while I was produced by Professor Afuwape. Furthermore, the first two reviews were given when Professors Ezeilo and Tejumola were alive unlike this very one that is in honour of the memory of Professor Afuwape.

In order not to overburden this review, I will, in what follows, give a very brief profile of Professor Afuwape.

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Anthony Uyi Afuwape was born in Ode-Erinje, Okitupupa, Ondo State on 7th February, 1948. Between 1953 and 1959, he had his early education at St. John's Catholic Primary School, Okitipupa, before proceeding to Stella Maris College, Okitipupa (1960-1962), and then St. Patrick College, Iwaro Oka (1962-1965) all in Ondo State. He had a basic certificate in Surveying in 1967 and thereafter was admitted to study Mathematics at the University of Ife. He graduated with a First Class Honours (Mathematics) in 1971. He was awarded M.A. (Mathematics), Duke University, U.S. in 1974, M.Sc. (Mathematics) Rutgers University, U.S. in 1976 and Ph.D. (Mathematics), University of Ibadan, Nigeria in 1983.

He joined the services of the then University of Ife in 1971 as a Graduate Assistant and rose through the ranks to become a Professor of Mathematics in 1989. He was the Head, Department of Mathematics, Obafemi Awolowo University, Ile-Ife, Nigeria (1997-2000; 2003-2005) and served on several committees. He was a member of American Mathematical Society, Nigerian Mathematical Society, Edinburgh Mathematical Society, Research Group in Mathematical Inequalities and Applications, Grupo Ecuaciones Diferenciales de Orden Mayor y Applicaciones (Higher Order Differential Equations and Applications Group), Colombia and Life Member of the Quality Control Society of Nigeria. He was also a member of the IMU (International Mathematical Union) nomination committee for 2009-2014. He participated in many mathematical conferences and workshops in many countries with a tradition of mathematical research.

Professor Afuwape was privileged to visit the following institutes either teaching at both the undergraduate and postgraduate levels or doing research in differential equations or both- Universidad de Antioquia, Columbia (2006 to 2016); University of Agriculture, Abeokuta, Nigeria (October 2000 - 2002); Middle East Technical University, Ankara, Turkey (1992-1993); Advanced Institute of Global Analysis and Applications, and University of Firenze, Firenze, Italy, (1985); Advanced Institute of Pure and Applied Mathematics, Louis Pasteur University, Strasbourg, France (1980). He was awarded certificates of meritorious service by the Council and Senate of Obafemi Awolowo University, Ile-Ife, Nigeria in 2007. He was a Senior Associate of the International Centre for Theoretical Physics (ICTP) Trieste, Italy from 2003 to 2009, a DAAD (Deutscher Akademischer Austauschdienst) Research Fellow in 1993 and a Regular Associate Member of the International

Centre for Theoretical Physics (ICTP) Trieste, Italy from 1985-1991. Professor Afuwape was a National Research Fellow of the Italian Council for National Research (CNR) in 1984 and won UNESCO Fellow for CIMPA Schools in 1980 and 1983. He was an African American Institute Graduate Fellow (AFGRAD) Schorlar between 1972 and 1976. He won over fifteen research grants at various times from Universidad de Antioquia, National Mathematical Centre, Abuja, Obafemi Awolowo University, Ile-Ife, University Research Grant, TWAS-ICTP Research grant and University of Ife, Ile-Ife, University Research Grant among others.

Professor Afuwape during his life time served as an assessor of academic colleagues at professorial levels both within and outside the country. Of course one cannot overlook his contributions as external examiner at various times for postgraduate students in universities in Nigeria and abroad. He successfully supervised over thirteen M.Sc. students and four Ph.D. students including this author. He was actively involved in the activities of the Nigerian Mathematical Society where he served as Treasurer between 1991 and 2006 and also as an Associate Editor of the Journal of the Nigerian Mathematical Society.

Professor Afuwape's research activities were principally on the qualitative properties of solutions of nonlinear differential equations both with and without delay terms. He was largely interested in the following properties of solutions of the nonlinear differential equations: boundedness, stability, periodicity, almost periodicity, dissipativity, oscillations, gradient-like behaviour and convergence of solutions. Other properties included limiting regime in the sense of Demidovich, resonant and non-resonant oscillations. His work had a large impact on a number of development in nonlinear differential equations of higher order, and in particular, on third order equations where he made significant contributions to the growth of qualitative behaviour of solutions. He retired from the services of Obafemi Awolowo University, Ile-Ife, Nigeria in 2006 and joined Instituto de Mathematicas, Universidad de Antioquia, Medellin, Colombia, as a Distinguished Professor of Mathematics-a position he held till his death.

He was a great mathematician, a faithful and a wonderful husband, a loving father and a prodigious mentor. He lived a fulfilled and contented life worthy of emulation.

The rest of this review is organised as follows. In the next section, we highlight Afuwape's contributions to nonlinear differential equations via Lyapunov's approach. Sections 3 is on his contributions to nonlinear differential equations via frequency domain approach. His contributions through functional analytical approach to nonlinear differential equations are considered in Section 4. Section 5 is dedicated to his contributions to Volterra integral operators. While his other contributions are given in Section 6. The conclusion is given in the last section.

