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HYPOMODULES AND AMENABILITY OF PSEUDO-COMPLETE LOCALLY CONVEX ALGEBRAS

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ABSTRACT. Given a Pseudo-complete locally convex algebra A. We define for A flat hypomodules and cyclic flat hypomodules in line with hypocontinuous multiplication in A. We generalize results available on amenability of Frechet algebras by using locally bounded approximate identity for pseudo-complete locally convex algebras endowed with the strict inductive limit topology.

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1. INTRODUCTION

The classical memoir of Johnson[11] in 1972 laid the foundation for amenable Banach algebras. Since the memoir of Johnson, a considerable number of notions of amenable Banach algebras have been developed and introduced by several authors. Notable among these notions include weak amenability, ideal amenability, character amenability and approximate amenability of Banach algebras, see [5, 7, 13, 15, 16, 17, 18]. Moreover, after the paper of Pirkovskii [19] where he extended amenability in Banach algebras to Frechet algebras, a number of work has been done to extend the aforementioned notions of amenability in Banach algebra to Frechet algebras. These work include approximate amenability for Frechet algebras by Lawson and Read [12], weak amenability in Frechet algebra by Fatemeh *et al* [8], Ideal amenability of Frechet algebras by Ranjbari1 and Rejali [20], character amenability of Frechet algebras by Abtahi1 *et al* [1] and n-ideal and n-weak amenability of Frechet algebras by Ranjbari1 and Rejali [21]. In [4], Ayinde and Agboola used point derivation to discuss character amenability in

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Frechet algebras. Furthermore, Ayinde *et al* [3] considered the concept of continuous derivation to extend amenable Frechet algebra to amenable Pseudo-complete locally convex algebras. In his paper, Pirkovskii [19] characterized amenable Frechet algebra in term of locally bounded approximate identity. He also gave a number of results that showed a relation between flat modules and amenable Frechet algebra. Among these results was the proof that shows that amenability of a Frechet algebra A is equivalent to flatness of unitization A_+ of A as a Frechet A-module.
In view of this, when considering the Frechet module structure setting, the symmetry between projective modules and injective modules fails when we move outside of Banach module structure setting (see [9], p.355). Given a Frechet algebra on which a Frechet module S is defined, the dual S' of S has no natural topological module structure (see [24], sec. 3). In what follows, Pirkowskii [19] defined a left Frechet module S over a Frechet algebra A as flat if the projective tensor product $(\cdot)\hat{\otimes}_A S$ takes "admissible" exact sequences of right Frechet A-modules to exact sequences of vector spaces (for details, see section 2).
In order to have a characterization of flat strict inductive limit modules, we need injective modules. In order to achieve this, we need to put restrictions on the inductive limit algebra (Pseudo-complete locally convex algebra (Pseudo-complete *lca*)) A and on the class of A-modules we intend to study. These restrictions include the introduction of hypocontinuous multiplication into our module multiplication. Also modules will be modules relative to complete inductive tensor product $\bar{\otimes}$ (see [24], sec.3).
Motivated by the work of Pirkovskii, we shall at this point give those results we intend to generalize.
Theorem 1.1 [19]**:** Suppose A is a Frechet-Arens-Michael algebra and $I \subset A_+$ (A_+ is the unitization of A) a closed left ideal. Set $S = A_+/I$. Given an Arens-Michael decomposition $A = \lim_{\longleftarrow}(A_\lambda, \sigma_\mu^\lambda)_{\lambda \in N}$ such that for each $\lambda \in N$, the ideal $I_\lambda = \overline{(\sigma_\lambda)_+(1)}$ is weakly complemented in $(A_\lambda)_+$. Then the following conditions are equivalent:

(i) I has a right locally bounded approximate identity (right *lbai*);
(ii) S is strictly flat Frechet A-module;
(iii) S is a flat Frechet A-module.

Theorem 1.2 [19]**:** Suppose A is a Frechet Arens-Michael algebra with a *lbai*. Then amenability of A is equivalent to biflatness of A.

In this paper, we aim to generalize Theorems 1.1 and 1.2 to pseudo-complete locally convex algebras. The paper is divided into three sections. Section 2 deals with preliminaries which contain definitions, notations, notions, results, and other materials that have direct bearing to our work. Section 3, subsection 3.1 contains results on flat and strictly flat hypomodules as they relate to amenability. In section 3, subsection 3.2, we give results on strictly flat cyclic hypomodules and amenable pseudo-complete locally convex algebras.

2. PRELIMINARY

This section contains three subsections.

2.1 Pseudo-complete locally convex algebras

A locally convex space (*lcs*) is a topological vector space that has 0-neighbourhoods base consisting of absolutely convex sets. Metrizable *lcs* that are complete are called Frechet spaces.
A normed linear space(*nls*) is a topological vector space whose topology is determined by a norm. Complete normed linear spaces are called Banach spaces. Completion of a *lcs* S is denoted by $S^{\sim}$.
A *lcs* S is called a DF-space if it possesses a basis sequence of bounded sets with a property that $U = \bigcap_{n=1}^{\infty} U_n$ absorbs bounded sets of S for which U is also a 0-neighbourhood where (U_n) is a sequence of closed absolutely convex 0-neighbourhood.
A barrel is a subset of a *lcs* that is absolutely convex, absorbent and closed. A *lcs* is called barrelled if every barrel is a neighbourhood.
If S, T and P are topological spaces, then a function $R : S\times T \longrightarrow P$ is said to be jointly continuous at $(s_0, t_0) \in S \times T$ if for each neighbourhood W of $R(s_0, t_0)$ there exists a product of open sets $U\times V \subseteq S \times T$ containing (s_0, t_0) such that $R(U \times V) \subset W$. The function R above is called separately continuous if for each (s_0, t_0) in $S \times T$, the map $s \longmapsto R(s, t_0) : S \longrightarrow P$ and $t \longmapsto R(s_0, t) : T \longrightarrow P$ are continuous.
A *lcs* S is called reflexive if it coincides with its continuous bidual space i.e., $S = S''$.
A is said to be a topological algebra over a field C if and only if it is an algebra endowed with a structure of *lcs* respect to which the product is separately continuous.
Let A be a Banach space. If A is an associative algebra such that the algebra multiplication satisfies

$$\|st\| \leq \|s\|\|t\| \quad (s, t \in A),$$

then we call A a Banach algebra.
A Frechet algebra is a topological algebra which is a Frechet space. Suppose A is a commutative algebra with identity. A bound structure for A is a non-empty collection of subsets of A such that

