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STABILITY AND BOUNDEDNESS PROPERTIES OF SOLUTIONS OF CERTAIN SYSTEM OF THIRD ORDER DELAY DIFFERENTIAL EQUATION

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ABSTRACT. This paper is concerned with certain system of third order delay differential equation. By using a suitable Lyapunov-Krasovskii functional as a tool, we investigate conditions for the stability, asymptotic stability, uniform stability of the trivial solution and uniform ultimate boundedness of all solutions of the equation considered. The results in this paper in some ways generalize and improve on some results found in literature.

1. INTRODUCTION

In this paper, we are interested in studying the conditions that guarantee the stability, asymptotic stability, boundedness and uniform ultimate boundedness of solutions of the following third order delay differential equation

$$X''' + H(X')X'' + G(X)X'(t-\tau) + b(t)X(t-\tau) = P(t, X, X', X''), \tag{1.1}$$

or its equivalent system

$$\begin{aligned} X' &= Y, \\ Y' &= Z, \\ Z' &= -H(Y)Z + b(t)\int_{t-\tau}^{t} Y(s)ds - b(t)X + G(X)\int_{t-\tau}^{t} Z(s)ds \\ &\quad -G(X)Y + P(t, X, Y, Z) \end{aligned} \tag{1.2}$$

where $\tau > 0$ is a fixed delay constant, $b(t)$ is a continuously differentiable function of t; $X,\ Y,\ Z \in \mathbb{R}^n$, G and H are $n \times n$ positive definite continuous symmetric matrix functions of the arguments displayed explicitly, the dots indicate differentiation with respect to t, $t \in \mathbb{R}^+ = [0, \infty)$ and $P : (\mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n) \to \mathbb{R}^n$. To ensure the existence and uniqueness of solutions of equation (1.1) or system (1.2), we assume that the functions G and H are continuous and also satisfy a Lipschitz condition with respect to their respective arguments.

In about five to six decades now, the study of qualitative behaviour (stability, boundedness, convergence, periodicity among others) of solutions of third order (even second and higher orders) scalar and vector, linear and non-linear differential equations with or without delay have received considerable attention from many notable researchers (see, [1] - [26]). In many of the papers, the second method of Lyapunov was used as a technique. To use this method, one needs to construct a suitable scalar function called Lyapunov functional (or Lyapunov-Krasovskii functional). The function is expected to be positive semi-definite and its derivative negative semi-definite along the solution path of the equation being studied. But unfortunately, to construct this function remains a difficult task especially for non-linear differential equations.

Going through the literature, we found that Omeike [14] used this direct method of Lyapunov to examine the stability and boundedness of solutions of differential system of third order with variable

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delay, $\tau(t)$, of the form

$$X''' + AX'' + BX' + H(X(t-\tau(t))) = P(t), \tag{1.3}$$

where A and B are real $n\times n$ constant symmetric matrices, $0 \le \tau(t) \le \gamma$, γ is a positive constant, $H : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous differentiable function with $H(0) = 0$. Later, Tunc [20] considered a more general third order delay differential equation of form

$$X''' + AX'' + G(X'(t-\tau(t))) + H(X(t-\tau(t))) = F(t,X,X',X''), \tag{1.4}$$

where $\tau(t)$ is a continuous differentiable function, with $0 \le \tau(t) \le \tau_0$, τ_0 is a positive constant and A is an $n\times n$ constant symmetric matrix, $G,H : \mathbb{R}^n \to \mathbb{R}^n$ are continuous differentiable functions with $G(0) = 0 = H(0)$. The author employed the direct method of Lyapunov and proved some interesting results on the asymptotic stability, uniform stability, boundedness and uniform boundedness of solutions to Eq. (1.4). The results in [20] generalize those in [14]. Particularly, Eq. (1.4) reduces to Eq. (1.3) on setting $G(X'(t-\tau(t))) = BX'$ and $F(t,X,X',X'') = P(t)$ in Eq. (1.4).

Furthermore, using a suitable Lyapunov-Krasovskii functional, the problem of stability and boundedness of solutions of certain third oder vector differential equation with constant delay ($\tau_1 > 0$) given by (1.5) was considered by Tunc and Mohammed [21].

$$X''' + \psi(X')X'' + BX'(t-\tau_1) + cX(t-\tau_1) = P(t), \tag{1.5}$$

where c is a positive constant, B is an $n\times n$ constant symmetric matrix, ψ is an $n\times n$ continuous differentiable symmetric matrix function. Equation (1.5) in some ways generalizes and improves on (1.3) and (1.4). Also, Tunc [23] used this direct method of Lyapunov in establishing some results on the asymptotic stability, boundedness and ultimate boundedness of solutions of the following vector differential equation

$$X''' + H(X')X'' + G(X'(t-\tau)) + cX(t-\tau) = F(t,X,X',X''), \tag{1.6}$$

where $\tau > 0$ is a fixed delay constant, c is a positive constant, $G : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous differentiable function with $G(0) = 0$ and H is an $n\times n$ continuous differentiable symmetric matrix function. The Jacobian matrices of both G and H are assumed to exist, symmetric and continuous. Obviously, Eq. (1.6) is more general when compared with Eq. (1.5).

Motivated by the works of Omeike [14], Tunc [20], Tunc and Mohammed [21] and Tunc [23], we shall employ the second method of Lyapunov to study certain conditions under which the third order delay differential equation (1.1) or system (1.2) has a stable trivial solution when $P(t,X,Y,Z) = 0$ and ultimately bounded solutions when $P(t,X,Y,Z) \neq 0$.

Remark It should be noted that Eq. (1.1) generalizes some equations found in the literature. For instance, if in Eq. (1.1), we let $b(t) = c$, $G(X)X'(t-\tau)) = G(X'(t-\tau))$ and $P(t,X,X',X'') = F(t,X,X',X'')$, we obtain Eq. (1.6) studied by Tunc [23]. Also, by taking $H(X') = \psi(X')$, $G(X) = B$, $b(t) = c$ and $P(t,X,X',X'') = P(t)$ in Eq.(1.1) we arrive at Eq. (1.5) examined by authors in [21].

2. Preliminary Results

The following definitions, theorems and lemmas obtained from [8], [19] and [24] are needed to establish our main results.