2. Afuwape's work on nonlinear differential equations via Lyapunov's approach

Like every other mathematician before Afuwape that worked in this area, the major difficulty in applying the second method of Lyapunov to the analysis of qualitative properties of solutions of nonlinear systems is the lack of a straightforward procedure for finding appropriate Lyapunov functions. The construction of Lyapunov functions is an art. But like any other art, there are guidelines to follow. One approach is to assume the Lyapunov function to be a Hermitian form or a quadratic form. This has a limitation because such assumption of a Hermitian form or a quadratic form is unnecessarily restrictive since a Hermitian form or a quadratic form may not exist for a given system. Another way is a systematic approach based on the fact that if a particular Lyapunov function exists which is capable of proving asymptotic stability of a given nonlinear system, then a unique gradient of this Lyapunov function also exists. In fact, for Euclidean spaces, this method is just another way of looking at the related Lyapunov theory for autonomous systems. However, the classical Lyapunov theory must be modified for systems in Banach spaces since the most important examples of such systems do not have all their trajectories differentiable, but only dense subsets of their trajectories have this property. Yet another way is to derive suitable Lyapunov functions for a general class of nonlinear systems expressed in state variables as n first order nonlinear differential equations. This method, which applies the integration by parts procedure, derives a Lyapunov function directly from the differential equation under study. For this, the integration is along the trajectories and the limits for the integral with respect to time are from zero to t. The derivatives of the Lyapunov function V and its time derivative $\dot{V}$ are based on the equation

$$V + \int_0^t -\dot{V} d\tau = 0 \tag{2.1}$$

and do not require the gradient of the scalar function V to be obtained.

Let us now try to catch the gist of how Afuwape handled the above problems. Afuwape in constructing a particular Lyapunov function for instance, chose a quadratic Lyapunov function candidate V with respect to the system

$$x^{'} = f(x,t), \tag{2.2}$$

where x and $f(x,t)$ are n-vectors with elements which are functions of $x_1, x_2, ..., x_n$ and t. The Lyapunov function candidate V would be assumed to be an explicit function of $x_1, x_2, ..., x_n$ but not an explicit function of t. Then

$$\dot{V} = \frac{\partial V}{\partial x_1}\dot{x_1} + \frac{\partial V}{\partial x_2}\dot{x_2} + ... + \frac{\partial V}{\partial x_n}\dot{x_n}. \tag{2.3}$$

$\dot{V}$ would then be constrained to be negative definite or at least negative semidefinite. With these, the Lyapunov function candidate V becomes a Lyapunov function. Every other verification then would become easy to handle. All Lyapunov functions constructed by him were given explicitly and they were devoid of the usual signum functions that characterized Lyapunov functions contsructed by others.

Equipped with these ideas, Afuwape began to use this approach to systematically study the behaviour of solutions of third and fourth order nonlinear differential equations. He did not only consider scalar differential equations but vector and matrix differential systems too. Properties of solutions such as boundedness, ultimate boundedness, uniform boundedness, convergence and limiting regime in the sense of Demidovich were all considered by him.

Paper [69] was the first paper of Afuwape in this direction. A fourth order nonlinear differential equation was considered. He defined two solutions $x_1(t)$, $x_2(t)$ of the equation

$$x^{(iv)} + ax^{'''} + bx^{''} + cx^{'} + h(x) = p(t, x, x^{'}, x^{''}, x^{'''}), \tag{2.4}$$

where $a > 0$, $b > 0$, and $c > 0$ are constants, h is continuous, and p has the form $p = q(t) + r(t, x, x^{'}, x^{''}, x^{'''})$, with q and r continuous to be convergent if $x_1 - x_2 \to 0$, $x_1{}^{'} - x_2{}^{'} \to 0$, $x_1{}^{''} - x_2{}^{''} \to 0$, $x_1{}^{'''} - x_2{}^{'''} \to 0$ as $t \to \infty$. The function h was assumed not to be differentiable but with the restriction

$$\frac{h(\zeta+\eta)-h(\zeta)}{\eta} \in I_0 \ \eta \neq 0,$$

where I_0 is a closed interval defined by

$$I_0 = \left[\Delta_0, K.\frac{(ab-c)c}{a^2}\right],$$

with $K < 1$ and Δ_0 as constants. By putting the equation (2.4) in an equivalent system

$$\begin{aligned} x' &= y, \\ y' &= z, \\ z' &= w + Q(t), \\ w' &= -aw - bz - cy - h(x) + r(t,x,y,z,w+Q) - aQ, \end{aligned} \tag{2.5}$$

he constructed a Lyapunov function as

$$\begin{aligned} 2V &= c^2\left[1-\epsilon\right]x^2 + \left[b^2 + ac(1-\epsilon)(D-1)\right]y^2 \\ &+ \left[a^2(1-\epsilon)D + \tfrac{b\delta}{ab-c-\delta}\right]z^2 + Dw^2 + 2bc(1-\epsilon)xy + 2ac(1-\epsilon)xz \\ &+ 2c(1-\epsilon)xw + 2(ab-\delta)Dyz + 2byw + 2a(1-\epsilon)Dzw, \end{aligned} \tag{2.6}$$

where $D = \frac{ab-c-c\epsilon}{ab-c-\delta}$ with $ab-c > \delta > 0$, $0 < \epsilon < 1$ and $2\epsilon - \epsilon^2 = \frac{\delta}{ab}$. He went on to prove that the solution of the equation (2.4) converges if

$$|r(t,y_1,y_2,y_3,y_4) - r(t,z_1,z_2,z_3,z_4| \leq \phi(t)\sum_{i=1}^{4}|y_i - z_i|$$

holds for arbitrary $y_i, z_i, i = 1,2,3,4$ and if there exists a constant M such that

$$\int_0^t \phi^\alpha(s)ds \leq M,$$

for some α, $1 \leq \alpha \leq 2$.