(a) is absolutely convex, $B^2 \subset B$, for each $B \in$;
(b) given B_1, B_2 in , there exists $B_3 \in$ and $\lambda > 0$ such that $B_1 \cup B_2 \subset \lambda B_3$.
$(A,)$ is then a bound algebra.
For every $B_n \in$, $A(B_n) = A_n = \{\lambda t | \lambda \in, \ t \in B_n\} =$ Span(B_n) for $n \in$. $A(B_n)$ is the subalgebra of A generated by B_n. A_n is a Banach algebra with respect to the gauge p_{B_n} of B_n:

$$p_{B_n}(t) = \inf\{|\lambda| > 0 | t \in \lambda B_n\} \quad (t \in B_n, \ B_n \in).$$

Let $\wedge$ be an index set directed upward by the relation $\leq$ defined by $n \leq m$ if and only if $B_n \subset \lambda B_m$ for some $|\lambda| > 0$. With $\|\cdot\|_n = P_{B_n}$ for the norm on A_n, for $n \leq m$, $A_n \subset A_m$ and the inclusion map R_{mn} is a continuous unital injection. Then $\{A_n; R_{mn}\}$ is an inductive system.
Let U_n be an absolutely convex 0-neighbourhood defined by the norm topology p_{B_n} in A_n. We define a map $R_n : A_n \longrightarrow A$, since R_n is continuous, $R_n(U_n) = U$, then, a base of 0-neighbourhood for the topology of A is given by W, the set of all absolutely convex subsets U of A, so that for each n, $R_n^{-1}(U) = U_n$ is a neighbourhood in A_n. This is an algebraic inductive topology and it coincides with the locally convex topology for A. (For detail see [2]).
Proposition 2.1.1 [2]**:** An algebra A is pseudo-complete with respect to some bounded structure if and only if A is isomorphic with the inductive limit of an inductive system $\{A_\alpha, R_{B_\alpha} : \alpha, \beta \in \wedge, \ \alpha \leq \beta\}$ of Banach algebras with identity and continuous unital monomorphisms.

We require that our inductive limit topology be strict. Let each A_n be given a topology τ_n defined by the norm $\| \ \|_n$ under which is a Banach algebra so that for each n, the topology induce on A_n by the norm topology $\| \ \|_{n+1}$ on A_{n+1} is the norm topology $\| \ \|_n$. This implies that each A_n is embedded algebraically and topologically in A_{n+1}. Furthermore, let τ be the inductive limit topology on $A = \lim_{\longrightarrow} A_n$ so that τ induces $\tau_n := \| \ \|_n$ on each A_n. Then $A = \lim_{\longrightarrow} A_n$ with τ is referred to as the strict inductive limit of the sequence of Banach algebra A_n. (For detail see [22], p. 127).
Let A be a pseudo-complete *lca*. If an identity were not present

in A, we can adjoin one as follows. Let us represent by A_+ the unitization of A and r an embedding of A into A_+. We define the homomorphism $t : A_+ \longrightarrow$ that vanishes on A with $t(1_+) = 1$ where 1_+ is the identity of A_+. By [2], the commutative pseudo-complete *lca* A with identity has a spectrum which is non-empty and compact in the weak* topology. Furthermore every maximal ideal is the kernel of a character. Moreover, the pseudo-complete *lca* homomorphism $\alpha : A \longrightarrow B$ is uniquely extended to a unital homomorphism $\alpha_+ : A_+ \longrightarrow B_+$. Hence, A_+ still carries the strict inductive limit topology.

Remark 2.1.2 ([24] **sec. 4):** For a sequence of Banach spaces $S_1 \supset S_2 \supset S_3 \supset \cdots$ we have a projective limit $\lim_{\longleftarrow} S_k = S$ $(k = 1, 2, 3, \dots)$ and also a space of linear functional S'_k for each Banach space S_k. Hence, $S = \lim_{\longleftarrow} S_k = \bigcap S_k$ is a Frechet space and $S' = \lim_{\longrightarrow} S_k = \bigcup S_k$ is a Pseudo-complete locally space and endowed with the strong topology makes it a DF-space. (see ([22], p. 161).

If $s = \lim_\alpha s\tau_\alpha$ (resp. $s = \lim_\alpha \tau_\alpha s$), then a net $(\tau_\alpha)_{\alpha\in I}$ is a right (resp. left) approximate identity (*ai*) in a topological algebra A for $s \in A$. If a net $(\tau_\alpha)_{\alpha\in I}$ is both a right and a left *ai*, then it is called an *ai*. If the set $(\tau_\alpha)_{\alpha\in I}$ is bounded, then we say an *ai* is bounded. If for each 0-neighbourhood $U \subset A$ we have $r > 0$ such that for each finite subset $K \subset A$ there is $t \in rU$ with $s - st \in U$ (respect. $s - ts \in U$) for all $s \in K$, then we say A has a right (resp. left) locally bounded approximate identity (*lbai*). If A has both right and left *lbai*, then we say A has a *lbai*. (see [19], p. 96).

Theorem 2.1.3: Suppose A is a pseudo-complete *lca* (i.e. a strict inductive limit algebra). Then A has a right *lbai* if each Banach algebra A_n has a right bounded *ai*.

Proof: Let $U \subset A$ be an absolutely convex 0-neighbourhood. Let consider the following composition map

$$A_i \xrightarrow{r_{ji}} A_j \xrightarrow{r_j} A \quad \text{for } i, j \in N,\ i < j$$

where $r_i(A_i) \subset r_j(A_j)$ and $r_j \circ r_{ji} = r_i$.