First, let us consider the general delay differential system

$$x' = f(x_t),\ x_t = x(t+\theta), -r \le \theta < 0,\ t \ge 0, \tag{2.1}$$

where $f : C_{\mathbf{H}} \to \mathbb{R}^n$ is a continuous mapping, $f(0) = 0$,

$$C_{\mathbf{H}} := \{\phi \in C([-r,0],\mathbb{R}^n) : \parallel \phi \parallel \le \mathbf{H}\},$$

and for $\mathbf{H_1} < \mathbf{H}$, there exists $L > 0$, with $|f(\phi)| \le L$ when $\parallel \phi \parallel \le \mathbf{H_1}$ and $C = C([-r,0],\mathbb{R}^n)$ denote the space of continuous vector function from $[-r,0]$ into $\mathbb{R}^n$. We say that $V : C \to \mathbb{R}$ is a Lyapunov function on a set $\mathbf{G} \subset C$ relative to f if V is continuous on $\tilde{\mathbf{G}}$, the closure of $\mathbf{G}$, V' is defined on $\mathbf{G}$ and $V' \le 0$ on $\mathbf{G}$.

Definition 1 [[8], [24]] *A solution $\phi(t)$ of Eq.* (2.1) *defined for $t \geq 0$, is said to be Lyapunov stable if given an $\varepsilon > 0$, there exists a $\delta > 0$ such that any solution $\varphi(t)$ of (2.1) with:*

$$\| \varphi(0) - \phi(0) \| < \delta,$$

satisfies

$$\| \varphi(t) - \phi(t) \| < \varepsilon,$$

for all $t \geq 0$, where $\| \,.\, \|$ stands for the usual Euclidean norm.

If in addition to the definition of stability above, we have:

$$\| \varphi(t) - \phi(t) \| \to 0 \text{ as } t \to \infty, \tag{2.2}$$

then we say the solution $\phi(t)$ is asymptotically stable.

Definition 2 [[8], [24]] *A solution $\phi(t)$ of Eq.* (2.1) *is said to be bounded if there exists a $\beta > 0$ and a constant $M > 0$ such that $\|\phi(t, t_0, x_0)\| < M$ whenever $\|x_0\| < \beta$, $t \geq t_0$.*

We will now provide some existing results on stability and boundedness of Eq. (2.1) for completeness sake.

Theorem 2.1[[8], [24]] *Suppose that there exists a Lyapunov function $V(t, \mathbf{X})$ defined on $0 \leq t < \infty$, $\| \mathbf{X} \| < \mathbf{H}$ which satisfies the following conditions:*

Theorem 2.2[[8], [24]] *If the condition (ii) in Theorem 2.1 is replaced by*

$$a(\| \mathbf{X} \|) \leq V(t, \mathbf{X}) \leq b(\| \mathbf{X} \|),$$

where $a(r)$ and $b(r)$ are continuous-increasing positive definite function(CIP). Then the zero solution of Eq. (2.1) *is uniformly stable.*

Theorem 2.3[[8], [24]] *Under the assumptions of the Theorem 2.1, if*
(iv) $V'_{(2.1)}(t, \mathbf{X}) \leq -c(\| \mathbf{X} \|)$,
where $c(r)$ is continuous on $[0, \varepsilon]$ and positive definite, and if $f(t, \mathbf{X})$ is bounded, then the zero solution of Eq. (2.1) *is asymptotically stable.*

Theorem 2.4[[8], [24]] *Under the assumptions of Theorem 2.1, if*

$$V'_{(2.1)}(t, \mathbf{X}) \leq -c(\| \mathbf{X} \|),$$

where $c(r)$ is continuous on $[0, \mathbf{H}]$ and is positive definite, then the zero solution of Eq. (2.1) *is uniformly-asymptotically stable.*

Theorem 2.5[[8], [24]] *Suppose that there exist a Lyapunov function $V(t, \mathbf{X})$ defined on $I \times \mathbb{R}^n$ which satisfies the following conditions:*

Theorem 2.6[[8], [24]] *Suppose that there exist a Lyapunov function $V(t, \mathbf{X})$ defined on $0 \leq t \leq R$, $\| \mathbf{X} \| \geq R$, (where R may be large) which satisfies:*

Theorem 2.7[[8], [24]] *Under the assumptions of Theorem 2.6, if $V'_{(2.1)}(t, \mathbf{X}) \leq -c(\| \mathbf{X} \|)$, where $c(r)$ is positive and continuous, then the solutions of Eq.* (2.1) *are uniformly ultimately bounded.*

Theorem 2.8(LaSalle's invariance principle)[[8], [24]]
If V is a Lyapunov function on a set $\mathbf{G}$ and $x_t(\phi)$ is a bounded solution such that $x_t(\phi) \in G$ for $t \geq 0$, then $\omega(\phi) \neq 0$ is contained in the largest invariant subset of $E \equiv \{\psi \in \mathbf{G}^* : V'(\psi) = 0\}$, where $\mathbf{G}^*$ is the closure of set $\mathbf{G}$ and ω denote the omega limit set of a solution.

Lemma 2.1[8] suppose $f(0) = 0$. Let V be a continuous functional defined on $C_{\mathbf{H}} = C$ with $V(0) = 0$, and let $u(s)$ be a function, non-negative and continuous for $0 \leq s < \infty$, $u(s) \to \infty$ as $u \to \infty$ with $u(0) = 0$. If for all $\phi \in C$, $U(|\phi(0)|) \leq V(\phi)$, $V(\phi) \geq 0$, $V'(\phi) \leq 0$, then the zero solution of $x' = f(x_t)$is stable.

If we define $Z = \{\phi : V'(\phi) = 0\}$, then the zero solution of $x' = f(x_t)$ is asymptotically stable, provided that the largest invariant set in Z is $Q = \{0\}$.

Lemma 2.2[19] *Let A be a real symmetric $n \times n$-matrix and*

$$\delta_a \leq \lambda_i(A) \leq \Delta_a, \ (i = 1, 2, ..., n),$$

where δ_a and Δ_a are constants representing the least and greatest eigenvalues of matrix A.
Then,

$$\delta_a \langle X, X \rangle \leq \langle AX, X \rangle \leq \Delta_a \langle X, X \rangle.$$

3. Main Results

In this section, we state the basic assumptions of our main results for the Eq. (1.1) or its equivalent system (1.2).

