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SHEHU TRANSFORM ADOMIAN DECOMPOSITION METHOD FOR THE SOLUTION OF LINEAR AND NONLINEAR INTEGRAL AND INTEGRO-DIFFERENTIAL EQUATIONS

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ABSTRACT. Shehu transform Adomian decomposition method (STADM) is proposed in this work to handle both linear and nonlinear integral and integro-differential equations.The Shehu transform is applied to both constant and variable coefficient initial value problems, just for the sake of completeness. Since Shehu transform generalizes the earlier two transforms, namely, Laplace and Sumudu transforms, as it conveniently replaces any of them with ease of computation. The so-called Adomian polynomials are adopted to overcome the nonlinearities whenever such is encountered in the problem. The proposed method is applied to selected problems in the literature and the results are comparatively good.

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1. INTRODUCTION

An integral part of analytical methods of solving initial value problems (IVPs) in ordinary differential equations (ODEs), integral equations (IEs) and integro-differential equations (IDEs) is the integral transform methods. Prominent among which are the Laplace and Fourier transforms [6]. Laplace transform was introduced in the late 18th century by Pierre-Simon laplace, while Joseph Fourier introduced Fourier transform in the early 19th century [21].
Many other integral transforms that were later introduced in succession are; Kamal transform [16], Mahgoub transform [17], Mohand transform [18, 19], Aboodh transform [4, 7, 10, 13, 15], Elzaki transform [14, 22], Sumudu transform [12] and lately, the Shehu

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transform [11, 20] which is implemented in this work. It is of interest that Shehu transform generalizes both Laplace transform and Sumudu transform when certain parameters are fixed [21].
Shehu transform, unlike some earlier transforms, is applicable to both constant and variable coeficients IVPs. This and the admissible of Shehu transform combined with the Adomian decomposition method in the solution of nonlinear integro-differential equation constitute the thrust of this research work.

2. THE SHEHU TRANSFORM

In this section, the preliminaries that serve as the basis for the results reported in this paper shall be discussed. It should be noted that the Shehu transform operator is denoted by $\mathbb{S}$

Definition [21]
The Shehu transform of the function $v(x)$ of exponential order is defined over the set of functions,

$$P=\left\{v(x):\exists N,\ \xi_1,\xi_2>0,|v(x)|<Nexp\left(\frac{|x|}{\xi_i}\right),\ \text{if } x\in(-1)^i\left[0,\infty\right)\right\}, \tag{2.1}$$

by the integral

$$\begin{aligned}\mathbb{S}\left[v(x)\right]=V(s,u)&=\int_0^\infty exp\left(\frac{-sx}{u}\right)v(x)dx\\ \mathbb{S}\left[v(x)\right]&=\lim_{\alpha\to\infty}\int_0^\alpha exp\left(\frac{-sx}{u}\right)v(x)dx;\quad s>0,u>0.\end{aligned} \tag{2.2}$$

The integral in (2) converges provided the limit of the integral exists, and diverges otherwise.
The inverse of Shehu transform is given by

$$\mathbb{S}^{-1}\left[V(s,u)\right]=v(x),\ \text{ for } x\geq 0. \tag{2.3}$$

Which can as well be stated as

$$v(x)=\mathbb{S}^{-1}\left[V(s,u)\right]=\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{1}{u}exp\left(\frac{sx}{u}\right)V(s,u)ds, \tag{2.4}$$

where s and u are the Shehu transform variables, and α is a real constant and the integral in (2) is taken along $s=\alpha$ in the complex plane $s=x+iy$.

2.1 APPLICATION OF SHEHU TRANSFORM TO SOME MATHEMATICAL EXPRESSIONS

Since the sufficient condition for the existence of Shehu transform had earlier been undertaken by [21], and Shehu Operator just like any other integral transforms that preceded it, is a linear operator, its linearity is also proved in the same work. Interested reader may check [5, 9, 21]. Therefore, in this section, its application to selected mathematical expression is demonstrated.

