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3-STEP BLOCK HYBRID LINEAR MULTISTEP METHODS FOR SOLUTION OF SPECIAL SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS

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ABSTRACT. This paper presents a set of two (2) Implicit Hybrid Block Methods which are derived through multistep collocation method using power series as a basis function for generating solution of special second order ordinary differential equations. The derived continuous forms which are evaluated at some grids and off-grid points of collocation and interpolation to form the block hybrid methods for step number $k = 3$. The discrete schemes obtained possess uniformly high order and found to be zero-stable, consistent and hence convergent. Some numerical examples are given to test the accuracy and efficiency advantages. The results of our evaluation show that our methods outperform reviewed work.

Keywords and phrases: Linear multistep method (LMM), Hybrid, Block, Implicit K-step, Convergence and Error Constant.
2020 Mathematical Subject Classification: A80

1. INTRODUCTION

Numerical methods are becoming more important in applications in engineering, science and social science due to the difficulties experienced in obtaining the analytical solution. Consider the special second order ordinary differential equation of the type

$$y'' = f(x, y), \qquad y(a) = y_0, \quad y'(a) = y'_0, \tag{1}$$

Numerical methods need to be developed for such problem and many researchers have worked extensively in this area see [1], [2], [3], and [4] to mention but few. The main aim of this research paper is to develop hybrid block linear multistep method when $k = 3$ with one off-grid point at both collocation and interpolation which can be used to solve special second order initial value problems of ordinary differential equations.

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Definition 1.1: Hybrid Method
Hybrid method is as a result of the desire to increase the order without increasing the step number and without reducing the stability interval. Therefore, a k-step hybrid method is defined as:

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h^2[\sum_{j=0}^{k} \beta_j f_{n+j} + \beta_v f_{n+v}] \tag{2}$$

where $\alpha_j = 1$, just to remove arbitrariness, α_0 and β_0 are not both zero and $v \notin [0, 1, 2, \ldots, k]$, $f_{n+v} = f(x_{n+v}, y_{n+v})$ which is the off grid function of evaluation, [1].

Definition 1.2: Linear Multistep Method (LMM)
Linear multistep method is the computational method which is used to determine the sequence $[y_n]$ and it is a linear relationship between y_{n+j} and f_{n+j}, $j = 0, 1, 2, \ldots, k$, see [1].
A linear $k-$ step method of order two is mathematically expressed as;

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h^2 \sum_{j=0}^{k} \beta_j f_{n+j} \tag{3}$$

where both α_k, $\beta_k \neq 0$, $\beta_k = 0$ is an explicit scheme and $\beta_k \neq 0$ is an implicit scheme. see [1]

Definition 1.3: Order and Error Constant
The linear multistep method of type (2) is said to be of order p if $C_0 = C_1 = \ldots C_{p+1} = 0$, but $C_{p+2} \neq 0$ and C_{p+2} is called the error constant, see [1].

Definition 1.4: Convergence
The necessary and sufficient conditions for linear multistep method of type (2) is said to be convergent if and only if it is consistent and zero-stable.

Definition 1.5: Stability Regions
The stability region of linear multistep method is part of the complex plane where the method when applied to the test equation $y'' = \lambda^2 y$ is absolutely stable whose resultant finite difference equation has characteristic equation $\pi(z, r) = \rho(r) - z^2\sigma(r)$, $z = i\lambda h$, see [5].

2. DERIVATION OF THE METHODS

The methods are derived for the special second order ordinary differential equation based on the multistep collocation. The general power series is used as basis function to produce an appropriate solution to (3) as follows

$$y(x) = \sum_{j=0}^{t+m-1} \alpha_j x^j \tag{4}$$

and

$$y''(x) = \sum_{j=2}^{t+m-1} j(j-1)\alpha_j x^{j-2} \tag{5}$$

where α_js are the parameters to be determined, t and m are the points of interpolation and collocation respectively. This process leads to $(t+m-1)$ nonlinear system of equations with $(t+m-1)$ unknown coefficients, which in this case are to be determined by the use of Maple 13 mathematical software.