Assumptions:

Further to the earlier assumptions on $G, H, b(t)$ appearing in (1.1) or (1.2), we assume that there exist some positive constants $\varepsilon, \alpha, a_0, a_1, b_0, b_1, c_0, c_1$ and a negative constant δ_1 such that the following conditions are satisfied:

(i) the eigenvalues $\lambda_i(H(Y))$ and $\lambda_i(G(X))$ of $H(Y)$ and $G(X)$ respectively satisfy:

$$a_0 + \varepsilon \leq \lambda_i(H(Y)) \leq a_1 \text{ and } b_0 \leq \lambda_i(G(X)) \leq b_1, (i = 1, 2, 3, ..., n),$$

(ii)

$$c_0 \leq b(t) \leq c_1, \ b'(t) = \frac{d}{dt} b(t) \leq \delta_1 = -\alpha,$$

(iii)

$$a_0 b_0 - c_1 > 0, 1 - \alpha a_0 > 0, \varepsilon > \frac{2(b_1 - b_0)^2}{a_0 b_0 - c_1},$$

(iv)

$$\tau < \min \Big\{ \frac{K_1}{2\alpha a_0 b_0 (b_1 + c_1)}; \frac{K_2}{a_0(b_1 + c_1) + (1 + \alpha a_0 b_0 + a_0)c_1}; \frac{K_3}{b_1 + c_1 + (1 + \alpha a_0 b_0 + a_0)b_1} \Big\},$$

with

$$K_1 = \alpha a_0 b_0 c_0 > 0,$$
$$K_2 = \left[(a_0 b_0 - c_1) - 2\alpha(a_0^2 b_0 + a_0^{-1})\right] > 0,$$
$$K_3 = \left[\varepsilon - 2\alpha a_0 b_0 c_0^{-1}(a_1 - a_0)^2\right] > 0.$$

Theorem 3.1. Suppose that conditions (i) - (iv) stated in the assumptions above are satisfied, then the trivial solution of Eq. (1.1) is uniformly asymptotically stable.

Proof. In proving this theorem, we make use of the following differentiable scalar function $V(t) \equiv V(t, X(t), Y(t), Z(t))$ given as

$$\begin{aligned} 2V(t) =& a_0 b(t) \langle X, X \rangle + 2a_0 \int_0^1 \langle \sigma H(\sigma Y) Y, Y \rangle d\sigma + 2\alpha a_0 b_0 \int_0^1 \langle \sigma G(\sigma X) X, X \rangle d\sigma \\ &+ b_0 \langle Y, Y \rangle + \langle Z, Z \rangle + 2\alpha a_0^2 b_0 \langle X, Y \rangle + 2\alpha a_0 b_0 \langle X, Z \rangle + 2a_0 \langle Y, Z \rangle \\ &+ 2b(t) \langle X, Y \rangle - \alpha a_0 b_0 \langle Y, Y \rangle + 2\lambda \int_{-\tau}^0 \int_{t+s}^t \| Y(\theta) \|^2 d\theta ds \\ &+ 2\eta \int_{-\tau}^0 \int_{t+s}^t \| Z(\theta) \|^2 d\theta ds, \end{aligned} \tag{3.1}$$

where

$$0<\alpha<\min\left\{\frac{1}{a_0};\frac{a_0}{b_0};\frac{b_1}{a_0b_0};\frac{a_0b_0-c_1}{2(a_0^2b_0+a_0^{-1})};\frac{\varepsilon c_0}{2a_0b_0(a_1-a_0)^2}\right\},$$

$a_1\neq a_0$; $\eta>0$ and $\lambda>0$ are constants whose values will be determined later. The Lyapunov function defined in (3.1) is similar to the one used in [23].

Obviously, Eq. (3.1) vanishes for $X=Y=Z=0$ and it can be shown to be positive definite when $X\neq 0, Y\neq 0, Z\neq 0$ as follows. From the basic assumptions on matrix functions $H(Y)$ and $G(X)$ and Lemma 2.2., we obtain

$$a_0(a_0+\varepsilon)\parallel Y\parallel^2\leq 2a_0\int_0^1\langle\sigma H(\sigma Y)Y,Y\rangle d\sigma\leq a_0a_1\parallel Y\parallel^2,$$

and

$$\alpha a_0b_0^2\parallel X\parallel^2\leq 2\alpha a_0b_0\int_0^1\langle\sigma G(\sigma X)X,X\rangle d\sigma\leq\alpha a_0b_0b_1\parallel X\parallel^2.$$