In another contribution on ultimate boundedness of solutions for third order nonlinear matrix differential equations, Afuwape (jointly with Omeike) in paper [16] considered equations of the form

$$\dddot{X} + A\ddot{X} + B\dot{X} + H(X) = P(t,X,\dot{X},\ddot{X}), \tag{2.7}$$

where A, B are constant symmetric $n\times n$ matrices, $X, H(X)$ and $P(t,X,\dot{X},\ddot{X})$ are real $n\times n$ matrices that are continuous in their respective arguments. Obtained results in this paper gave a matrix analogue of earlier results of Afuwape [67], and other earlier results in the literature for the case in which $H(X)$ was not required to be differentiable were extended. The main result of the paper is the

following.
Let $H(0) = 0$ and suppose that

(i) there exists an $n \times n$ real continuous matrix operator $C(X, Y)$ for any $X, Y \in M$;

(ii) the matrices $\tilde{A}, \tilde{B}, \tilde{C}(X, Y)$ are associative and commute pairwise. The eigenvalues $\lambda_i(\tilde{A})$ of $\tilde{A}$, $\lambda_i(\tilde{B})$ of $\tilde{B}$ and $\lambda_i(\tilde{C}(X, Y))$ of $\tilde{C}(X, Y)$ $(i = 1, 2, \cdots, n^2)$ satisfy

$$0 < \delta_a \leq \lambda_i(\tilde{A}) \leq \Delta_a$$

$$0 < \delta_b < \lambda_i(\tilde{B}) \leq \Delta_b$$

$$0 < \delta_c < \lambda_i(\tilde{C}(X, Y)) \leq \Delta_c$$

where $\delta_a, \delta_b, \delta_c, \Delta_a, \Delta_b, \Delta_c$ are finite constants. Furthermore,

$$\Delta_c \leq k\delta_a\delta_b,$$

where $$k = \min \left\{ \frac{\alpha(1-\beta)\delta_b}{\delta_a(\alpha+\Delta_a)^2} \; ; \; \frac{\alpha(1-\beta)\delta_a}{2(\delta_a+2\alpha)^2} \right\}$$

$\alpha > 0, 0 < \beta < 1$ are some constants,

(iii) P satisfies

$$\|P(t, X, Y, Z)\| \leq \delta_0 + \delta_1(\|X\| + \|Y\| + \|Z\|)$$

for all arbitrary $X, Y, Z \in M$, where $\delta_0 \geq 0, \delta_1 \geq 0$ are constants and δ_1 is sufficiently small.

Then every solution $X(t)$ of the considered equation satisfies

$$\|X(t)\| \leq \Delta_1, \quad \|\dot{X}(t)\| \leq \Delta_1, \|\ddot{X}(t)\| \leq \Delta_1$$

for all t sufficiently large, where Δ_1 is a positive constant the magnitude of which depends only on $\delta_0, \delta_1, A, B, H$ and P.

Afuwape also laid his hands on fourth order differential equations that can be reduced to system of the form

$$x' = f(x, t), \tag{2.8}$$

where x and f are real n dimensional column vectors and t is real independent variable. He obtained results on the limiting regime in the sense of Demidovich. Hitherto, the boundedness of solutions of various equations of this form had been of intense investigation by many authors but most of the work done in this direction have been restricted to the boundedness of solutions of t in some semi infinite range $(t_0, +\infty)$ and comparatively little was done to investigate whether there were solutions which are bounded for all t in the infinite range $(-\infty, +\infty)$. Afuwape in his exposition [58] showed that the solutions of the considered fourth order nonlinear differential equations under some boundedness restrictions on the incremental ratio of the nonlinear function in the limiting regime are almost periodic or periodic in t, uniformly in x, x', x'', x'''. He utilized the idea of Demidovich by showing that if the partial derivatives $\frac{\partial F_i}{\partial x_j}$ $(1 \leq i, j \leq n)$ exist and continuous and if the characteristic roots $\lambda_k (k = 1, 2, ..., n)$ of the symmetric matrix

$$\left(\frac{1}{2} \left[\frac{\partial F_i}{\partial x_j} + \frac{\partial F_j}{\partial x_i} \right] \right)$$

satisfy $\lambda_k \leq -\delta < 0$, then there is precisely one solution $x^*(t)$, the limiting regime of the system which satisfies

$$\|x^*(t)\| \leq m_2, \ -\infty < t < +\infty,$$

where m_2 $(0 < m_2 < \infty)$ is a constant.

Afuwape (jointly with Ogundare) in paper [3] concerned themselves with the problem of global asymptotic stability and boundedness of solutions of the Lienard equation

$$\ddot{x} + f(x)\dot{x} + g(x) = p(t, x, \dot{x}), \tag{2.9}$$

where functions f, g, and p are continuous and depend (at most) only on the arguments displayed explicitly. They formulated the following:

(i)

$$\frac{g(x) - g(0)}{x} = G_0 \leq \alpha = \frac{\nu(\epsilon - \beta)}{(a + \nu\beta)} \in I_0 \ (x \neq 0 \text{ and } g(0) = 0),$$

where $I_0 = [0, a\alpha]$,

(ii)

$$\frac{F(x) - F(0)}{x} = F_0 \leq \beta \in I_0, \ (x \neq 0 \text{ and } F(0) = K \neq 0);$$

(iii)

$$\frac{\phi(y) - \phi(0)}{y} = \Phi_0 \leq \gamma \in I_0, \ (y \neq 0 \text{ and } \phi(0) = 0);$$

where a, α, β, γ and ν are all positive with $\dfrac{(\nu\epsilon - a)}{2\nu} < \beta < \epsilon$. I_0 is a closed subinterval of the Routh-Hurwitz interval. They then proved the following theorems.