Example 1
Obtain the Shehu transform of $v(x) = a$, where a is a constant
Solution

So, when $v(x) = a$, a being a constant, we have:

$$\mathbb{S}\{a\} = \int_0^\infty e^{-\frac{s}{u}x} a dx$$

Evaluating the integral on the right hand side, we have

$$\begin{aligned}\mathbb{S}\{a\} &= -\frac{au}{s}\left[\frac{1}{e^{\frac{s}{u}x}}\right]_0^\infty \\ &= -\frac{au}{s}\left[\frac{1}{e^\infty} - \frac{1}{e^0}\right] \\ &= -\frac{au}{s}(-1) \\ &= \frac{au}{s}\end{aligned}$$

Example 2
Find the Shehu Transform of $sin(ax)$.
Solution

$$\mathbb{S}\{sin(ax)\} = \int_0^\infty e^{-\frac{s}{u}x} sin(ax) \mathrm{d}x$$

Using Euler's equation, we have

$$sin(ax) = \frac{1}{2i}\left(e^{iax} - e^{-iax}\right)$$

We have

$$
\begin{aligned}
\mathbb{S}\left\{sin(ax)\right\} &= \frac{1}{2i}\int_0^\infty e^{-\frac{s}{u}x}\left(e^{iax}-e^{-iax}\right)\mathrm{d}x \\
&= \frac{1}{2i}\left[\int_0^\infty e^{-\left(\frac{s}{u}-ia\right)x}\mathrm{d}x - \int_0^\infty e^{-\left(\frac{s}{u}+ia\right)x}\mathrm{d}x\right] \\
&= \frac{1}{2i}\left[-\frac{1}{\left(\frac{s}{u}-ia\right)}e^{-\left(\frac{s}{u}-ia\right)x} + \frac{1}{\left(\frac{s}{u}+ia\right)}e^{-\left(\frac{s}{u}+ia\right)x}\right]_0^\infty \\
&= \frac{1}{2i}\left[-\frac{1}{\left(\frac{s}{u}-ia\right)e^{\left(\frac{s}{u}-ia\right)x}} + \frac{1}{\left(\frac{s}{u}+ia\right)e^{\left(\frac{s}{u}+ia\right)x}}\right]_0^\infty \\
&= \frac{1}{2i}\left[0-\left(-\frac{1}{\left(\frac{s}{u}-ia\right)}+\frac{1}{\left(\frac{s}{u}+ia\right)}\right)\right] \\
&= \frac{1}{2i}\left[\frac{1}{\left(\frac{s}{u}-ia\right)}-\frac{1}{\left(\frac{s}{u}+ia\right)}\right] \\
&= \frac{1}{2i}\left[\frac{\frac{s}{u}+ia-\frac{s}{u}+ia}{\frac{s^2}{u^2}+a^2}\right] \\
&= \frac{1}{2i}\left[\frac{2ia}{\frac{s^2}{u^2}+a^2}\right] \\
&= \frac{a}{\frac{s^2}{u^2}+a^2} \\
&= \frac{au^2}{s^2+a^2u^2}.
\end{aligned}
$$

Example 3
Find the Shehu Transform of $\cosh(ax)$.
Solution

$$
\mathbb{S}\left\{\cosh(ax)\right\} = \int_0^\infty e^{-\frac{s}{u}x}\cosh(ax)\mathrm{d}x
$$