Therefore,

$$\begin{aligned}
2V(t)\geq & a_0b(t)\langle X,X\rangle+2a_0\int_0^1\langle\sigma H(\sigma Y)Y,Y\rangle d\sigma+\alpha a_0b_0^2\langle X,X\rangle+b_0\langle Y,Y\rangle\\
&+\langle Z,Z\rangle+2\alpha a_0^2b_0\langle X,Y\rangle+2\alpha a_0b_0\langle X,Z\rangle+2a_0\langle Y,Z\rangle+2b(t)\langle X,Y\rangle\\
&-\alpha a_0b_0\langle Y,Y\rangle+2\lambda\int_{-\tau}^0\int_{t+s}^t\parallel Y(\theta)\parallel^2 d\theta ds+2\eta\int_{-\tau}^0\int_{t+s}^t\parallel Z(\theta)\parallel^2 d\theta ds,\\
=&\parallel\alpha a_0b_0X+a_0Y+Z\parallel^2+a_0b_0\parallel a_0^{-\frac{1}{2}}b_0^{-1}b(t)X+a_0^{-\frac{1}{2}}Y\parallel^2-b_0\parallel Y\parallel^2\\
&+\alpha a_0b_0^2\langle X,X\rangle-\frac{b^2(t)}{b_0}\parallel X\parallel^2+2a_0\int_0^1\langle\{\sigma H(\sigma Y)-Ia_0\}Y,Y\rangle d\sigma\\
&-\alpha^2a_0^2b_0^2\parallel X\parallel^2-\alpha a_0b_0\parallel Y\parallel^2+b_0\langle Y,Y\rangle+a_0^2\parallel Y\parallel^2\\
&+a_0b(t)\parallel X\parallel^2+2\eta\int_{-\tau}^0\int_{t+s}^t\parallel Z(\theta)\parallel^2 d\theta ds+2\lambda\int_{-\tau}^0\int_{t+s}^t\parallel Y(\theta)\parallel^2 d\theta ds,\\
\geq&\parallel\alpha a_0b_0X+a_0Y+Z\parallel^2+a_0b_0\parallel a_0^{-\frac{1}{2}}b_0^{-1}b(t)X+a_0^{-\frac{1}{2}}Y\parallel^2\\
&+\{\alpha a_0b_0^2(1-\alpha a_0)+\frac{c_1}{b_0}(a_0b_0-c_1)\}\parallel X\parallel^2+[\varepsilon a_0+a_0(a_0-\alpha b_0)]\parallel Y\parallel^2\\
&+2\eta\int_{-\tau}^0\int_{t+s}^t\parallel Z(\theta)\parallel^2 d\theta ds+2\lambda\int_{-\tau}^0\int_{t+s}^t\parallel Y(\theta)\parallel^2 d\theta ds,\\
\geq&\parallel\alpha a_0b_0X+a_0Y+Z\parallel^2+a_0b_0\parallel a_0^{-\frac{1}{2}}b_0^{-1}b(t)X+a_0^{-\frac{1}{2}}Y\parallel^2\\
&+\{\alpha a_0b_0^2(1-\alpha a_0)+\frac{c_1}{b_0}(a_0b_0-c_1)\}\parallel X\parallel^2+[\varepsilon a_0+a_0(a_0-\alpha b_0)]\parallel Y\parallel^2.
\end{aligned}$$

Thus, it is obvious from the above that we can find some positive constants K_4, K_5, K_6 such that

$$2V(t)\geq K_4\parallel X\parallel^2+K_5\parallel Y\parallel^2+K_6\parallel Z\parallel^2.$$

By letting

$$K_7=\min\{K_4;K_5;K_6\},$$

we have

$$2V(t)\geq K_7\{\parallel X\parallel^2+\parallel Y\parallel^2+\parallel Z\parallel^2\}. \tag{3.2}$$

Hence, the function $V(t)$ is positive definite at all points (X,Y,Z) and zero only at point $X=Y=Z=0$. In addition, $V(t)=0$ if and only if $\parallel X(t)\parallel^2+\parallel Y(t)\parallel^2+\parallel Z(t)\parallel^2=0$ and $V(t)>0$ if and only if $\parallel X(t)\parallel^2+\parallel Y(t)\parallel^2+\parallel Z(t)\parallel^2\neq 0$. It follows then that

$$V(t)\to+\infty \text{ as } \parallel X(t)\parallel^2+\parallel Y(t)\parallel^2+\parallel Z(t)\parallel^2\to\infty. \tag{3.3}$$

Also,

$$\begin{aligned}
2V(t) \leq & a_0 b(t)\langle X,X\rangle + 2a_0\int_0^1 \langle \sigma H(\sigma Y)Y,Y\rangle d\sigma + \alpha a_0 b_0 b_1\langle X,X\rangle + b_0\langle Y,Y\rangle \\
& + \langle Z,Z\rangle + 2\alpha a_0^2 b_0\langle X,Y\rangle + 2\alpha a_0 b_0\langle X,Z\rangle + 2a_0\langle Y,Z\rangle + 2b(t)\langle X,Y\rangle \\
& - \alpha a_0 b_0\langle Y,Y\rangle + 2\lambda\int_{-\tau}^0\int_{t+s}^t \| Y(\theta) \|^2 \, d\theta ds + 2\eta\int_{-\tau}^0\int_{t+s}^t \| Z(\theta) \|^2 \, d\theta ds, \\
= & \| \alpha a_0 b_0 X + a_0 Y + Z \|^2 + a_0 b_0 \| a_0^{-\frac{1}{2}} b_0^{-1} b(t) X + a_0^{-\frac{1}{2}} Y \|^2 - b_0 \| Y \|^2 + b_0 \| Y \|^2 \\
& + \alpha a_0 b_0 b_1 \| X \|^2 - \frac{b^2(t)}{b_0} \| X \|^2 + 2a_0\int_0^1 \langle \{\sigma H(\sigma Y) - I a_0\} Y,Y\rangle d\sigma + a_0 \| Y \|^2 \\
& - \alpha^2 a_0^2 b_0^2 \| X \|^2 - \alpha a_0 b_0 \| Y \|^2 + a_0 b(t) \| X \|^2 + 2\eta\int_{-\tau}^0\int_{t+s}^t \| Z(\theta) \|^2 \, d\theta ds \\
& + 2\lambda\int_{-\tau}^0\int_{t+s}^t \| Y(\theta) \|^2 \, d\theta ds + a_0^2 \| Y \|^2, \\
\leq & \| \alpha a_0 b_0 X + a_0 Y + Z \|^2 + a_0 b_0 \| a_0^{-\frac{1}{2}} b_0^{-1} b(t) X + a_0^{-\frac{1}{2}} Y \|^2 \\
& + \{\alpha a_0 b_0 (b_1 - \alpha a_0 b_0) + \frac{1}{b_0}(a_0 b_0 c_1 - c_0^2)\} \| X \|^2 \\
& + a_0(a_1 - \alpha b_0) \| Y \|^2 + 2\eta\int_{-\tau}^0\int_{t+s}^t \| Z(\theta) \|^2 \, d\theta ds \\
& + 2\lambda\int_{-\tau}^0\int_{t+s}^t \| Y(\theta) \|^2 \, d\theta ds,
\end{aligned}$$

clearly, $a_0 b_0 c_1 - c_0^2 > 0$ since $a_0 b_0 - c_1 > 0$.