Theorem 2.1. *Suppose that conditions (i)-(iii) are satisfied with* $p(t, x, \dot{x}) \equiv 0$, *then the trivial solution of the equation is globally asymptotically stable.*

Theorem 2.2. *In addition to conditions (i)-(iii) being satisfied, suppose that*

(iv)

$$p(t, x, \dot{x}) \equiv p(t) \text{ and } |p(t)| \leq M,$$

for all $t \leq 0$*, then there exists a constant* $\sigma, (0 < \sigma < \infty)$ *depending only on the constants* α, β *and* γ *such that every solution of the equation* (2.9) *satisfies*

$$x^2(t) + \dot{x}^2(t) \leq e^{-\sigma t} \left\{ A_1 + A_2 \int_{t_0}^{t} |p(\tau)| \, e^{\frac{1}{2}\sigma\tau} d\tau \right\}^2$$

for all $t \geq t_0$*, where the constant* $A_1 > 0$*, depends on* α*,* β *and* γ *as well as on* $t_0, x(t_0), \dot{x}(t_0)$*; and the constant* $A_2 > 0$ *depends on* α*,* β *and* γ*.*

In another interesting joint contribution, Afuwape (and Omeike) [12] considered the stability and boundedness of solutions of a nonlinear third-order delay differential equation

$$\begin{aligned} x^{'''} + h(x^{'})x^{''} &+ g(x^{'}(t - r(t))) + f(x(t - r(t))) \\ &= p(t, x(t), x^{''}(t), x(t - r(t)), x^{'}(t - r(t)), x^{''}(t)) \end{aligned}$$

or its equivalent system

$$\begin{aligned} x^{'} &= y \\ y^{'} &= z \\ z^{'} &= -h(y)z - g(y) - f(x) + \textstyle\int_{t-r(t)}^{t} g'(y(s))z(s)ds \\ &\quad + \textstyle\int_{t-r(t)}^{t} f'(x(s))y(s)ds + p(t, x, y, x(t - r(t)), y(t - r(t)), z), \end{aligned}$$

where $0 \leq r(t) \leq \gamma, r'(t) \leq \beta, 0 < \beta < 1, \beta$ and γ are some positive constants, $f(x), g(y), h(y), p(t, x, y, x(t - r(t)), y(t - r(t)), z)$ are continuous in their respective arguments. Besides, it is supposed that the derivatives $f'(x), g'(y)$ are continuous for all x, y with $f(0) = g(0) = 0$. In addition, it is also assumed that the functions $f(x(t - r(t))), g(y(t - r(t)))$ and $p(t, x, y, x(t - r(t)), y(t - r(t)), z)$ satisfy a Lipschitz condition in $x, y, x(t - r(t)), y(t - r(t))$ and z;. Sufficient conditions for the stability and boundedness of solutions for the equations considered are obtained by constructing a Lyapunov functional. The following was proved.

Theorem 2.3. *Consider the system above with* $p(t,x,y,x(t-r(t)),y(t-r(t)),z) \equiv 0, f(x), f'(x), g(y), h(y)$ *continuous in their respective arguments. Suppose further that*

(i) for some $a > 0, \epsilon_0 > 0, h(y) \geq a + \epsilon_0$ *for all* y;

(ii) for some $b > 0, \frac{g(y)}{y} \geq b$ *for all* $y \neq 0$;

(iii) for some $c_0, \frac{f(x)}{x} \geq c_0$ *for all* $x \neq 0$;

(iv) for some $c > 0, f'(x) \leq c$ *for all* x, *where* $ab - c > 0$;

(v) for some constants $L, M, |f'(x)| \leq L, |g'(y)| \leq M$, *for all* x, y.

Then the zero solution of the system is asymptotically stable, provided that

$$\gamma < \min\left\{\frac{2\epsilon_0(1-\beta)}{(L+M)(1-\beta)+(1+a)M}, \frac{2a(ab-c)(1-\beta)}{a(L+M)(1-\beta)+(1+a)L}\right\}.$$

Contributions of Afuwape under this section are the paper [3] on stability and boundedness of solutions and paper [36] on convergence of solutions for second order nonlinear differential equations. On third order nonlinear differential equations (scalar, vector and matrix) with and without delay terms, papers [7-8], [13], [16], [19], [39-40], [52], [62] and [67] are on ultimate boundedness of solutions, while papers [12], [32] and [25] dwell on boundedness and stability of solutions. Furthermore, Afuwape published his contributions to convergence of solutions in papers [14], [21], [24], [28], [34] and [66] on third order nonlinear differential equations. His contributions to limiting regimes can be found in papers [17] and [58] on third and fourth order nonlinear differential equations respectively. Lastly, he contributed three papers in [5], [59] and [69] to convergence of solutions of fourth order nonlinear differential equations.

3. Afuwape's work on nonlinear differential equations via frequency domain approach

Afuwape was fortunate to have met Professor Aristide Halanay at a mathematical workshop in Perugia, Italy in 1977. It was Halanay that introduced him to the frequency domain approach. Influenced by the works of Professors Aristide Halanay, V. A. Yacubovich, I. Barbalat, V.I. Rasvan and their collaborators from the Romanian and Russian block, on frequency domain approach to study qualitative behaviour of solutions of nonlinear differential equations, Afuwape in the late seventies started systematic extentions to higher order nonlinear diffferntial equations and answered many of the basic questions evoked by these pioneering contributors which had opened up a new area in the qualitative study of nonlinear differential equations as many results that had hitherto been obtained with the Lyapunov's second method were improved with the use of this technique.