For U_i a 0-neighbourhood in A_i we have U_j a 0-neighbourhood in A_j such that

$$U_i \subseteq \Gamma\left(\bigcup_{j\in N} r_i^{-1}\left(r_j(U_j)\right)\right) = r_i^{-1}(U) \text{ (see [10], p. 114)}$$

Since each A_n has a right bounded ai. Let $\{t_i\}$ be a right bounded ai for A_i, then $r_i(t_i)$ exists in A. We find $t \in A$ such that $r_i(t_i) = t$ in A which implies that $r_i^{-1}(t) = t_i \in A_i$ from $U_i = r_i^{-1}(U)$. So, also for a finite subset K_i in A_i, find a finite subset $K \subset A$ such that given $s \in K_i$ we find $s' \in K$ with $r(s) = s' \in K$. By ([6], p. 58) we have $\|t_i\| < c$ $(t_i \in cU_i)$ where $c > 0$, this implies that $\|r_i^{-1}(t)\| < c$ for $c > 0$. Hence, $t \in cU$ since the topology is strict. So also for a finite set $K_i \subset A_i$ there is a finite set $K \subset A$ so that $\|r_i^{-1}(s') - r_i^{-1}(s')r_i^{-1}(t)\| < 1$ by ([6], p 58). This implies that $s' - s't \in U$. Hence t is a right $lbai$ in A. Left $lbai$ for A can also be shown analogously. Therefore A has a $lbai$. □

2.2 Inductive Tensor Product

Let S and T be lcs, by inductive tensor product topology on $S \otimes T$, we mean the finest locally convex topology on $S \otimes T$ where the canonical bilinear map $g : S \times T \longrightarrow S \otimes T$ is separately continuous. Its completion is denoted by $S \bar{\otimes} T$. Hence, the inductive tensor product topology is called a compatible tensor product topology on $S \bar{\otimes} T$ and it is the finest topology. (see ([9] p. 370)).

Definition 2.2.1 [10]**.** The inductive tensor product $S \bar{\otimes} T$ of two lcs completes the algebraic tensor product $S \otimes T$ endowed with the finest compatible tensor product topology. The following are the properties of inductive tensor product.

(i) Given a complete topological vector space P, there is an isomorphism between the space $L(S \bar{\otimes} T, P)$ of continuous linear maps from $S \bar{\otimes} T$ to P and the space of separately continuous bilinear maps from $S \times T$ to P; in particular, there exists an isomorphism between the dual $(S \bar{\otimes} T)'$ and the space of separately continuous bilinear functionals on $S \times T$;

(ii) the relation $(S \bar{\otimes} T)' = L(S, T')$ for inductive tensor product corresponds to the relation $(S \hat{\otimes} T) = B(S, T')$ for projective tensor product where $B(S, T')$ denotes the bounded linear transformation, from S to T'. (see [14], p.369 & 370) and [9], p. 350).

(iii) if S and T are Frechet spaces, separate continuity of bilinear forms and joint continuity of bilinear forms are equal, hence $S \bar{\otimes} T$ is equal to the projective tensor product $S \hat{\otimes} T$.

(iv) if $S = \lim_{\longrightarrow} S_i$ and $T = \lim_{\longrightarrow} T_j$ are inductive limits on which the inductive limit topology is defined, then $S \bar{\otimes} T =$

$\lim\limits_{\underset{i,j}{\longrightarrow}} S_i\bar{\otimes}T_j$ also carries the inductive limit topology

where S_i and S_j are sequences of Banach spaces.

2.3 Flat Hypomodules

Let S, T and P be topological vector spaces. A bilinear map $R : S \times T \to P$ is referred to as hypocontinuous if given a bounded set $K \subset S$ (resp. $K \subset T$) and 0-neighbourhood $U \subset P$ there is a 0-neighbourhood $V \subset T$ (resp. $V \subset S$) for which $R(K, V) \subset U$ (resp. $f(V, K) \subset U$). If S and T are barreled spaces it follows that separately continuous bilinear map $R : S \times T \longrightarrow P$ is hypocontinuous. The above conditions are also satisfied if our algebra A is a barreled DF-space. See [24].

Let A be a topological algebra and S a topological vector space. S is called a left (resp. right) A-module, if there exists a bilinear map $A \times S \longrightarrow T$ (resp. $S \times A \longrightarrow S$) such that $(a, s) \longmapsto as$, (resp. $(s, a) \longmapsto sa$) are continuous for $a \in A$ and $s \in S$. If S is both a left A-module and a right A-module, then it is referred to as an A-bimodule. A left (resp. right) A-module S for a topological algebra A is a hypomodule if the map $(a, s) \longmapsto as : A \times S \to S$ [resp. $(s, a) \longmapsto sa : S \times A \to S$] is hypocontinuous. We have the notion of bi-hypomodule if the left and right module operations are both hypocontinuous. (For detail see [24]). A set S is called a summand of T if there are homomorphisms $\pi : T \longrightarrow S$ and $i : S \longrightarrow T$ such that $\pi \circ i = id_T$.

Definition 2.3.1 [24]**.** A hypomodule T (left, right or bi) is referred to as injective, if given a hypomodule S and a submodule P which is a *lcs* direct summand of S, for which each module homomorphism $P \longrightarrow T$ extends to a module homomorphism $S \longrightarrow T$.

Throughout, our algebra A is barreled and complete. So also, we have on A a hypocontinuous multiplication thereby making A to be an A-bi-hypomodule.

However, $L(T, S)$ which is a linear space of all continuous linear maps from T to S will also have hypocontinuous multiplication. The operations in $L(A, S)$ are defined by $(aR)(b) = R(ba)$ and $R(a)(b) = R(ab)$. The map $\phi : T \longrightarrow L(A, S)$ is continuous as a result of hypocontinuity.

We require that $L(A, S)$ be complete and the topology of uniform convergence on bounded sets ensures this. With this, S is complete and the map $Q : A \longrightarrow S$ is continuous provided it is continuous on bounded sets of A.

Proposition 2.3.2 [24]**.** Let $L(A, S)$ be a left A-hypomodule. Given a left hypomodule T, then $hom_A(T, L(A, S))$ is isomorphic to $L(T, S)$.

Proposition 2.3.3 [24]**.** A left hypomodule is injective if and only if it is a left module direct summand of $L(A, S)$ for some complete locally convex space S.