Then, for some positive constants K_8, K_9, K_{10} we have from the above that

$$2V(t) \leq K_8 \| X \|^2 + K_9 \| Y \|^2 + K_{10} \| Z \|^2 .$$

Letting $K_{11} = \max\{K_8; K_9; K_{10}\}$ we obtain

$$2V(t) \leq K_{11}\{\| X \|^2 + \| Y \|^2 + \| Z \|^2\}. \tag{3.4}$$

On combining inequalities (3.2) and (3.4), we get

$$K_7\{\| X \|^2 + \| Y \|^2 + \| Z \|^2\} \leq 2V(t) \leq K_{11}\{\| X \|^2 + \| Y \|^2 + \| Z \|^2\}. \tag{3.5}$$

Given that $(X,Y,Z) = (X(t),Y(t),Z(t))$ is any solution of system (1.2). Differentiating the Lyapunov-Krasovskii function $V(t) = V(X(t),Y(t),Z(t))$ defined by (3.1) with respect to t along the trajectory of system (1.2), we get

$$\begin{aligned}
V'(t) = & -\alpha a_0 b_0 b(t)\langle X,X\rangle - \langle Z, G(X)Y\rangle - a_0\langle Y, G(X)Y\rangle + \alpha a_0^2 b_0\langle Y,Y\rangle \\
& - \alpha a_0 b_0\langle X, H(Y)Z\rangle + \alpha a_0^2 b_0\langle X,Z\rangle - \langle Z, H(Y)Z\rangle + a_0\langle Z,Z\rangle \\
& + b_0\langle Y,Z\rangle + \langle Z, b(t)\int_{t-\tau}^t Y(s)ds\rangle + \langle Z, G(X)\int_{t-\tau}^t Z(s)ds\rangle \\
& + \alpha a_0 b_0\langle X, G(X)\int_{t-\tau}^t Z(s)ds\rangle + \alpha a_0 b_0\langle X, b(t)\int_{t-\tau}^t Y(s)ds\rangle \\
& + a_0\langle Y, G(X)\int_{t-\tau}^t Z(s)ds\rangle + b(t)\langle Y,Y\rangle + a_0\langle Y, b(t)\int_{t-\tau}^t Y(s)ds\rangle \\
& + \lambda\tau \| Y \|^2 + \eta\tau \| Z \|^2 - \lambda\int_{t-\tau}^t \| Y(\theta) \|^2 \, d\theta - \eta\int_{t-\tau}^t \| Z(\theta) \|^2 \, d\theta \\
& + \frac{1}{2}a_0 b'(t)\langle X,X\rangle + b'(t)\langle X,Y\rangle + \langle \alpha a_0 b_0 X + a_0 Y + Z, P(t,X,Y,Z)\rangle.
\end{aligned} \tag{3.6}$$

Setting $P(t,X,Y,Z)=0$, we obtain

$$\begin{aligned}V'(t)=&-\alpha a_0b_0b(t)\langle X,X\rangle-\langle Z,G(X)Y\rangle-a_0\langle Y,G(X)Y\rangle+\alpha a_0^2b_0\langle Y,Y\rangle\\&-\alpha a_0b_0\langle X,H(Y)Z\rangle+\alpha a_0^2b_0\langle X,Z\rangle-\langle Z,H(Y)Z\rangle+a_0\langle Z,Z\rangle\\&+b_0\langle Y,Z\rangle+\langle Z,b(t)\int_{t-\tau}^{t}Y(s)ds\rangle+\langle Z,G(X)\int_{t-\tau}^{t}Z(s)ds\rangle\\&+\alpha a_0b_0\langle X,G(X)\int_{t-\tau}^{t}Z(s)ds\rangle+\alpha a_0b_0\langle X,b(t)\int_{t-\tau}^{t}Y(s)ds\rangle\\&+a_0\langle Y,G(X)\int_{t-\tau}^{t}Z(s)ds\rangle+b(t)\langle Y,Y\rangle+a_0\langle Y,b(t)\int_{t-\tau}^{t}Y(s)ds\rangle\\&+\lambda\tau\parallel Y\parallel^2+\eta\tau\parallel Z\parallel^2-\lambda\int_{t-\tau}^{t}\parallel Y(\theta)\parallel^2d\theta\\&-\eta\int_{t-\tau}^{t}\parallel Z(\theta)\parallel^2d\theta+\frac{1}{2}a_0b'(t)\langle X,X\rangle+b'(t)\langle X,Y\rangle.\end{aligned}$$

This we can re-write as

$$V'(t)=V_1'(t)+V_2'(t), \tag{3.7}$$

where

$$\begin{aligned}V_1'(t)=&-\alpha a_0b_0b(t)\langle X,X\rangle-\langle Z,G(X)Y\rangle-a_0\langle Y,G(X)Y\rangle+\alpha a_0^2b_0\langle Y,Y\rangle\\&-\alpha a_0b_0\langle X,H(Y)Z\rangle+\alpha a_0^2b_0\langle X,Z\rangle-\langle Z,H(Y)Z\rangle+a_0\langle Z,Z\rangle+b_0\langle Y,Z\rangle\\&+\frac{1}{2}a_0b'(t)\langle X,X\rangle+b'(t)\langle X,Y\rangle+b(t)\langle Y,Y\rangle,\end{aligned}$$

and

$$\begin{aligned}V_2'(t)=&\langle Z,b(t)\int_{t-\tau}^{t}Y(s)ds\rangle+\langle Z,G(X)\int_{t-\tau}^{t}Z(s)ds\rangle+\alpha a_0b_0\langle X,G(X)\int_{t-\tau}^{t}Z(s)ds\rangle\\&+\alpha a_0b_0\langle X,b(t)\int_{t-\tau}^{t}Y(s)ds\rangle+a_0\langle Y,G(X)\int_{t-\tau}^{t}Z(s)ds\rangle+a_0\langle Y,b(t)\int_{t-\tau}^{t}Y(s)ds\rangle\\&+\lambda\tau\parallel Y\parallel^2+\eta\tau\parallel Z\parallel^2-\lambda\int_{t-\tau}^{t}\parallel Y(\theta)\parallel^2d\theta-\eta\int_{t-\tau}^{t}\parallel Z(\theta)\parallel^2d\theta.\end{aligned}$$