Using Euler's equation, we have

$$
\cosh(ax) = \frac{1}{2}\left(e^{ax}+e^{-ax}\right)
$$

We have,

$$
\begin{aligned}
\mathbb{S}\left\{\cosh(ax)\right\} &= \frac{1}{2}\int_0^\infty e^{-\frac{s}{u}x}\left(e^{ax}+e^{-ax}\right)\mathrm{d}x\\
&= \frac{1}{2}\left[\int_0^\infty e^{-\left(\frac{s}{u}-a\right)x}\mathrm{d}x + \int_0^\infty e^{-\left(\frac{s}{u}+a\right)x}\mathrm{d}x\right]\\
&= \frac{1}{2}\left[-\frac{1}{\frac{s}{u}-a}e^{-\left(\frac{s}{u}-a\right)x} - \frac{1}{\frac{s}{u}+a}e^{-\left(\frac{s}{u}+a\right)x}\right]_0^\infty\\
&= \frac{1}{2}\left[-\frac{1}{\left(\frac{s}{u}-a\right)e^{\left(\frac{s}{u}-a\right)x}} - \frac{1}{\left(\frac{s}{u}+a\right)e^{\left(\frac{s}{u}+a\right)x}}\right]_0^\infty\\
&= \frac{1}{2}\left[0-\left(-\frac{1}{\left(\frac{s}{u}-a\right)} - \frac{1}{\left(\frac{s}{u}+a\right)}\right)\right]\\
&= \frac{1}{2}\left[\frac{1}{\left(\frac{s}{u}-a\right)} + \frac{1}{\left(\frac{s}{u}+a\right)}\right]\\
&= \frac{1}{2}\left[\frac{\frac{s}{u}+a+\frac{s}{u}-a}{\frac{s^2}{u^2}-a^2}\right]\\
&= \frac{1}{2}\left[\frac{2\frac{s}{u}}{\frac{s^2}{u^2}-a^2}\right]\\
&= \frac{\frac{s}{u}}{\frac{s^2}{u^2}-a^2}\\
&= \frac{us}{s^2-a^2u^2}.
\end{aligned}
$$

Example 4
Obtain the Shehu transform of $f(t) = e^{ax}$, where a is a constant.
Solution

Here, $v(x) = e^{ax}$, so that

$$\mathbb{S}\left\{e^{ax}\right\} = \int_0^\infty e^{-\frac{s}{u}x}e^{ax}dx$$

Evaluating the integral on the right hand side, we get

$$
\begin{aligned}
&= -\frac{1}{\frac{s}{u}-a}e^{-\left(\frac{s}{u}-a\right)x}\Big|_0^\infty\\
&= -\frac{1}{\frac{s}{u}-a}\left(\frac{1}{e^\infty}-\frac{1}{e^0}\right)
\end{aligned}
$$

Simplifying the above, we have

$$\mathbb{S}\left\{e^{ax}\right\} = \frac{u}{s-au}.$$

Table of Applications of Shehu Transform

S\N	Expression $v(x)$	Shehu Transform $\mathbb{S}[v(x)]$
1	a	$\frac{au}{s}$
2	x	$\frac{u^2}{s^2}$
3	e^{ax}	$\frac{u}{s-au}$
4	$\sin(ax)$	$\frac{au^2}{s^2+a^2u^2}$
5	$\cos(ax)$	$\frac{us}{s^2+a^2u^2}$
6	$\frac{x^n}{n!}$, $n = 0, 1, ...$	$\left(\frac{u}{s}\right)^{n+1}$
7	$\sinh(ax)$	$\frac{au^2}{s^2-a^2u^2}$
8	$x\sin(ax)$	$\frac{2au^3s}{(s^2+a^2u^2)^2}$
9	xe^{ax}	$\frac{u^2}{(s-u)^2}$
10	$x\cos(ax)$	$\frac{u^2(s^2-a^2u^2)^2}{(s^2+a^2u^2)^2}$

2.2 APPLICATION OF SHEHU TRANSFORM TO EXPRESSIONS INVOLVING CONVOLUTION OF PRODUCTS

Theorem: Let the Shehu transforms of two functions $F_1(x)$ and $F_2(x)$ be $\mu_1(v, u)$ and $\mu_2(v, u)$ respectively. Then, their convolution $F_1(x) * F_2(x)$ can be defined as

which equally implies that

where $F_1(x) * F_2(x)$ is defined by

$$\begin{aligned} F_1(x) * F_2(x) &= \int_0^x F_1(x-t)F_2(t)dt \\ &= \int_0^x F_1(t)F_2(x-t)dt, \end{aligned}$$

Since $F_1(x) * F_2(x) = F_2(x) * F_1(x)$, that is, the convolution of functions is commutative. See [11] for the proof.

2.3 APPLICATION OF SHEHU TRANSFORM TO THE SOLUTION OF CONSTANT COEFFICIENT IVPS

To be able to apply Shehu transform successfully while solving an IVP, a thorough understanding of the Shehu transform of derivatives is paramount.