As stated earlier, universal problems for bilinear maps that are separately continuous are solved by inductive tensor products. Hence, we consider the following construction. By property (ii) of 2.2.1 each Banach space S_i and T_j are Frechet spaces, hence, $S_i\bar{\otimes}T_j = S_i\hat{\otimes}T_j$. For each strict inductive limits $S = \lim_{\longrightarrow} S_i$ and $T = \lim_{\longrightarrow} T_j$, we have by property (iv) of 2.2.1 that $S\bar{\otimes}T = \lim\limits_{\substack{\longrightarrow \\ i,j}} (S_i\bar{\otimes}T_j) = \lim\limits_{\substack{\longrightarrow \\ i,j}} (S_i\hat{\otimes}T_j)$. Hence, $S\bar{\otimes}T$ is the strict inductive limit of Banach spaces and it is a complete *lcs*. We also note here that, if each S_i and T_j are Banach module, then each S and T are inductive limit hypomodules. So also if $S_i\hat{\otimes}T_j$ is a Banach module, then, we record that $S\bar{\otimes}T$ is an inductive limit hypomodule.

Definitions 2.3.4 [9]**:** Given a Frechet algebra A.

(i) Let S be a right A-module and T be a left A-module, the projective (inductive) module tensor product $S\hat{\otimes}_A T = S\bar{\otimes}_A T$ of S and T is defined as the quotient of $S\hat{\otimes}T$ (or $S\bar{\otimes}T$), by M the closed linear span of $\{s \cdot a \otimes t - s \otimes a \cdot t : s \in S, t \in T, a \in A\}$ i.e. $S\hat{\otimes}T/M$(or $S\bar{\otimes}T/M$).

(ii) A homomorphism $S \longrightarrow T$ of A-module is an admissible monomorphism if there exists an A-module map $T \longrightarrow S$ which is left inverse to $S \longrightarrow T$;

(iii) A homomorphism $T \longrightarrow P$ of A-modules is an admissible epimorphism if there exists an A-module map $P \longrightarrow T$ which is right inverse to $T \longrightarrow P$.

(iv) A short exact sequence of modules S, T and P is given by two maps $\alpha : S \longrightarrow T$ and $\beta : T \longrightarrow P$ written as $0 \longrightarrow S \xrightarrow{\alpha} T \xrightarrow{\beta} P \longrightarrow 0$ where α is injective and β is surjective. Furthermore, the image of α is equal to the kernel of β, or equivalently if β equals coker α.

(v) A short exact sequence in (iv) is said to split if there is a map $\gamma : P \longrightarrow T$ such that $\beta \circ \gamma$ is the identity on P.

(vi) A short exact sequence of A-module is an admissible short exact sequence if it is split as a sequence A-module.

Definition 2.3.4(i) is applicable to the strict inductive limit algebra and its hypomodules with respect to the inductive tensor product. Furthermore, we employ the definitions of flatness and strictly flatness of Banach and Frechet modules for strict inductive limit hypomodules, since by (Proposition 2.3.3), we have enough injective hypomodules for flatness.

Definition 2.3.5 ([cf. [6]]) & ([cf. [23]] p.134): Let A be a pseudo-complete *lca* (resp. Barreled algebra).

(i) A left A-hypomodules T is referred to as flat if given an admissible short exact sequence $0 \longrightarrow S_2 \longrightarrow S_1 \longrightarrow S_1/S_2 \longrightarrow 0$ of right A-hypomodules the sequence $0 \longrightarrow S_2 \bar{\otimes}_A T \longrightarrow S_1 \bar{\otimes}_A T \longrightarrow (S_1/S_2 \bar{\otimes}_A T) \longrightarrow 0$ is exact.

(ii) A left A-hypomodule T is referred to as strictly flat if given a short exact sequence $0 \longrightarrow S_2 \longrightarrow S_1 \longrightarrow S_1/S_2 \longrightarrow 0$ of right A-hypomodules the sequence $0 \longrightarrow S_2 \bar{\otimes}_A T \longrightarrow S_1 \bar{\otimes}_A T \longrightarrow (S_1/S_2 \bar{\otimes} T) \longrightarrow 0$ is exact.

We shall consider in this paper strict inductive limit hypomodules, as our domain as against the Frechet modules considered by Pirkouskii [19].

Let I be a closed left ideal in a topological (unitization) algebra A_+. S is called a cyclic module (hypomodule) when $S := A_+/I$.

A subspace S_0 in a topological vector space S is called complemented, if S decomposes into a topological direct sum of S_0 and other subspace.

I is said to be weakly complemented in A_+ if the space $I^\perp := \{R \in A'_+ : R|_I = 0\}$ for all $s \in I$ is complemented as a subspace in the dual A'_+ of A_+.

The following condition is called almost criterion for a cyclic module to be flat.

Condition 2.3.6 ([9], p. 351): Let I be a closed left ideal in A_+. Then for A_+/I to be flat it is sufficient and, if $I^\perp := \{R \in A'_+ : R|_I = 0\}$ is complemented in A'_+, it is also necessary that I have a right locally bounded approximate identity.

Lemma 2.3.7 [25]**:** A short sequence $0 \longrightarrow T \longrightarrow P \longrightarrow W \longrightarrow 0$ in Frechet spaces is exact if and only if the dual short sequence of the strong dual $(\beta(S', S))$ where S is a Frechet space is exact.

Proposition 2.3.8([22], p. 121, Prop. 16): The bidual of a metrizable space is a Frechet space under its strong topology.

3. MAIN RESULTS

3.1 Flat and Strictly Flat Hypomodules and Amenability In this sub-section we prove some results on flat and strictly flat pseudo-complete locally convex hypomodules and link them with amenability of pseudo-complete locally convex algebras.

Theorem 3.1.1: Let A be a complete and barreled topological algebra and P a reflexive Frechet space. If P is flat as a left A-module, then there exist a strict inductive limit space T that is also flat as a left A-hypomodule.

Proof: We split the proof into two steps.