In a series of published papers, he presented results concerning the following qualitative properties of solutions: boundedness, uniform dissipativity, exponential stability, periodicity and almost periodicity for mostly third order (and some few fourth order) nonlinear differential equations with different combinations of nonlinear terms. In the most general setting, he considered equations of the form

$$x''' + f(x^{''}) + g(x') + h(x) = p(t, x, x^{'}, x^{''}) \tag{3.1}$$

and

$$x^{(iv)} + \psi(x''') + f(x^{''}) + g(x') + h(x) = p(t, x, x^{'}, x^{''}, x^{'''}), \tag{3.2}$$

where the functions ψ, f, g, h and ψ are continuous in their respective arguments. In some cases, he introduced retarded arguments in the considered equations whenever feasible. It should be noted that these equations with various combinations of the nonlinear and forcing terms are not only of theoretical but also of big practical importance. For instance, such equations can be applied in modeling for automatic control in T.V. systems realized by means of R-C filters; they also have applications in electric circuit problems and control theory.

In particular, the paper [73] which was Afuwape's first paper dealt with equations of the form:

$$x''' + ax^{''} + g(x') + h(x) = p(t, x, x^{'}, x^{''}) \tag{3.3}$$

and

$$x''' + ax^{''} + g_1(x)x' + h(x) = p(t, x, x^{'}, x^{''}), \tag{3.4}$$

where $a > 0$, g, g_1 and h are continuous functions with further conditions that g, h and $\int_0^x g_1(s)ds$ are almost linear in their arguments. It was shown that the equation (3.3) was uniformly dissipative if the function $p(t, x^{'}, x^{''})$ was bounded for all t and if there exist constants $b > 0, c > 0, ab > c, \mu_1 \geq 0$, $\mu_2 \geq 0$ such that

$$b \leq \frac{g(z)}{z} \leq b + \mu_1,\ (z \neq 0)\ ; b \leq g^{'}(z) \leq b + \mu_1$$

$$c \leq \frac{h(z)}{z} \leq c + \mu_2,\ (z \neq 0)\ ; \leq h^{'}(z) \leq c + \mu_2.$$

For the second equation (3.4), Afuwape cleverly used a duality idea to conclude that both equations satisfy the same equivalent criteria for uniformly dissipative solutions by utilizing a non singular matrix transformation with an additional condition that

$$b \leq \frac{\int_0^z g_1(s)ds}{z} + \mu_1.$$

In another exposition, Afuwape in paper [72] considered the same equations as those contained in paper [73] with $p(t, x, x^{'}, x^{''}) = p(t)$ and under different conditions obtained criteria that guarantee the existence of bounded solutions that are globally exponentially stable, periodic or almost periodic. Further research in this direction by Afuwape on uniformly dissipative solutions and existence of bounded solutions that are globally exponentially stable, periodic or almost periodic were on more complicated third and fourth order nonlinear differential equations. These can be found in papers [18], [20], [27], [31], [33], [37], [48], [57], [60-61], [63-65] and [68]. In all these papers, and in many others, Afuwape showed remarkable insight into the qualitative behaviour of solutions of the considered equations. Worthy of note here are papers [18] and [48] where delayed terms were introduced to the nonlinear equations. Specifically, paper [18] considered third order nonlinear delayed differential equations of the forms:

$$\begin{aligned} x'''(t) &+ ax''(t) + [b_1x'(t) + b_2x'(t-\tau)] + [c_1x(t) + c_2x(t-\tau)] \\ &+ h(x(t)) = p(t) \end{aligned}$$

$$\begin{aligned} x'''(t) &+ ax''(t) + [b_1x'(t) + b_2x'(t-\tau)] + [c_1x(t) + c_2x(t-\tau)] \\ &+ h_1(x(t-\tau)) = p(t) \end{aligned}$$

$$\begin{aligned} x'''(t) &+ ax''(t) + [b_1x'(t) + b_2x'(t-\tau)] + [c_1x(t) + c_2x(t-\tau)] \\ &+ g(x'(t)) = p(t) \end{aligned}$$

and

$$\begin{aligned} x'''(t) &+ ax''(t) + [b_1x'(t) + b_2x'(t-\tau)] + [c_1x(t) + c_2x(t-\tau)] \\ &+ g_1(x'(t-\tau)) = p(t), \end{aligned}$$

where a, b_1, b_2, c_1, c_2, are constants and h, h_1, g, g_1 and $p(t)$ are real valued continuous functions depending only on the arguments displayed. It was shown for the considered equations that

(i) the zero solution is asymptotically stable, (when there is no forcing term, i.e. $p(t) = 0$); and

(ii) the existence of a bounded solution which is exponentially stable (when there is a bounded forcing term, i.e. $p(t) \neq 0$ but bounded). Obtained results in this paper generalised results in [48] and extend the ideas in [20] to the case where delay terms are present.

One of the papers of Afuwape where the elegance of his work can be found is [60] where he introduced the idea of dual systems of the frequency domain method for uniform dissipativity. He solved a longstanding problem by proving the equivalence of the frequency domain conditions for dual systems and applied it to a third order nonlinear differential equation arising from the vacuum tube circuit problem studied by L. L. Rauch. An earlier result of I. Barbalat and A. Halanay [Rev. Roumaine Sci. Tech. Ser. Electrotech. Energet. 19 (1974), 321-341; MR0352613 (50 5100)] was obtained as a particular case.

Later on in his research, Afuwape and this author initiated the use of the frequency domain technique to study extensively, more challenging fifth order nonlinear differential equations ([41], [43] and [45]). For instance in paper [41], one of the considered equations is the following:

$$x^{(v)}+ax^{(iv)}+bx^{'''}+cx^{''}+g(x^{'})+h(x) = p(t,x,x^{'},x^{''},x^{'''},x^{(iv)}), \quad (3.5)$$

where a, b, c are positive constants, functions h, g and p are real valued and continuous in their respective arguments such that the uniqueness theorem is valid, and the solutions are continuously dependent on the initial conditions. They proved that further to the basic assumptions on the functions g, h and p, suppose that the following conditions are satisfied.