2.1 Derivation of the Hybrid Block Method when $k = 3$ with one Off-grid Point at Collocation

Using equations (4) and (5) with $t = 2$,$m = 5$. The degree of our polynomial is $(t+m-1)$. Equation (4) is interpolated at $x = x_{n+j}$, $j = 0, 1$ and (5) collocated at $x = x_{n+j}$, $j = 0, 1, \frac{6}{5}, 2, 3$ which gives the following non-linear system of equation of the form:

$$\sum_{j=0}^{t+m-1} \alpha_j x_{n+i}^j = y_{n+i}, \qquad i = 0, 1 \tag{6}$$

$$\sum_{j=2}^{t+m-1} j(j-1)\alpha_j x_{n+i}^{(j-2)} = f_{n+i}, \qquad i = 0, 1, \frac{6}{5}, 2, 3 \tag{7}$$

With maple 13 software, we obtain the continuous formulation of equations (6) and (7) as follows

$$y(x) = \left(\frac{x_{n+1}}{h} - \frac{x}{h}\right) y_n + \left(\frac{-x_{n+1}+h}{h} + \frac{x}{h}\right) y_{n+1}$$

$$-\frac{1}{2160}\frac{(-x_n+1+h)x_n n + 1(13h^4 = 143h^3 x_{n+1} + 123h^2 x_{n+1}^2 + 58x_{n+1}^3 + 10x_{n+1}^4)}{h^4}$$

$$+\frac{1}{2160}\left(143h^5 - 60h^3 x_{n+1}^2 - 260h^2 x_{n+1}^3 - 240hx_{n+1}^4 - 60x_{n+1}^5\right)x$$

$$+\frac{1}{72}\frac{x_{n+1}(2h^3 + 13h^2 x_{n+1} + 16hx_{n+1}^2 + 5x_{n+1}^3)}{h^4}x^2$$

$$-\frac{1}{108}\frac{(h^3 + 13h^2 x_{n+1} + 24hx_{n+1}^2 + 10x_{n+1}^3)}{h^4}x^3$$

$$+\frac{1}{432}\frac{(13h^2 + 48hx_{n+1} + 30x_{n+1}^2)}{h^4}x^4 - \frac{1}{180}\frac{(4h + 5x_{n+1})}{h^4}x^5 + \frac{1}{216}\frac{x^6}{h^4}]f_n$$