Let P be a flat left Frechet A-module, $S_\bullet := \{0\} \longrightarrow S_2 \longrightarrow S_1 \longrightarrow S_1/S_2 \longrightarrow \{0\}$ be an admissible short exact sequence of right Frechet A-module and $W_\bullet := \{0\} \longrightarrow W_2 \longrightarrow W_1 \longrightarrow W_1/W_2 \longrightarrow \{0\}$ be an admissible sequence of right inductive limit A-hypomodule. We want to show that $W_\bullet \bar{\otimes}_A T$ is exact. We first show that $(W_\bullet \bar{\otimes}_A T)'$ is exact. Let $\alpha \in \hom_A(W_\bullet, T')$, we define $\beta : W_\bullet \times T \longrightarrow$ by $\beta(s,t) = \alpha(s)t$. By definition 2.2.1 condition (i) there exists $R \in (W_\bullet \bar{\otimes}_A T)'$ such that $R(s \otimes t) = \beta(s,t) = \alpha(s)t$. Hence, $(W_\bullet \bar{\otimes}_A T)' = \hom_A(W_\bullet, T')$. T' is the dual of strict inductive limit of T, endowed with strong topology makes it a Frechet space (see [22], p.85 and Prop. 2.3.8). The dual of P i.e.; P' is the inductive limit of sequence of Banach spaces (see [22], p.161). Hence, since P is reflexive, i.e., $P = P''$, we can have $P' = T$ which implies that $P = P'' = T'$. However, since P is flat it implies that T' is flat and injective as a Frechet right A-module in the admissible exact sequence of right Frechet A-module. It follows that T' is a co-retract of $L(A_+, T')$ in the admissible sequence of right Frechet A-module see ([9], p.355) and Prop. 2.3.3. We also have that $\hom_A(W_\bullet, L(A_+, T') \simeq L(W_\bullet, T')$ by Prop. 2.3.2. Therefore by 2.2.1 condition (ii) $L(W_\bullet, T') = (W_\bullet \bar{\otimes}_A T)'$ and as a result of injectivity of T', $(W_\bullet \bar{\otimes}_A T)'$ is exact. By Lemma 2.3.7 $(W_\bullet \bar{\otimes}_A T)$ is exact. Hence, T is flat. □

Theorem 3.1.2: Let A be a complete and barreled topological algebra and P a Frechet space. If P is strictly flat left Frechet A-module, then there exist a strict inductive limit space T that is also strictly flat as a left A-hypomodule.

Proof: Suppose that P is a strictly flat left A-module, $P_\bullet := \{0\} \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_1/P_2 \longrightarrow \{0\}$ is a short exact sequence of right Frechet A-module and $W_\bullet := \{0\} \longrightarrow\longrightarrow W_2 \longrightarrow W_1 \longrightarrow W_1/W_2 \longrightarrow \{0\}$ is a short exact sequence of right inductive limit A-hypomodule. Since P is strictly flat, it is then strictly injective. This implies that T' is also strictly flat since T' is Frechet. Hence, let F be an injective Frechet module and $R : T' \longrightarrow F$ an embedding. We define the following composition and call it g.
$T' \xrightarrow{\alpha} L(A_+, T') \xrightarrow{\beta} L(A_+, F)$. α is an embedding and β is induced by R. Since α and β are 1-to-1, $g = \beta \circ \alpha$ is also 1-to-1. Strict injectivity of T' implies that there exists a left inverse of g i.e., $L(A_+, F) \longrightarrow T'$. Hence, by Proposition (2.3.3) there exists a coretract $L(A_+, F)$ in the short exact sequence of right modules. Hence, $\hom_A(W_\bullet, L(A_+, F) \simeq L(W_\bullet, F)$ is exact, this implies that $(W_\bullet \bar{\otimes}_A F')'$ is exact. Hence, $(W_\bullet \bar{\otimes}_A T)'$ is strictly exact and by Lemma 2.3.7 $W_\bullet \bar{\otimes}_A T$ is strictly exact, therefore T is strictly flat. □

Remark 3.1.3: As inspired by the definition of amenable Frechet algebra by Pirkovskii [19], which goes thus: A Frechet algebra A is amenable if A_+ ($A_+ = A\oplus$) is a flat Frechet A-bimodule. Using theorem 3.1.1 which implies that there exists flat strict inductive limit A-hypomodules. We can conveniently state the following corollary.
Corollary 3.1.4: Let A be a complete and barreled topological algebra. If A is amenable as a Frechet algebra, then A is amenable when considered as a pseudo-complete locally convex algebra.

3.2 Cyclic Flatness and Amenability.

This subsection considers the generalization of theorems 1.1 and 1.2.

Lemma 3.2.1: Let A be a pseudo-complete lca and $S := \left(\lim_{\longrightarrow} S_i, \tau_i^j\right)$ be a strict inductive limit of sequence of Banach spaces S_i. Suppose that S is a right A-hypomodule and also $I \subset A$ is a closed left ideal. Consider the following maps for each i.

$$\hat{\tau}_i : S_i/\overline{S_i \cdot I} \longrightarrow S/\overline{S \cdot I}, \ \ \tau_i(s) + \overline{S_i \cdot I} \longrightarrow s + \overline{S \cdot I}$$

Then, we have a topological isomorphism

$$S/\overline{S \cdot I} \cong \lim_{\longrightarrow} \left(S_i/\overline{S_i \cdot I}, \ \hat{\tau}_i^j\right)$$

where the structure maps $\hat{\tau}_i^j$ $(i > j)$ are defined by

$$\hat{\tau}_i^j : S_j/\overline{S_j \cdot I} \longrightarrow S_i/\overline{S_i \cdot I}$$