From the definition of $V_1'(t)$, we have the following

$$\frac{1}{2}a_0b'(t)\langle X,X\rangle+b'(t)\langle X,Y\rangle=\frac{1}{2}b'(t)\parallel a_0^{\frac{1}{2}}X+a_0^{-\frac{1}{2}}Y\parallel^2-\frac{b'(t)}{a_0}\parallel Y\parallel^2;$$

$$\begin{aligned}&-\frac{1}{2}\alpha a_0b_0b(t)\langle X,X\rangle-\alpha a_0b_0\langle X,H(Y)Z\rangle+\alpha a_0^2b_0\langle X,Z\rangle\\&=-\frac{1}{2}\alpha a_0b_0\parallel b^{\frac{1}{2}}(t)X+b^{-\frac{1}{2}}(t)(H(Y)-Ia_0)Z\parallel^2+\frac{\alpha a_0b_0}{b(t)}\parallel(H(Y)-Ia_0)Z\parallel^2;\end{aligned}$$

and

$$\begin{aligned}-\frac{1}{2}\langle Z,(H(Y)-a_0I)Z\rangle-\langle Z,(G(X)-b_0I)Y\rangle=&-\frac{1}{2}\parallel(H(Y)-a_0I)^{\frac{1}{2}}Z+(H(Y)-a_0I)^{-\frac{1}{2}}(G(X)-b_0I)Y\parallel^2\\&+\parallel(H(Y)-a_0I)^{-\frac{1}{2}}(G(X)-b_0I)Y\parallel^2.\end{aligned}$$

From the above estimates and condition (i) of the theorem, we obtain

$$
\begin{aligned}
V_1'(t) \le & -\frac{1}{2}\alpha a_0 b_0 c_0 \parallel X \parallel^2 - \{a_0 b_0 - c_1 - \alpha a_0^2 b_0\} \parallel Y \parallel^2 - \frac{1}{2}\langle \{H(Y) - I a_0\} Z, Z\rangle \\
& + \frac{1}{2} b'(t) \parallel a_0^{\frac{1}{2}} X + a_0^{-\frac{1}{2}} Y \parallel^2 - \frac{b'(t)}{a_0} \parallel Y \parallel^2 \\
& - \frac{1}{2}\alpha a_0 b_0 \parallel b^{\frac{1}{2}}(t) X + b^{-\frac{1}{2}}(t)(H(Y) - I a_0) Z \parallel^2 + \frac{\alpha a_0 b_0}{b(t)} \parallel (H(Y) - I a_0) Z \parallel^2 \\
& - \frac{1}{2} \parallel (H(Y) - a_0 I)^{\frac{1}{2}} Z + (H(Y) - a_0 I)^{-\frac{1}{2}} (G(X) - I b_0) Y \parallel^2 + \frac{(G(X) - I b_0)^2}{(H(Y) - a_0 I)} \parallel Y \parallel^2,
\end{aligned}
$$

and under the assumptions of Theorem 3.1, we have

$$
\begin{aligned}
V_1'(t) \le & -\frac{1}{2}\alpha a_0 b_0 c_0 \parallel X \parallel^2 - \{a_0 b_0 - c_1 - \alpha a_0^2 b_0\} \parallel Y \parallel^2 + \frac{(b_1 - b_0)^2}{\varepsilon} \parallel Y \parallel^2 \\
& - \frac{1}{2}\langle \{H(Y) - I a_0\} Z, Z\rangle - \frac{\delta_1}{a_0} \parallel Y \parallel^2 + \frac{\alpha a_0 b_0}{c_0} \parallel (H(Y) - I a_0) Z \parallel^2 .
\end{aligned} \tag{3.8}
$$

Similarly, from the definition of $V_2'(t)$, we have the following.

$$
\begin{aligned}
\langle Z(t), G(X) \int_{t-\tau}^{t} Z(s) ds \rangle & \le \parallel Z(t) \parallel \parallel G(X) \parallel \int_{t-\tau}^{t} \parallel Z(s) \parallel ds, \\
& \le b_1 \parallel Z(t) \parallel \int_{t-\tau}^{t} \parallel Z(s) \parallel ds, \\
& \le \frac{b_1}{2} \int_{t-\tau}^{t} \{\parallel Z(t) \parallel^2 + \parallel Z(s) \parallel^2\} ds, \\
& = \frac{1}{2} b_1 \tau \parallel Z(t) \parallel^2 + \frac{1}{2} b_1 \int_{t-\tau}^{t} \parallel Z(s) \parallel^2 ds.
\end{aligned}
$$

In a similar way, we have the following

$$
\alpha a_0 b_0 \langle X(t), G(X) \int_{t-\tau}^{t} Z(s) ds \rangle \le \frac{1}{2}\alpha a_0 b_0 b_1 \tau \parallel X(t) \parallel^2 + \frac{1}{2}\alpha a_0 b_0 b_1 \int_{t-\tau}^{t} \parallel Z(s) \parallel^2 ds;
$$

$$
\langle Z(t), b(t) \int_{t-\tau}^{t} Y(s) ds \rangle \le \frac{1}{2} c_1 \tau \parallel Z(t) \parallel^2 + \frac{1}{2} c_1 \int_{t-\tau}^{t} \parallel Y(s) \parallel^2 ds;
$$

$$
a_0 \langle Y(t), G(X) \int_{t-\tau}^{t} Z(s) ds \rangle \le \frac{1}{2} a_0 b_1 \tau \parallel Y(t) \parallel^2 + \frac{1}{2} a_0 b_1 \int_{t-\tau}^{t} \parallel Z(s) \parallel^2 ds;
$$

$$
\alpha a_0 b_0 \langle X(t), b(t) \int_{t-\tau}^{t} Y(s) ds \rangle \le \frac{1}{2}\alpha a_0 b_0 c_1 \tau \parallel X(t) \parallel^2 + \frac{1}{2}\alpha a_0 b_0 c_1 \int_{t-\tau}^{t} \parallel Y(s) \parallel^2 ds;
$$

and lastly,

$$
a_0 \langle Y(t), b(t) \int_{t-\tau}^{t} Y(s) ds \rangle \le \frac{1}{2} a_0 c_1 \tau \parallel Y(t) \parallel^2 + \frac{1}{2} a_0 c_1 \int_{t-\tau}^{t} \parallel Y(s) \parallel^2 ds.
$$