$$
\begin{aligned}
&+[-\frac{1}{120}\frac{(-x_{n+1}+h)x_{n+1}(142h^4+82h^3x_{n+1}+28h^2x_{n+1}^2-43hx_{n+1}^3-10x_{n+1}^4)}{h^4}\\
&+\frac{1}{120}\frac{(142h^5-120hx_{n+1}^4-330h^3x_{n+1}^2-60h^2x_{n+1}^3+165hx_{n+1}^4+60x_{n+1}^5)}{h^4}x\\
&\frac{1}{4}\frac{(-x_{n+1}+h)(2h^4+13h^2x_{n+1}+16hx_{n+1}^2+5x_{n+1}^3)}{h^4}x^2\\
&-\frac{1}{12}\frac{(11h^3+6h^2x_{n+1}-33hx_{n+1}^2-20x_{n+1}^3)}{h^4}x^3\\
&+\frac{1}{8}\frac{(h^2-11hx_{n+1}-10x_{n+1}^2)}{h^4}x^4+\frac{1}{40}\frac{(11h+20x_{n+1})x^5}{h^4}-\frac{1}{12}\frac{x^6}{h^4}]f_{n+1}\\
&+[-\frac{1}{480}\frac{(-x_{n+1})x_{n+1}(37h^4+37h^3x_{n+1}+17h^2x_{n+1}^2+28hx_{n+1}^3-10x_{n+1}^4)}{h^4}\\
&+\frac{1}{450}\frac{37h^5-60h^3x_{n+1}^2-180hx_{n+1}^2+90hx_{n+1}^4+60x_{n+1}^5}{h^4}x\\
&-\frac{1}{108}\frac{(-x_{n+1}+h)x_{n+1}(h^2+6hx_{n+1}+5x_{n+1}^2)}{h^4}x^2 \qquad (8)\\
&-\frac{1}{24}\frac{(h^3+9h^2x_{n+1}-10x_{n+1}^3)}{h^4}x^3+\frac{1}{32}\frac{(3h^2-6hx_{n+1}-10x_{n+1}^2)}{h^4}x^4\\
&+\frac{1}{80}\frac{(3h+10x_{n+1})}{h^4}x^5-\frac{1}{48}\frac{x^6}{h^4}]f_{n+2}\\
&+[\frac{1}{3240}\frac{(-x_{n+1}+h)x_{n+1}(22h^4+22h^3x_{n+1}+12h^2x_{n+1}^2-13hx_{n+1}^3-10x_{n+1}^4)}{h^4}\\
&-\frac{1}{3240}\frac{(22h^5-30h^3x_{n+1}^2-100h^2x_{n+1}^3+15hx_{n+1}^4+60x_{n+1}^5)}{h^4}x\\
&-\frac{1}{108}\frac{(-x_{n+1}+h)x_{n+1}(h^2+6hx_{n+1}+5x_{n+1}^2)}{h^4}x^2\\
&+\frac{1}{324}\frac{(h^3+10h^2x_{n+1}-3hx_{n+1}^2-20x_{n+1}^3)}{h^4}x^3\\
&-\frac{1}{648}\frac{(5h^2-3hx_{n+1}-30x_{n+1}^2)}{h^4}x^4-\frac{1}{1080}\frac{(h+20x_{n+1})}{h^4}x^5\\
&+\frac{1}{324}\frac{x^6}{h^4}]f_{n+3}\\
&+[\frac{125}{2592}\frac{(-x_{n+1}+h)x_{n+1}(17h^4+17hx_{n+1}^2-3h^2x_{n+1}^3-2x_{n+1}^4)}{h^4}\\
&-\frac{125}{2592}\frac{(17h^5-60h^3x_{n+1}^2-20h^2x_{n+1}^3+30hx_{n+1}^4+12x_{n+1}^5)}{h^4}x\\
&-\frac{625}{432}\frac{(-x_{n+1}+h)x_{n+1}(20h^2+3hx_{n+1}+x_{n+1}^2)}{h^4}x^2
\end{aligned}
$$

$$+\frac{625}{648}\frac{h^3+h^2x_{n+1}-3hx_{n+1}^2-2x_{n+1}^3}{h^4}x^3$$

$$-\frac{625}{2592}\frac{(h^2-6hx_{n+1}-6x_{n+1}^2)}{h^4}x^4-\frac{125}{432}\frac{(h+2x_{n+1})}{h^4}x^5$$

$$+\frac{125}{1296}\frac{x^6}{h^4}]f_{n+\frac{6}{5}}$$

when equation (8) is evaluated at $x=x_{n+j}$, where $j=\frac{6}{5},2,3$ and its first derivative also evaluated at $x=x_n$, we obtain the following set of discrete schemes which forms the first hybrid block method when $k=3$ with one off-grid point at collocation.

$$y_{n+3}-3y_{n+1}+2y_n=\frac{h^2}{2}\left[\frac{1}{4}f_n+5f_{n+1}-\frac{125}{72}f_{n+\frac{6}{5}}+\frac{19}{8}f_{n+2}+\frac{1}{9}f_{n+3}\right]$$

$$y_{n+2}-2y_{n+1}+y_n=\frac{h^2}{12}\left[\frac{5}{6}f_n+13f_{n+1}-\frac{125}{36}f_{n+\frac{6}{5}}+\frac{7}{4}f_{n+2}-\frac{1}{9}f_{n+3}\right]$$

$$y_{n+\frac{6}{5}}-\frac{6}{5}y_{n+1}+\frac{1}{5}y_n=\frac{h^2}{500}\left[\frac{1651}{250}f_n+\frac{15601}{125}f_{n+1}-\frac{2821}{36}f_{n+\frac{6}{5}}\right. \tag{9}$$