Proof: Injectivity of the map $\tau_i : S_i \longrightarrow S$ implies that the map $\hat{\tau}_i$ is also injective. We only need to show that there exists an isometry between $S_i/\overline{S_i \cdot I}$ and $S/\overline{S \cdot I}$ via their quotient topologies. Let the seminorm defining the topology of S be denoted by p_i and the norm topology on S_i be denoted by $\|\cdot\|_i$. Since the inductive limit is strict, hence, the topology defined by the seminorm on $S = \lim_{\longrightarrow} S_i$ coincides with the norm topology on S_i. Likewise, the topology the quotient seminorm p_i' defined on $S/\overline{S \cdot I}$ also coincides with the quotient norm on $S_i/\overline{S_i \cdot I}$. Hence, we have

$$\begin{aligned} p_i'\left(\hat{\tau}_i(s) + \hat{\tau}_i(\overline{S \cdot I})\right) &= \left\{p_i'(\hat{\tau}_i(s) + \hat{\tau}_i(t)) : t \in \overline{S_i \cdot I}\right\} \\ &= \inf\left\{p_i(\hat{\tau}_i(s)) + p_i(\hat{\tau}_i(t)) \cdot t \in \overline{S_i \cdot I}\right\} \\ &= \inf\left\{\|s\|_i + \|t\|_i : t \in S_i \cdot I\right\} \\ &= \inf\left\{\|s\|_i + \|\overline{s_i \cdot I}\|_i\right\} = (\|s + \overline{S_i \cdot I}\|) \end{aligned}$$

where p_i and $\|\cdot\|_i$ are the seminorm and norm on S and S_i respectively. □

Remark 3.2.2: (i) By Proposition 3.1 of [19], we can have the multiplication replaced by separately continuous one, where the topological isomorphism remains $S\bar{\otimes}_A(A_+/I)^\sim \cong (S/\overline{S \cdot I})^\sim$ which is uniquely determined by
$s \otimes (a + I) \longmapsto s \cdot a + \overline{S \cdot I}$ where A is a pseudo-complete lca and $S = \lim_{\longrightarrow} S_i$ for sequence of Banach spaces S_i.
(ii) Proposition 4.1 of [19] also remains valid for pseudo-complete lca.

Proposition 3.2.3: Let A be a pseudo-complete lca. Then A with hypocontinuous multiplication is m-convex.

Proof: The multiplication we have given to pseudo-complete lca so far is hypocontinuous multiplication. We shall show that algebra A can be said to be m-convex. Given bounded sets $B_1, B_2 \subset A$ and a 0-neighbourhood $U \subset A$, we may find 0-neighbourhoods $V_1 \subset A$ and $V_2 \subset A$ such that $V_1B_2 \subset U$ and $V_2B_1 \subset U$ and $\lambda > 0$ such that $B_1 \subset \lambda V_1$ and $B_2 \subset \lambda V_2$ respectively. Then $B_1B_2 \subset \lambda U$ and $B_2B_1 \subset \lambda U$. Hence, $B_1B_2 \cdot B_2B_1 \subset \lambda V_1B_2 \cdot \lambda V_2B_1 \subset \lambda U \cdot \lambda U$ which implies $\lambda^2 V_1B_2 \cdot V_2B_1 \subset \lambda^2 U^2$. Therefore $V_1B_2 \cdot V_2B_1 \subset U^2$. Hence, pseudo-complete lca with hypocontinuous multiplication is m-convex. □

Theorem 3.2.4: Let A be a pseudo-complete lca and let $0 \longrightarrow S_1 \xrightarrow{R} S_2 \xrightarrow{Q} S_3 \longrightarrow 0$ be an exact sequence of A-hypomodules, that is, $\text{coker}R = Q$. Let T be an A-hypomodule, then the sequence

$0 \longrightarrow S_1\bar{\otimes}_A T \stackrel{R\otimes 1_T}{\longrightarrow} S_2\bar{\otimes}_A T \stackrel{Q\otimes 1_T}{\longrightarrow} S_3\bar{\otimes}_A T \longrightarrow 0$, is an exact sequence of locally convex spaces.

Proof: We first show that $0 \longrightarrow (S_3\bar{\otimes}_A T)' \longrightarrow (S_2\bar{\otimes}_A T)' \longrightarrow (S_1\bar{\otimes}_A T)' \longrightarrow 0$ is exact. We have that $0 \longrightarrow Bil_A(S_3 \times T,) \longrightarrow Bil_A(S_2 \times T,) \longrightarrow Bil_A(S_1 \times T,) \longrightarrow 0$ is isomorphic to $0 \longrightarrow (S_3\bar{\otimes}_A T)' \longrightarrow (S_2\bar{\otimes}_A T)' \longrightarrow (S_1\bar{\otimes}_A T)' \longrightarrow 0$ by Definition 2.2.1(i). By ([19], Prop. 3.5) and ([22], sec.4) $0 \longrightarrow Bil_A(S_3 \times T,) \longrightarrow Bil_A(S_2 \times T,) \longrightarrow Bil_A(S_1 \times T,) \longrightarrow 0$ is exact which implies that $0 \longrightarrow (S_3\bar{\otimes}_A T) \longrightarrow (S_2\bar{\otimes}_A T)' \longrightarrow (S_1\bar{\otimes}_A T)' \longrightarrow 0$ is exact. By Lemma 2.3.7 $0 \longrightarrow (S_1\bar{\otimes}_A T) \longrightarrow (S_2\bar{\otimes}_A T) \longrightarrow S_3\bar{\otimes}_A T \longrightarrow 0$ is exact. □

Theorem 3.2.5: Let A be a pseudo-complete *lca* and $I \subset A_+$ a closed left ideal. Set $S = (A_+/I)^{\sim}$. Then, the following conditions are equivalent.

(i) I has a right *lbai*
(ii) given a short exact sequence $0 \longrightarrow T_1 \longrightarrow T_2 \longrightarrow T_3 \longrightarrow 0$ of right A-hypomodule, the sequence $0 \longrightarrow T_1\bar{\otimes}_A S \longrightarrow T_2\bar{\otimes}_A S \longrightarrow T_3\bar{\otimes}_A S \longrightarrow 0$ is exact.
(iii) given a short exact sequence $0 \longrightarrow T_1 \longrightarrow T_2 \longrightarrow T_3 \longrightarrow 0$ of right Banach A-module, the sequence $0 \longrightarrow T_1\bar{\otimes}_A S \longrightarrow T_2\bar{\otimes}_A S \longrightarrow T_3\bar{\otimes}_A S \longrightarrow 0$ is exact.