(i)The functions g and h are differentiable.

(ii)The homogeneous linear part of the equation given as

$$x^{(v)} + ax^{(iv)} + bx''' + cx^{''} + dx' + ex = 0, \quad (3.6)$$

where a, b, c, d, and e are constants, and such that all solutions of the equation tend to the trivial solution, as $t \to \infty$, provided that the Routh-Hurwitz conditions

$$\begin{aligned} &a > 0, (ab-c) > 0, (ab-c)c-(ad-e)a > 0, \\ &(ab-c)(cd-be)-(ad-e)^2 > 0, e > 0 \end{aligned}$$

are satisfied.

(iii)There exist positive constants d, e and $\mu_1 > 0$, $\mu_2 > 0$ such that

$$d \leq \frac{g(z)}{z} \leq d+\mu_1, \;\; d \leq g^{'}(z) \leq d+\mu_1, \;\; (z \neq 0).$$

$$e \leq \frac{h(z)}{z} \leq e + \mu_2,\ e \leq h'(z) \leq e + \mu_2,\ (z \neq 0).$$

(iv)There is a positive constant $\rho_0 > 0$ such that for all t, x, x', x'', x''', $x^{(iv)}$,

$$|p(t, x, x', x'', x''', x^{(iv)})| \leq \rho_0.$$

(v)

$$\lim_{|z|\to\infty} \frac{1}{z^2}\left[\int_0^z g(\zeta)d\zeta - \frac{1}{2}zg(z)\right] \leq 0.$$

(vi)

$$\lim_{|z|\to\infty} \frac{1}{z^2}\left[\int_0^z h(\zeta)d\zeta - \frac{1}{2}zh(z)\right] \geq 0.$$

If

$$e^2 + \frac{\mu_1}{\tau_1}(\tau_1 + \nu_1)\left[e - \frac{\mu_2}{4\tau_2}\right] > \frac{c^2}{2}\left[\theta_1\mu_2(\tau_2 + \nu_2) + \theta_2\mu_1(\tau_1 + \nu_1),\right.$$

for all positive parameters τ_1, τ_2, θ_1, θ_2, ν_1 and μ_2, then every solution of the equation (3.5) is uniformly dissipative.

Afuwape [48] dealt with the existence of bounded solutions which are exponentially stable for third order non-linear delayed differential equations of the form

$$x'''(t) + ax''(t) + bx'(t) + cx(t) + dx(t-\tau) + h(x(t)) = p(t) \quad (3.7)$$

and the equation with delay in the non-linear term of the form

$$x'''(t) + ax''(t) + bx'(t) + cx(t) + dx(t-\tau) + h(x(t-\tau)) = p(t), \quad (3.8)$$

where a, b, c and d are positive constants, $\tau > 0$ and the functions $h(x(t))$, $h(x(t-\tau))$ and $p(t)$ are continuous. It was shown that if
$ab > c > |d|$;
p satisfies $|p(t)| \leq \rho_o \quad for\ all\ t\ in\ (-\infty, \infty)$;
and
$h(0) = 0$ and for some $\delta > 0$, there exists $\mu > 0$ such that

$$0 \leq \frac{h(x(t)) - h(\bar{x}(t))}{x(t) - \bar{x}(t)} \leq \mu$$

satisfying

$$\frac{1}{\mu} > \delta + \{\frac{a^2d^2(c - |d|)}{c(2|d| - c)(a^2d^2 - a^2 + 2b) + d^2(a^2 - 2b)}\},$$

then, there exists a bounded exponentially stable solution for the equation which is periodic (or almost periodic) whenever $p(t)$ is periodic (or almost periodic). The proof was based on the generalized

Theorem of Kurzweil on the invariance of manifolds of flows via the frequency domain method.

Afuwape (jointly with the author) considered the issue of bounds for the mean-values of solutions of some third order nonlinear differential equations whose interesting feature advertised these bounds not to be independent of the solutions. Bounds for the mean values of solutions of third-order differential equations of the form

$$\dddot{x} + a\ddot{x} + g(\dot{x}) + h(x) = q(t) \tag{3.9}$$

and

$$\dddot{x} + a\ddot{x} + g(x)\dot{x} + h(x) = q(t) \tag{3.10}$$

where $a > 0$ and functions g, g_1, h and q are continuous in their respective arguments, were considered. A natural question on how these bounds affect the solutions and the forcing term $q(t)$ of the considered equations were answered under the condition that the solutions are globally exponentially stable.

For the solution $x(t)$ of the given equation (3.9), the mean-values of $x(t)$, $\dot{x}(t)$, $\ddot{x}(t)$ and $\dddot{x}$ for $T > 0$, were given as

$$\begin{array}{ll} K^2 & = \limsup_{T\to\infty} \frac{1}{T}\int_0^T x^2(t)dt; \\ L^2 & = \limsup_{T\to\infty} \frac{1}{T}\int_0^T \dot{x}^2(t)dt; \\ M^2 & = \limsup_{T\to\infty} \frac{1}{T}\int_0^T \ddot{x}^2(t)dt; \\ N^2 & = \limsup_{T\to\infty} \frac{1}{T}\int_0^T \dddot{x}^2(t)dt. \end{array}$$

Similarly, equation (3.10) has mean-values

$$\begin{array}{ll} {K_1}^2 & = \limsup_{T\to\infty} \frac{1}{T}\int_0^T x^2(t)dt; \\ {L_1}^2 & = \limsup_{T\to\infty} \frac{1}{T}\int_0^T \dot{x}^2(t)dt; \\ {M_1}^2 & = \limsup_{T\to\infty} \frac{1}{T}\int_0^T \ddot{x}^2(t)dt; \\ {N_1}^2 & = \limsup_{T\to\infty} \frac{1}{T}\int_0^T \dddot{x}^2(t)dt. \end{array}$$