On putting these estimates in $V_2'(t)$ we obtain

$$
\begin{aligned}
V_2'(t) \le & \frac{1}{2}\alpha a_0 b_0 \tau \{b_1 + c_1\} \parallel X(t) \parallel^2 \\
& + \frac{1}{2} a_0 \{b_1 + c_1\} \tau \parallel Y(t) \parallel^2 + \frac{1}{2}\{b_1 + c_1\} \tau \parallel Z(t) \parallel^2 \\
& + \lambda \tau \parallel Y(t) \parallel^2 + \eta \tau \parallel Z(t) \parallel^2 - \frac{1}{2}\{2\lambda - c_1 - a_0 c_1 (\alpha b_0 + 1)\} \int_{t-\tau}^{t} \parallel Y(s) \parallel^2 ds \\
& - \frac{1}{2}\{2\eta - b_1 - a_0 b_1 (\alpha b_0 + 1)\} \int_{t-\tau}^{t} \parallel Z(s) \parallel^2 ds.
\end{aligned}
$$

By taking $\lambda = \frac{1}{2}(1+\alpha a_0 b_0 + a_0)c_1$ and $\eta = \frac{1}{2}(1+\alpha a_0 b_0 + a_0)b_1$, we have

$$\begin{aligned} V_2'(t) \le & \frac{1}{2}\alpha a_0 b_0 \tau\{b_1+c_1\} \parallel X(t) \parallel^2 \\ & + \frac{1}{2}a_0\{b_1+c_1\}\tau \parallel Y(t) \parallel^2 + \frac{1}{2}\{b_1+c_1\}\tau \parallel Z(t) \parallel^2 \\ & + \frac{1}{2}(1+\alpha a_0 b_0 + a_0)c_1\tau \parallel Y(t) \parallel^2 + \frac{1}{2}(1+\alpha a_0 b_0 + a_0)b_1\tau \parallel Z(t) \parallel^2 . \end{aligned} \tag{3.9}$$

Combining (3.8) and (3.9) and noting that $\delta = -\alpha$, we have,

$$\begin{aligned} V'(t) \le & -\frac{1}{2}\alpha a_0 b_0 \Big\{c_0 - 2\tau(b_1+c_1)\Big\} \parallel X \parallel^2 \\ & -\frac{1}{2}\Big\{\Big(2a_0b_0 - 2c_1 - 2\alpha a_0^2 b_0 - \frac{2\alpha}{a_0} - \frac{2(b_1-b_0)^2}{\varepsilon}\Big) \\ & - [(a_0(b_1+c_1) + (1+\alpha a_0 b_0 + a_0)c_1)]\tau\Big\} \parallel Y \parallel^2 \\ & -\frac{1}{2}\Big\{\big[(H(Y) - Ia_0) - 2\alpha a_0 b_0 c_0^{-1}(H(Y)-Ia_0)^2)\big] \\ & - [b_1 + c_1 + (1+\alpha a_0 b_0 + a_0)b_1]\tau\Big\} \parallel Z \parallel^2 \\ \le & -\frac{1}{2}\alpha a_0 b_0 \Big\{c_0 - 2\tau(b_1+c_1)\Big\} \parallel X \parallel^2 \\ & -\frac{1}{2}\Big\{\big[(a_0b_0 - c_1) - 2\alpha(a_0^2 b_0 + a_0^{-1})\big] + \big[\varepsilon - 2(a_0b_0 - c_1)^{-1}(b_1-b_0)^2\big]\Big) \\ & - [(a_0(b_1+c_1) + (1+\alpha a_0 b_0 + a_0)c_1)]\tau\Big\} \parallel Y \parallel^2 \\ & -\frac{1}{2}\Big\{\big[\varepsilon - 2\alpha a_0 b_0 c_0^{-1}(a_1-a_0)^2\big] - \big[b_1 + c_1 + (1+\alpha a_0 b_0 + a_0)b_1\big]\tau\Big\} \parallel Z \parallel^2 \\ \le & -\frac{1}{2}\Big\{\alpha a_0 b_0 c_0 - 2\alpha a_0 b_0 \tau(b_1+c_1)\Big\} \parallel X \parallel^2 \\ & -\frac{1}{2}\Big\{\big[(a_0b_0 - c_1) - 2\alpha(a_0^2 b_0 + a_0^{-1})\big] \\ & - \big[(a_0(b_1+c_1) + (1+\alpha a_0 b_0 + a_0)c_1)\big]\tau\Big\} \parallel Y \parallel^2 \\ & -\frac{1}{2}\Big\{\big[\varepsilon - 2\alpha a_0 b_0 c_0^{-1}(a_1-a_0)^2\big] - \big[b_1 + c_1 + (1+\alpha a_0 b_0 + a_0)b_1\big]\tau\Big\} \parallel Z \parallel^2 . \end{aligned}$$

Taking

$$K_1 = \alpha a_0 b_0 c_0 > 0,$$

$$K_2 = \big[(a_0b_0 - c_1) - 2\alpha(a_0^2 b_0 + a_0^{-1})\big] > 0,$$

$$K_3 = \big[\varepsilon - 2\alpha a_0 b_0 c_0^{-1}(a_1-a_0)^2\big] > 0.$$

Then,

$$\begin{aligned} V'(t) \le & -\frac{1}{2}[K_1 - 2\alpha a_0 b_0 (b_1+c_1)\tau] \parallel X \parallel^2 \\ & -\frac{1}{2}[K_2 - \{(a_0(b_1+c_1) + (1+\alpha a_0 b_0 + a_0)c_1)\}\tau] \parallel Y \parallel^2 \\ & -\frac{1}{2}[K_3 - \{b_1+c_1+(1+\alpha a_0 b_0 + a_0)b_1\}\tau] \parallel Z \parallel^2 . \end{aligned}$$

Let

$$\tau \quad < \quad \min\Big\{\frac{K_1}{2\alpha a_0 b_0 (b_1+c_1)}; \frac{K_2}{(a_0(b_1+c_1) + (1+\alpha a_0 b_0 + a_0)c_1)}; \frac{K_3}{b_1+c_1+(1+\alpha a_0 b_0 + a_0)b_1}\Big\}.$$

It is possible to find some positive constants, K_4, K_5, K_6 such that

$$V'(t) \leq -K_4 \parallel X(t) \parallel^2 - K_5 \parallel Y(t) \parallel^2 - K_6 \parallel Z(t) \parallel^2, \\ \leq -K_7\{\parallel X(t) \parallel^2 + \parallel Y(t) \parallel^2 + \parallel Z(t) \parallel^2\} \leq 0, \tag{3.10}$$

where

$$K_7 = \min\{K_4; K_5; K_6\}.$$

So far, from inequalities (3.2), (3.10) and (3.5), (3.10) the trivial solution of Eq. (1.1) is stable and uniformly stable respectively.