Proof: (i) $\Longrightarrow$ (ii). We only need to show that for each T as an A-hypomodule and each closed sub A-hypomodule $P \subset T$, the map $P\bar{\otimes}_A S \longrightarrow T\bar{\otimes}_A S$ is topologically injective. Let us choose the direct limit $T = \lim_{\longrightarrow} T_i$ and for each i, let $\bar{P} = \tau_i(P_i)$ where P_i is closed in T_i with $\tau_i : T_i \longrightarrow T$, we then have $P = \lim_{\longrightarrow} P_i$ a direct limit of P_i. Since by Proposition 3.2.3, A is m-convex, then, there exists m-compatible seminorm p_i on T and a continuous seminorm p_j on A such that $P_i(ta) \leq p_i(t)p_j(a)$ for each $t \in T$ and $a \in A$ since the topology is strict p_j coincide with the norm topology on A_i. It is obvious that T_i is a right Banach A_i-module and P_i a closed A_i-submodule of P_i. Consider the canonical map $R_i : A_i \longrightarrow A$ and set $\bar{I}_i = (R_i)_+(I_i)$. It is clear that I_i is a closed left ideal in $(A_i)_+$. By (Thm. 2.1.3), I_i has a right bounded *ai* and so $S_i = (A_i)_+/I_i$ is a strictly flat Banach A_i-module ([19], Theorem 1.1). Therefore, the map $P_i\bar{\otimes}_{A_i} S_i \longrightarrow T_i\bar{\otimes}_{A_i} S_i$ is topologically injective by ([9], p.350).

By the use of

$$\begin{array}{ccc} P_i\bar{\otimes}_{A_i}S_i & \longrightarrow & T_i\bar{\otimes}_{A_i}S_i \\ \| & & \| \\ P_i/\overline{P_i\cdot I_i} & \longrightarrow & T_i/\overline{T_i\cdot I_i} \\ \| & & \| \\ P_i/\overline{P_i\cdot I} & \longrightarrow & T_i/\overline{T_i\cdot I} \end{array}$$

and taking the inductive limit, we get a topologically injective map $\lim_{\longrightarrow} P_i/\overline{P_i\cdot I} \longrightarrow \lim_{\longrightarrow} T_i/\overline{T_i\cdot I}$.

By Lemma 3.2.1 and Remark 3.2.2(i), we have the following result.

$$\begin{array}{ccc} \lim_{\longrightarrow} P_i/\overline{P_i\cdot i} & \longrightarrow & \lim_{\longrightarrow} T_i/\overline{T_i\cdot I} \\ \| & & \| \\ P/\overline{P\cdot I} & \longrightarrow & T/(\overline{T\cdot I}^{\sim} \\ \| & & \cdot\| \\ P\hat{\otimes}_A S & \longrightarrow & T\bar{\otimes}_A S \end{array}$$

This shows (ii).

(ii) $\Longrightarrow$ (iii). Since the topology is strict, the norm topology on the Banach algebra coincides with the strict topology induced on the Banach algebra by the pseudo-complete *lca*. Hence, (iii) is satisfied.

(iii) $\Longrightarrow$ (i). Let $\varphi : B \longrightarrow A$ be a homomorphism of Banach algebra B to A. By Remark 3.2.2(i) $B_+ \otimes_A S \cong B_+/\overline{B_+\cdot I}$ is strictly flat. Then, $\overline{B_+\cdot I}$ of B_+ has right bounded *ai*. Let choose the direct limit $A = \lim_{\longrightarrow} A_i$ with the canonical homomorphism $R_i : A_i \longrightarrow A$ the ideal $I = A_+ \cdot I$ is equal to the closure $\bar{I} = (R_i)_+(\overline{(A_i)_+\cdot I})$ as that $I = \lim_{\longrightarrow} I_i$.

By applying (Thm. 2.1.3) I has a right *lbai*. □

Corollary 3.2.6: Let A be a pseudo-complete *lca* and $I \subset A_+$ a closed left ideal. Set $S = A_+/I$. Then the following conditions are equivalent

(i) I has a right *lca*;

(ii) S is a strictly flat pseudo-complete locally convex A-hypomodule.

Theorem 3.2.7: Let A be a pseudo-complete *lca* and $I \subset A_+$ a closed left ideal. Set $S = (A_+/I)^{\sim}$. Suppose that there exists a strict inductive limit $A = \lim_{\longrightarrow}(A_i\tau_\alpha^i)_{i\in\wedge}$ such that for each $i \in \wedge$ the ideal $I_i = R_i^{-1}(I)$ is weakly complemented in $(A_i)_+$. Then, the following conditions are equivalent.

(i) I has a right *lca*;

(ii) given a short exact sequence $0 \longrightarrow T_1 \longrightarrow T_2 \longrightarrow T_3 \longrightarrow 0$ of right A-hypomodule, the sequence $0 \longrightarrow T_1\bar{\otimes}_A S \longrightarrow T_2\bar{\otimes}_A S \longrightarrow\longrightarrow T_3\bar{\otimes}_A S \longrightarrow 0$ is exact;

(iii) given a short exact sequence $0 \longrightarrow T_1 \longrightarrow T_2 \longrightarrow T_3 \longrightarrow 0$ of right Banach A-module the sequence $0 \longrightarrow T_1\bar{\otimes}_A S \longrightarrow T_2\bar{\otimes}_A S \longrightarrow T_3\bar{\otimes}_A S$ is exact;

(iv) given an admissible short exact sequence $0 \longrightarrow T_1 \longrightarrow T_2 \longrightarrow T_3 \longrightarrow 0$ of right A-hypomodule, the sequence $0 \longrightarrow T_1\bar{\otimes}_A S \longrightarrow T_2\bar{\otimes}_A S \longrightarrow T_3\bar{\otimes}_A S$ is exact;

(v) given an admissible short exact sequence $0 \longrightarrow T_1 \longrightarrow T_2 \longrightarrow T_3 \longrightarrow 0$ of right Banach A-module the sequence $0 \longrightarrow T_1\bar{\otimes}_A S \longrightarrow T_2\bar{\otimes}_A S \longrightarrow T_3\bar{\otimes}_A S$ is exact.