$$\left.+\frac{3811}{500}f_{n+2}-\frac{757}{1125}f_{n+3}\right]$$

$$hz_0-y_{n+1}+y_n=\frac{h^2}{12}\left[-\frac{53}{20}f_n-\frac{143}{10}f_{n+1}-\frac{875}{72}f_{n+\frac{6}{5}}+\frac{53}{40}f_{n+2}+\frac{11}{90}f_{n+3}\right]$$

Equation (9) has uniform order five (5) with error constant: $\left(\frac{1}{750},\frac{11}{6000},\frac{28373}{93750000},\frac{617}{252000}\right)^T$

2.2 Derivation of the Second Hybrid Block Method when $k=3$ with one Off-grid Point at Interpolation

Equation (6) is interpolated at $x=x_{n+j}$, $j=0,1,\frac{6}{5}$ and equation (7) collocated at $x=x_{n+j}$, $j=0,1,2,3$ which gives the system of non-linear equations of the form:

$$\sum_{j=0}^{t+m-1}\alpha_j x_{n+u}^j=y_{n+u},\qquad u=0,1,\frac{6}{5} \tag{10}$$

$$\sum_{j=2}^{t+m-1}j(j-1)\alpha_j x_{n+v}^{(j-2)}=f_{n+v},\qquad v=0,1,2,3 \tag{11}$$

Adopting the previous procedure in the first block to generate the continuous formula and this continuous formula is evaluated at $x=x_{n+j}$, $j=2,3$. Its second derivative is evaluated at $x=x_{n+\frac{6}{5}}$

and the first derivative evaluated at $x = x_n$ which gives the second hybrid block method when $k = 3$ with one off-grid point at interpolation as follows:

$$
\begin{aligned}
&y_{n+3} - \frac{15625}{2821} y_{n+\frac{6}{5}} + \frac{10287}{2821} y_{n+1} + \frac{2517}{2821} y_n \\
&\quad = \frac{h^2}{62}\left[\frac{45}{14} f_n + \frac{12609}{182} f_{n+1} + \frac{12447}{182} f_{n+2} + \frac{711}{182} f_{n+3}\right] \\
&y_{n+2} - \frac{15625}{8463} y_{n+\frac{6}{5}} + \frac{608}{2821} y_{n+1} + \frac{5338}{8463} y_n \\
&\quad = \frac{h^2}{7}\left[\frac{88}{279} f_n + \frac{1756}{403} f_{n+1} + \frac{332}{403} f_{n+2} - \frac{172}{3627} f_{n+3}\right] \\
&\frac{18000}{2821} y_{n+\frac{6}{5}} - \frac{21600}{2821} y_{n+1} + \frac{3600}{2821} y_n \\
&= \frac{h^2}{875}\left[\frac{2286}{31} f_n + \frac{561636}{403} f_{n+1} - 875 f_{n+\frac{6}{5}} + \frac{34299}{403} f_{n+2} \right. \\
&\qquad \left. - \frac{3028}{403} f_{n+3}\right] \\
&hz_0 + \frac{15625}{2418} y_{n+\frac{6}{5}} - \frac{3528}{403} y_{n+1} + \frac{5543}{2418} y_n = \frac{h^2}{5}\left[-\frac{21}{31} f_n \right. \\
&\qquad \left. + \frac{849}{403} f_{n+1} - \frac{24}{403} f_{n+2} - \frac{3}{403} f_{n+3}\right]
\end{aligned}
\tag{12}
$$

Equation (12) has uniform order five (5), with error constants $\left(\frac{387}{1128400}, \frac{809}{634725}, \frac{85119}{44078125}, \frac{139}{282100}\right)^T$

3. ANALYSIS OF NUMERICAL METHODS

3.1 Convergence Analysis

The convergence analysis of all the block hybrid methods is done, using [6] approach with the block method presented as a single block $r-$point multistep method as follows

$$
A^{(0)} Y_m = \sum_{i=1}^{k} A^{(i)} Y_{m-1} + h^2 \sum_{i=0}^{k} B^{(i)} F_{m-i} \tag{13}
$$

where h is a fixed mesh size within a block $A^{(i)}$, $B^{(i)}$, $i = 0(1)k$ are $r \times r$ matrix coefficients and $A^{(0)}$ is $r \times r$ identity matrix, Y_m, Y_{m-1}, F_m and F_{m-1} are vectors of numerical estimates.