Now, let us consider a set defined by

$$W \equiv \{(X,Y,Z) : V'(X,Y,Z) = 0\}.$$

By applying the well-known LaSalle's invariance principle (see, Theorem 2.8), we note that $(X,Y,Z) \in W$ implies that $X = Y = Z = 0$, i.e, $(X,Y,Z) = (0,0,0)$. This fact shows that the largest invariant set contained in W is $(0,0,0) \in W$. Now, by Lemma 2.1 and Theorems 2.1 - 2.4, the trivial solution of the equation (1.1) is uniformly asymptotically stable. □

The following is our theorem on uniform ultimate boundedness of solutions of Eq. (1.1) when $P(X,X',X'') \neq 0$.

Theorem 3.2. Suppose that all the conditions of Theorem 3.1 hold and in addition, there exist a positive constant δ_2 such that

$$P(X,X',X'') \leq \delta_2 \text{ for all } t \geq 0, \tag{3.11}$$

then all the solutions of Eq. (1.1) are uniform-ultimately bounded provided that

$$\tau < \min\left\{\frac{K_1}{2\alpha a_0 b_0(b_1+c_1)}; \frac{K_2}{(a_0(b_1+c_1)+(1+\alpha a_0 b_0+a_0)c_1)}; \frac{K_3}{b_1+c_1+(1+\alpha a_0 b_0+a_0)b_1}\right\}.$$

with,

$$K_1 = \alpha a_0 b_0 c_0 > 0,$$

$$K_2 = \left[(a_0 b_0 - c_1) - 2\alpha(a_0^2 b_0 + a_0^{-1})\right] > 0,$$

$$K_3 = \left[\varepsilon - 2\alpha a_0 b_0 c_0^{-1}(a_1 - a_0)^2\right] > 0.$$

Proof. The starting point in the proof of this theorem is the inequality (3.6) since estimates (3.5) is still valid. From (3.6), we have

$$\begin{aligned}
V'(t) =& -\alpha a_0 b_0 b(t)\langle X,X\rangle - \langle Z,G(X)Y\rangle - a_0\langle Y,G(X)Y\rangle + \alpha a_0^2 b_0\langle Y,Y\rangle\\
&-\alpha a_0 b_0\langle X,H(Y)Z\rangle + \alpha a_0^2 b_0\langle X,Z\rangle - \langle Z,H(Y)Z\rangle + a_0\langle Z,Z\rangle\\
&+b_0\langle Y,Z\rangle + \langle Z,b(t)\int_{t-\tau}^{t} Y(s)ds\rangle + \langle Z,G(X)\int_{t-\tau}^{t} Z(s)ds\rangle\\
&+\alpha a_0 b_0\langle X,G(X)\int_{t-\tau}^{t} Z(s)ds\rangle + \alpha a_0 b_0\langle X,b(t)\int_{t-\tau}^{t} Y(s)ds\rangle\\
&+a_0\langle Y,G(X)\int_{t-\tau}^{t} Z(s)ds\rangle + b(t)\langle Y,Y\rangle + a_0\langle Y,b(t)\int_{t-\tau}^{t} Y(s)ds\rangle\\
&+\lambda\tau \parallel Y\parallel^2 + \eta\tau\parallel Z\parallel^2 - \lambda\int_{t-\tau}^{t}\parallel Y(\theta)\parallel^2 d\theta - \eta\int_{t-\tau}^{t}\parallel Z(\theta)\parallel^2 d\theta\\
&+\frac{1}{2}a_0 b'(t)\langle X,X\rangle + b'(t)\langle X,Y\rangle + \langle\alpha a_0 b_0 X + a_0 Y + Z, P(t,X,Y,Z)\rangle,\\
\le& -\frac{1}{2}\Big\{\alpha a_0 b_0 c_0 - 2\alpha a_0 b_0\tau(b_1+c_1)\Big\}\parallel X\parallel^2\\
&-\frac{1}{2}\Big\{\big[(a_0 b_0 - c_1) - 2\alpha(a_0^2 b_0 + a_0^{-1})\big]\\
&-\big[(a_0(b_1+c_1) + (1+\alpha a_0 b_0 + a_0)c_1)\big]\tau\Big\}\parallel Y\parallel^2\\
&-\frac{1}{2}\Big\{\big[\varepsilon - 2\alpha a_0 b_0 c_0^{-1}(a_1-a_0)^2\big] - \big[b_1 + c_1 + (1+\alpha a_0 b_0 + a_0)b_1\big]\tau\Big\}\parallel Z\parallel^2\\
&+(\alpha a_0 b_0\parallel X\parallel + a_0\parallel Y\parallel + \parallel Z\parallel)\parallel P(t,X,Y,Z)\parallel .
\end{aligned}$$

By letting $K_7 \le \min\{K_4; K_5; K_6\}$ and $\delta_3 = \max\{\alpha a_0 b_0; a_0, 1\}$, we obtain

$$V'(t) \le -K_7\{\parallel X(t)\parallel^2 + \parallel Y(t)\parallel^2 + \parallel Z(t)\parallel^2\} + \delta_2\delta_3(\parallel X\parallel + \parallel Y\parallel + \parallel Z\parallel).$$

The remaining part of the proof can be obtained by following the same procedure as highlighted in Afuwape [4] and Meng [10]. Therefore, we omit this part of the proof. □

4. Conclusion

We have studied in this paper, some qualitative properties of solutions to a certain third order delay differential equation by employing Lyapunov-Krasovskii approach. Sufficient conditions for the stability, asymptotic stability, uniform stability and uniform ultimate boundedness of solutions to the systems of equations considered were given. The results contained in this paper complement and improve on the existing results found in the literature.

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