Proof: (i) $\Longleftrightarrow$ (ii) $\Longleftrightarrow$ (iii); this is Theorem 3.2.5.
(ii) $\Longrightarrow$ (iv) $\Longrightarrow$ (v): this is clear as a result of coincidence of the norm topology and strict inductive limit topology.
(v) $\Longrightarrow$ (i): This proof is similar to the proof of implication (iii) $\Longrightarrow$ (i) in Theorem 3.2.5. The equivalence of (i) and (iii) in ([17], Thm 1.1) is used. □

Corollary 3.2.8: Let A be a pseudo-complete *lca* and $I \subset A_+$ a closed left ideal. Set $S = A_+/I$. Suppose that there exists a strict inductive limit $A = \lim_{\longrightarrow} A_i \quad i \in$, the ideal $I_i = \overline{f_i^{-1}(I)}$ is weakly complemented in $(A_i)_+$. Then the following conditions are equivalent.

(i) I has a right *lbai*;
(ii) S is a strictly flat strict inductive limit A-hypomodule;
(iii) S is a flat strict inductive limit A-hypomodule

The following notations will be convenient and are recalled from ([24], p.164). By our assumption A is a barreled DF-space, so is $A_0^e = A\bar{\otimes}A^{op}$. Then, $A^e = A_+\bar{\otimes}A_+^{op}$ is the enveloping algebra of A. We define the product map $\pi : A^e \longrightarrow A_+$ and $I^{\triangle} = \ker\pi = \{(a_1a_2, b_2b_1) \longrightarrow (a_1a_2)\cdot(b_2b_1) : A_+\bar{\otimes}A_+^{op} \longrightarrow A_+\}$ a closed left ideal of A^e. A_+ behaves as a left A^e-hypomodule, since it is injective, then it is isomorphic to $A^e/I^{\triangle}$. This implies that $I^{\triangle}$ is complemented in A^e. Moreover, from $A_0^e = A\bar{\otimes}A^{op}$ we define a product map $\pi_0 : A_0^e \longrightarrow A$ and set $I_0^{\triangle} = \ker\pi_0$. We define $M = A_+\otimes A + A\bar{\otimes}A_+$ as $M = \ker(\epsilon_A \otimes \epsilon_A) \subset A^e$, $\epsilon_A \otimes \epsilon_A$;
Given that $A = \lim_{\longrightarrow} A_i \quad i \in \wedge$.
The following symbols are employed in the Banach algebra setting A_i^e, $(A_0^e)_i$, π_i, $(\pi_0)_i$, $I_i^{\triangle}$, $(I_0^{\triangle})_i$ and M_i.

So also for each $i > j$, we have

$$\begin{aligned} \tau_i^j &= (R_i^j)_+ \otimes (R_i^j)_+ : A_j^e \longrightarrow A_i^e \\ \tau_i &= (R_i)_+ \otimes (R_i)_+ : A_i^e \longrightarrow A^e \end{aligned}$$

Lemma 3.2.9: For each $i \in \wedge$ we have

$$I_0^{\triangle} = \overline{\tau_\lambda^{-1}(I^{\triangle})}, \quad M_\lambda = \tau_\lambda^{-1}(M), \tag{I}$$

and there is an inductive limit

$$A^e \cong \lim_{\longrightarrow} A_i^e, \quad A_0^e \cong \lim_{\longrightarrow}(A_0^e)_i, \tag{II}$$

$$I^{\triangle} \cong \lim_{\longrightarrow} I_i^{\triangle}, \quad M \cong \lim_{\longrightarrow} M_i \tag{III}$$

where the linking maps are injective maps τ_i^j of the sub-algebras of A_i^e. If, in addition A has a one-sided *lbai* then

$$(I_0^{\triangle})_i = f_i^{-1} \otimes f_i^{-1}(I_0^{\triangle}) \tag{IV}$$

and there exists an inductive limit

$$I_0^{\triangle} \cong \lim_{\longrightarrow}(I_0^{\triangle})_i \tag{V}$$

Proof: From the fact that $M_i = (A_\lambda)_+ \otimes A_+ + A_+ \otimes (A_i)_+$, and since $(f_i) : A_i \longrightarrow A$ is injective i.e., there exists M in A such that $f^{-1}(M) = M_i$ in A_i, ([14], p.114) the second formula in I is satisfied. The same thing goes for the first formula, if $I_i^{\triangle}$ is the smallest closed left ideal of A_i^e containing all elements of the form $1_+ \otimes a - a \otimes 1_+ \ (a \in (A_i)_+)$. Let A has a one sided *lbai*. By (Thm. 2.1.3), each A_i has one-sided *bai* and so the product map $A_i \bar{\otimes}_{A_i} A_i \longrightarrow A_i$ is an isomorphism ([19], Lemma 9.3). Therefore, $(I_0^{\triangle})_i$ coincides with the kernel of the map $\hat{\pi}_i : A_i \bar{\otimes} A_i \longrightarrow A_i \bar{\otimes}_{A_i} A_i \quad a \otimes b \longmapsto a \otimes b$. By ([19], Lemma 9.3) $\ker\hat{\pi}_i$ is equal to the closure of the linear span of all elements of the form $ab \otimes c - a \otimes bc \quad (a, b, c \in A_i)$. Since $f_i : A_i \longrightarrow A$ is injective and $\ker\hat{\pi}_i$ exists in $A_i \bar{\otimes} A_i$ and also the topology is a strict inductive topology, (IV) holds.
By the properties of inductive tensor product (Def. 2.2.1 iv), the isomorphism (II) follows.
Consequently, (III) and (V) follow from (I), (II) and (IV). □

Theorem 3.2.10: Let A be a pseudo-complete *lca* with *lbai*. Then A is amenable if and only if A is biflat.

Proof: Let $A = \lim_{\longrightarrow} A_i$ be a pseudo-complete *lca* and let A have a *lbai*, by theorem 2.1.3, each A_i has a *bai*. By ([19], Lemma 9.4), the ideal M_i for each $i \in$ has a bounded *ai*. By using Lemma 3.2.9

M has a *lbai*. By Corollary 3.2.8 $= A^e/M$ is a flat strict inductive limit A^e-hypomodule. We finally apply ([19], Prop. 4.2) and Theorem 3.1.1 to the sequence $0 \longrightarrow A \longrightarrow A_i \longrightarrow\longrightarrow 0$ of A^e-hypomodules to get the result. □

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