The method in (9) is presented in matrix form as

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ \frac{6}{5} & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ -3 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{n+1} \\ y_{n+\frac{6}{5}} \\ y_{n+2} \\ y_{n+3} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -\frac{1}{5} \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} y_{n-2} \\ y_{n-\frac{9}{5}} \\ y_{n-1} \\ y_n \end{pmatrix}$$
$$+ h^2 \begin{pmatrix} -\frac{143}{120} & \frac{875}{864} & -\frac{53}{480} & \frac{11}{1080} \\ \frac{15601}{62500} & -\frac{2821}{18000} & \frac{3811}{250000} & -\frac{757}{562500} \\ \frac{13}{12} & -\frac{125}{432} & \frac{7}{48} & -\frac{1}{108} \\ \frac{5}{2} & -\frac{125}{144} & \frac{19}{16} & \frac{1}{8} \end{pmatrix} \begin{pmatrix} f_{n+1} \\ f_{n+\frac{6}{5}} \\ f_{n+2} \\ f_{n+3} \end{pmatrix}$$
$$+ h^2 \begin{pmatrix} 0 & 0 & 0 & -\frac{53}{240} \\ 0 & 0 & 0 & -\frac{1651}{125000} \\ 0 & 0 & 0 & \frac{5}{72} \\ 0 & 0 & 0 & \frac{1}{8} \end{pmatrix} \begin{pmatrix} f_{n-2} \\ f_{n-\frac{9}{5}} \\ f_{n-1} \\ f_n \end{pmatrix} \tag{14}$$

We normalize the above block method (14) by multiplying matrices $A^{(0)}$, $A^{(1)}$, $B^{(0)}$, and $B^{(1)}$ with inverse of $A^{(0)}$ to obtain the below method as follow;

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{n+1} \\ y_{n+\frac{6}{5}} \\ y_{n+2} \\ y_{n+3} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{n-2} \\ y_{n-\frac{9}{5}} \\ y_{n-1} \\ y_n \end{pmatrix}$$
$$+ h^2 \begin{pmatrix} -\frac{143}{120} & -\frac{875}{864} & \frac{53}{480} & -\frac{11}{1080} \\ \frac{26244}{15625} & -\frac{343}{250} & \frac{4617}{31250} & -\frac{212}{15625} \\ \frac{52}{15} & -\frac{125}{54} & \frac{11}{30} & -\frac{4}{135} \\ \frac{243}{40} & -\frac{125}{32} & \frac{243}{160} & \frac{1}{40} \end{pmatrix} \begin{pmatrix} f_{n+1} \\ f_{n+\frac{6}{5}} \\ f_{n+2} \\ f_{n+3} \end{pmatrix}$$
$$+ h^2 \begin{pmatrix} 0 & 0 & 0 & -\frac{53}{240} \\ 0 & 0 & 0 & \frac{4347}{15625} \\ 0 & 0 & 0 & \frac{23}{45} \\ 0 & 0 & 0 & \frac{63}{80} \end{pmatrix} \begin{pmatrix} f_{n-2} \\ f_{n-\frac{9}{5}} \\ f_{n-1} \\ f_n \end{pmatrix} \tag{15}$$