They showed that if the solutions of the equations are bounded and globally exponentially, then the mean-values K, L, M, N, K_1, L_1, M_1 and N_1 of any solution $x(t)$ are bounded independent of $x(t)$, with bounds given by

$$\begin{array}{ll} K & \leq \frac{A}{c}\left(1 + \left[\mu_1 + (ac)^{\frac{1}{2}}\right]\omega\right); \\ L & \leq A\omega; \\ M & \leq \frac{A}{2a}\omega_1; \text{ and} \\ N & \leq \frac{A}{2a}\left(a + [a^2 + 2(b+\mu_1)(\omega_1 + 2a(c+\mu_2)\omega^2)]^{\frac{1}{2}}\right) \end{array}$$

where

$$\begin{aligned} A^2 &= \limsup_{T\to\infty} \tfrac{1}{T} \int_0^T q^2(t)dt; \\ \omega &= \frac{\left(a+[\frac{c+\mu_1}{a}]^{\frac{1}{2}}+\left[a^2+b+2(a(c+\mu_2))^{\frac{1}{2}}+\frac{3(ab-c-\mu_2)}{a}\right]^{\frac{1}{2}}\right)}{2(ab-c-\mu_2)} \text{ and} \\ \omega_1 &= (1+[1+4a(c+\mu_2)\omega^2]) \end{aligned}$$

and

$$\begin{aligned} K_1 &\leq \tfrac{A}{2c}\left(1+[1+4ac\omega^2]^{\frac{1}{2}}\right); \\ L_1 &\leq A\omega; \\ M_1 &\leq \tfrac{A}{2a}\omega_1; \text{ and} \\ N_1 &\leq \tfrac{A}{2a}\left(a+[a^2+2(b+\mu_1)(\omega_1+2a(c+\mu_2)\omega^2)]^{\frac{1}{2}}\right). \end{aligned}$$

Much later in his career, Afuwape used the frequency domain approach to treat pendulum-like and oscillatory systems. A major result in this direction was obtained by him in the paper [30] where he modified the Barbashin-Ezeilo problem as:
When will a general third-order differential equation of the form

$$\dddot{x} + a\ddot{x} + g(\dot{x}) + \varphi(x) = 0, \tag{3.11}$$

where a is a constant, $g(\dot{x})$ is a continuous bounded function, and $\varphi(x)$ is a $2\pi-$periodic C^1 function, having zeros $0,\ x_o$ in $[0, 2\pi)$, and at any point $x \in [0, 2\pi)$, we have $\varphi^2(x) + [\varphi'(x)]^2 \neq 0$, have non-trivial periodic solutions?

A similar problem was also studied in 2007 in a joint work by Afuwape and Castellanos [26], for equations of the form:

$$\dddot{x} + a\ddot{x} + g_1(x)\dot{x} + \varphi(x) = 0 \tag{3.12}$$

using a non-local reduction method.

In summary, under this section Afuwape used the frequency domain approach to obtain effective criteria that guarantee the existence of solutions that are bounded, exponentially stable, periodic, almost periodic and uniformly dissipative for higher order nonlinear differential equations with or without delay terms. Second order results can be found in [4] on dissipative systems. Furthermore, for third order equations, papers [18], [20], [31], [33], [48], [61], [65] contributed to boundedness, exponentially stability, periodicity and almost periodicity of solutions while for dissipativity of solutions, we have papers [60], [64] and [68]. It must be pointed out here that papers [18] and [48] contained delay terms. Papers [37], [57] and [63] treated fourth order equations while fifth order nonlinear differential equations were considered in papers [41], [43] and [45].

Afuwape's papers on pendulum-like and oscillatory systems can be found in [2], [9], [10], [26] and [30].

4. Afuwape's work on nonlinear differential equations via functional analytical approach

In a research collaboration with Professors P. Omari and F. Zanolin and facilitated by the International Centre for Theoretical Physics, Trieste, Italy in the mid-eighties, Afuwape digressed into the study of nonlinear differential equations through a coincidence degree approach. He and his collaborators were concerned with the problem of periodic solutions to forced ordinary differential equations of various types. In particular, paper [56] was on the equation

$$Lx = EGx + f, \tag{4.1}$$

where $f \in ImL$ with L Fredholm of index zero, ImL orthogonal to $KerL$, E linear and G nonlinear. E, G and L are all continuous. Obtained results in this paper generalized earlier outstanding results in the literature on periodic solutions. Nonresonace case ($KerL = 0$) for the abstract equationwas also treated.

Later on, Afuwape introduced this method to his then Ph.D. student, A. S. Ukpera and together they obtained vector version results on existing scalar results. In paper [38], they considered the forced third-order nonlinear vector differential system

$$X''' + AX'' + BX' + sH(t, X) = P(t) \tag{4.2}$$

subject to periodic boundary conditions

$$X(0) = X(T) = X' - X'(T) = X''?X'' = 0, \tag{4.3}$$

where X is an n-dimensional vector $X : [0, T] \to R^n$. A and B are constant, symmetric $n \times n$-matrices, and $H = H(t, X)$ satisfies the Caratheodory conditions i.e. measurable in X, continuous a.e. in t, and bounded from above in the norm by an $L^1[0, T]$ function. Existence of T-periodic solutions of system was proved by the use of Leray-Schauder fixed point principle with additional hypotheses to establish a relationship between nonlinearities caused by the presence of the nonlinear term $H(s, X)$ and spectral properties of the linear operator

$$X''' + AX'' + BX' = \frac{d}{dt}\{X''' + AX'' + BX\}.$$

In summary, Afuwape contributed four works-[38], [42], [50] and [54] under the heading of this section.