The block method (15) is the normalized form of the above schemes for the block hybrid method (9). The first characteristics polynomial of the block method (9) is defined as:

$$\rho(\lambda) = det\left[\lambda I - A_1^{(1)}\right] \tag{16}$$

substituting for λI and $A_1^{(1)}$ in (16) above, gives

$$\left| \lambda \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right| = \begin{pmatrix} \lambda & 0 & 0 & -1 \\ 0 & \lambda & 0 & -1 \\ 0 & 0 & \lambda & -1 \\ 0 & 0 & 0 & \lambda - 1 \end{pmatrix}$$

solving the above determinant yield the following solution

$$\rho(\lambda) = \lambda^3(\lambda - 1) = 0 \tag{17}$$

since $\lambda_1 = \lambda_2 = \lambda_3 = 0$, $\lambda_4 = 1$. That is to say the block method is zero stable and consistent and it is of order $(5, 5, 5, 5)^T > 1$, as stated by [6] and [7]. Hence, the block method is convergent. The same analysis holds for the second block method (12). Thus, they are zero-stable, consistent and convergent.

3.2 Region of Absolute Stability (RAS)

The regions of absolute stability of all the block methods derived are determine by reformulating them as general linear method (see [8]) expressed as follows

$$\begin{pmatrix} Y \\ Y_{i+1} \end{pmatrix} = \begin{pmatrix} A & U \\ B & V \end{pmatrix} \begin{pmatrix} h^2 f(Y) \\ Y_{i-1} \end{pmatrix} \qquad i = 1, 2, \ldots, N \tag{18}$$

Applying (17) to the test equation $y'' = \lambda^2 y$ leads to a recursion formula

$$M(z) = V + zB(1 - zA)^{-1}U \tag{19}$$

where $z = \lambda h$, equation (18) is the stability matrix and the stability function is

$$\rho(\eta, z) = det[\eta I - M(z)] \tag{20}$$

Computing the stability function, we obtain the stability polynomial of the methods which is plotted to produce the required absolute stability region of the method. To plot the absolute stability region of equation (9) is expressed in the form of equation (18) and the values of the matrices A, B, U and V are substituted into equations (19) and (20), with the aid of maple software, to obtain the characteristics polynomial and the stability function.

These values of the stability function and characteristics polynomial are used in matlab programme to obtain the region of absolute stability as shown in figure 1.

The same analysis holds for the second block methods (12), as shown in figure 2 below.

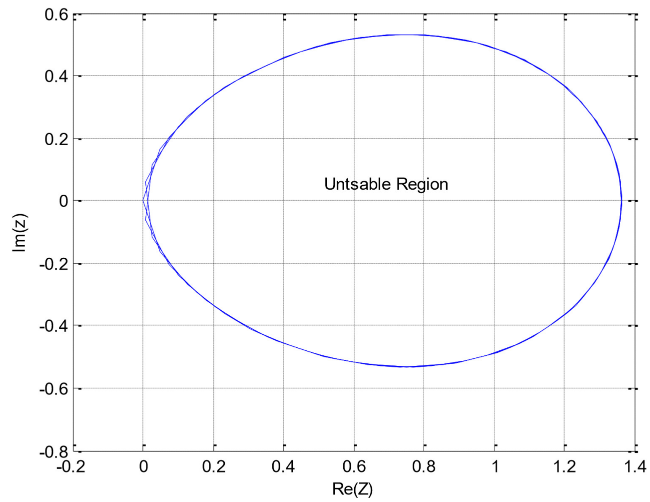

FIGURE 1. Region of absolute stability when $k = 3$ with one off-grid point at collocation and the block method is A-stable