5. Afuwape's contributions to Volterra integral operators

In paper [9], Afuwape (jointly with others) considered the third order nonlinear differential equation

$$x''' + \alpha x'' + g(x') + \phi(x) = 0, \tag{5.1}$$

where α is an arbitrary constant, g is a continuous bounded function and ϕ is a 2π-periodic odd function that equals zero at 0 and x_0 in $[0, 2\pi)$ and at any point $x \in [0, 2\pi)$ satisfies

$$\phi^2 + [\phi']^2 \neq 0. \tag{5.2}$$

Using the ideas of nonlocal reduction method, and transforming equation into an appropriate system which itself was transformed into an equivalent Volterra integral equation form and verifying that the resulting operator is an asymptotically pseudo contractive map, they used the Osilike-Akuchu Theorem [M. O. Osilike and B. G. Akuchu, Fixed Point Theory Appl. 2004, no. 2, 81-88; MR2086707 (2005c:47087)] and the Igbokwe-Udo-Utun Theorem [D.I.Igbokwe and X.Udo-Utun, J.Pure Appl. Math. Adv. Appl. 3(2010), no.1, 105- 121; MR2681551 (2011g:47172)], to show that the resulting feedback control system has a fixed point in a Banach space for all α (and in a Hilbert space for $\alpha > 0$). Using this, the authors formulate conditions for finding approximate cycles of the second kind to the problem given in the form of the considered equation.

In paper [11], Afuwape (jointly with others) contributed to the the solvability of the problem of representing causal operators with classical Volterra integral operators. They combined notions from Nemytskii operator theory and frequency domain methods to achieve their aims.

6. Other contributions of Afuwape

Afuwape [70, 71] jointly with C. O. Imoru contributed to Jensen inequality by obtaining new bounds for the Beta function $B(x, y)$. Afuwape also made important contributions to other areas, in particular to epidemiology where he and his collaborators were concerned with such problems as modeling and simulating measles and rubella elimination [15], [22] and [23].

In paper [35] jointly written with D. Oluwade, a novel approach to the study of the qualitative equivalence of first order autonomous ordinary differential equation

$$x' = f_i(x),$$

where $f_i(i = 1, 2, ...)$ is a transcendental function was presented. In the approach, $f(x)$ was approximated as a polynomial $h(x)$ via the power series expansion. A description of the qualitative classes of the equation was then given using the first few terms of $h(x)$. Cases such as the qualitative equivalence of $x' = p(x)$ when p is a polynomial of the second degree to that of $x' = g_i(x)$ where $g_i(x)$ is the polynomial formed from the terms of $h(x)$ up to the second degree were considered. The focus was on the exponential function e^x and the hyperbolic function $coshx$. A necessary condition for the qualitative equivalence of $x' = f_1(x)$ and $x' = f_2(x)$ to imply the qualitative equivalence of $x' = g_1(x)$ and $x' = g_2(x)$ was given.

Lastly, Professor Afuwape contributed to reviewing research publications of notable Nigerian Mathematicians such as Professor H. O. Tejumola in papers [6] and [49], Professor J. O. C. Ezeilo in paper[1] and Professor R. F. A. Abiodun in paper [47].

7. Conclusion

Aside from all the above published works of Professor Afuwape, some of the manuscripts that he submitted for publications before his death were:

1. Afuwape, A.U. Gradient-like solutions of Pendulum-like systems of a class of nonlinear third-order differential equations.
2. Afuwape, A.U. Frequency domain approach to the Sunflower equation with small delays.

Hopefully, these manuscripts will be published.

In his long, arduous but fruitful career, Professor Afuwape had achieved plenty to talk about. He had contributed his fair and important share to teaching, research and service, and he took legitimate pride in his achievements. During his lifetime, he made exceptional achievements with his huge and extensive research works which were characterized not only by originality and speed, but by the fact that they were done almost single-handedly in the absence of the much needed academic luxury enjoyed nowadays. His contributions to the theory and applications of nonlinear differential equations of higher order have been fundamental and are a milestone in the development of this area of Mathematics. He was always willing with enthusiasm to share certain parts of his profound and insightful knowledge in Mathematics with colleagues and students. He did not only introduce his students to the world of scientific research, but he also created an excellent atmosphere and condition in his research group with his everlasting optimism and enthusiasm. With tact and patience he guided and provided his students with his shoulders to stand on. Most of the students and colleagues that he mentored are now Professors. I have probably and unarguably personally benefited more than anyone else in this connection.

Professor Afuwape was fond of *harassing* his students and colleagues on the need to do quality research and publish their results in good outlets. The caption 'publish or perish' was boldly written at one corner on the big blackboard in his office at the 'White House'. For a long time, I wondered what this caption meant until I gathered enough courage to ask the respected Professor what it was all about. He took his time to explain to me comprehensively, what research concerned and in particular how to be a good researcher in Mathematics. He then stood up gently from his chair to lecture me with matchless pedagogical finesse and exposed me to some trends as at then in differential equations especially on the qualitative behaviour of solutions via Lyapunov's second method and frequency domain techniques. Third order differential equations were his focal point. I kept this wonderful exposition intensely in my memory. He conceived his ideas in such a way that it created the impression that I was directly involved in the discovery of the considered results and their proofs. I had earlier observed during this exposition that Professor Afuwape concentrated on third order equations through the frequency domain technique in his research work and when I asked him for the reasons, he gave some answers that had since been our *little secret*. It is now time for me to let out some of those secrets.

Adieu Professor Anthony Uyi Afuwape. May your soul continue to rest in peace.

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