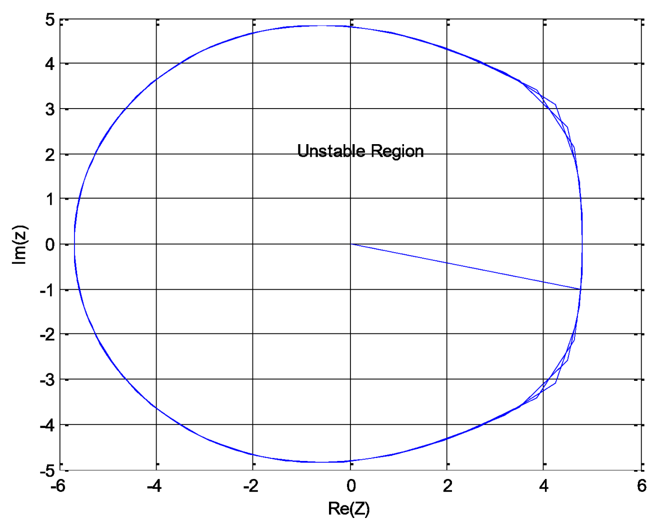

FIGURE 2. The block method is $A(\alpha)$ - stable

4. NUMERICAL EXPERIMENTS

This section deals with numerical experiment by considering the derived discrete schemes in block form for solution of stiff and non-stiff initial value problems of second order ordinary differential equation for case when $k = 3$.

Problem 1

Consider the problem solved by [9]

$$y'' = -y, \qquad y(0) = 1, y'(0) = 1, \qquad h = 0.1, \quad 0.1 \leq x \leq 0.4$$

Exact Solution:

$$y(x) = \sin(x) + \cos(x)$$

Problem 2

We consider the stiff differential equation

$$y'' = 2y^3, \qquad y(1) = 1, y'(1) = -1, \qquad h = 0.1, \quad 0.1 \leq x \leq 0.4$$

Exact Solution:

$$y(x) = \frac{1}{x}$$

Table 1. Comparison of Errors when $k = 3$ at Collocation for Problem 1

x	Exact Solution	Error from [9]	Error of Proposed Method
0.1	1.094837582	1.47 e(-07)	1.0 e(-09)
0.2	1.178735909	1.99 e(-07)	3.0 e(-09)
0.3	1.250856696	4 e(-09)	1.0 e(-09)
0.4	1.310479336	4.6 e(-08)	9.0e(-08)

In Table 1, the results produced from the proposed method for solving problem 1 has outperformed the results of [9] when compared.

Table 2. Comparison of Errors when $k = 3$ at Interpolation for Problem 1

x	Exact Solution	Error from [9]	Error of Proposed Method
0.1	1.094837582	1.47 e(-07)	0.0 e(-00)
0.2	1.178735909	1.99 e(-07)	1.0 e(-09)
0.3	1.250856696	4 e(-09)	2.0 e(-09)
0.4	1.310479336	4.6 e(-08)	1.0e(-09)

In Table 2, the results obtained from the proposed method for solving problem 1 has minimum error than Table 1 and outperformed the results of [9] when compared.

Table 3. Results for the derived method (12) at Collocation with One Point Off-Grid when $k = 3$ for Problem 2

x	Exact Solution	Computed Solution	Error of Proposed Method
0.1	0.909090909	0.909091412	5.03 e(-07)
0.2	0.833333333	0.833334723	1.39 e(-06)
0.3	0.769230769	0.769232741	1.972 e(-06)
0.4	0.714285714	0.714482411	1.96697 e(-04)

In Table 3, the result of the computed solution is compared with the exact solution and the new method is accurate with minimum error.

Table 4. Results for the derived method (12) at Interpolation with One Point Off-Grid when $k = 3$ for Problem 2

x	Exact Solution	Computed Solution	Error of Proposed Method
0.1	0.909090909	0.909091410	5.01 e(-07)
0.2	0.833333333	0.833334721	1.388 e(-06)
0.3	0.769230769	0.769232737	1.968 e(-06)
0.4	0.714285714	0.714023971	2.61743 e(-04)

In Table 4, the result of the computed solution is compared with the exact solution and the new method is accurate with minimum error.

5. CONCLUSION

All the derived hybrid block methods developed for the step number when $k = 3$ can be used for the solution of special second order ordinary differential equation of type (1). The derived methods are implemented in block mode which have the advantages of being self-starting, uniformly of order five (5) and do not need predictors. The stability domains of the methods are presented in figures 1 and 2. Maple13 and Matlab 2013 software packages are employed to generate the schemes and results. The examples displayed the superiority of the method over [9].

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