2.3.1 SHEHU TRANSFORM OF DERIVATIVES

Let the function whose derivative is being sought be $F(x)$ with the Shehu transform of $\mathbb{S}\{F(x)\} = \mu(v,u)$, then

$$\mathbb{S}\left\{F'(x)\right\} = \frac{v}{u}\mu(v,u) - F(0);$$

and that of the nth order term

$$\mathbb{S}\left\{F^{(n)}(x)\right\} = \frac{v^n}{u^n}\mu(v,u) - \sum_{i=0}^{n-1}\left(\frac{v}{u}\right)^{n-(i+1)} F^{(i)}(0).$$

Consider the following problems to demonstrate the convolution theorem in Shehu's transform.

Example 1

Find the Shehu transform of the expression with the convolution product given as

$$x^2 + \int_0^x e^{x-t}y(t)dt.$$

Solution

Given the expression

$$x^2 + \int_0^x e^{x-t}y(t)dt,$$

we take the Shehu transform of the terms that constitute the expression as follows

$$\begin{aligned}\mathbb{S}\{x^2\} + \mathbb{S}\{e^x\}\mathbb{S}\{y(t)\} &= 2!\left\{\frac{u}{s}\right\}^3 + \frac{u}{s-u}Y(s,u)\\ &= 2\left(\frac{u}{s}\right)^3 + \frac{u}{s-u}Y(s,u).\end{aligned}$$

Example 2

Obtain the Shehu transform of $\int_0^x (x-4)y(t)dt$.

Solution

Taking the Shehu transform while applying the convolution theorem

$$\begin{aligned}\mathbb{S}\left\{\int_0^x (x-4)y(t)dt\right\} &= \mathbb{S}\{x\}\mathbb{S}\{y(t)\}\\ &= \frac{u^2}{s^2}Y(s,u)\end{aligned}$$

2.4 AN EXTENSION OF THE APPLICATION OF SHEHU TRANSFORM TO VARIABLE COEFFICIENT IVPS

The method demonstrated in section 2.3 above is extended to the variable coefficient problems, where the shift theorem of integral transforms is invoked.

Example 1 Consider the differential equation

$$y'' + xy' - y = 0, \qquad y(0) = 0,\ y'(0) = 1 \tag{i}$$

Solution

Taking the Shehu transform of both sides of (i), we have

where

$$\mathbb{S}\{y''\} = \frac{s^2}{u^2}Y(s,u) - \frac{s}{u}y(0) - y'(0) \tag{ii}$$
$$= \frac{s^2}{u^2}Y(s,u) - 1$$
$$\mathbb{S}\{xy'\} = -u\frac{d}{ds}\left(\frac{s}{u}Y(s,u) - y(0)\right)$$

and

$$\mathbb{S}\{y\} = Y(s,u) \tag{iii}$$

Now using (iii) in (ii), we have

$$\frac{s^2}{u^2}Y(s,u) - 1 - Y(s,u) - s\frac{d}{ds}Y(s,u) - Y(s,u) = 0$$
$$Y(s,u)\left(\frac{s^2}{u^2} - 2\right) - s\frac{d}{ds}Y(s,u) = 1$$
$$\frac{d}{ds}Y(s,u) - \frac{s^2 - 2u^2}{u^2 s}Y(s,u) = -\frac{1}{s} \tag{iv}$$

Now, solving the first order linear differential equation (iv) by integrating factor method to get

And with $C = 0$, (v) reduces to

Applyication of inverse Shehu transformto both sides of (vi) gives

$$y(x) = x. \tag{vi}$$

Example 2

Consider the variable coefficient IVP

$$y'' + xy' - 4y = 6, \qquad y(0) = 0,\ y'(0) = 0. \tag{vii}$$

Solution

Taking the Shehu transform of both sides of (vii), we have

where

$$\mathbb{S}\{y''\} = \frac{s^2}{u^2}Y(s,u) - \frac{s}{u}y(0) - y'(0) \tag{viii}$$
$$= \frac{s^2}{u^2}Y(s,u)$$
$$\mathbb{S}\{xy'\} = -u\frac{d}{ds}\left(\frac{s}{u}Y(s,u) - y(0)\right)$$

and

$$\mathbb{S}\left\{y\right\} = Y(s,u) \tag{ix}$$

Substituting (ix) in $(viii)$, we get

$$\begin{aligned} \frac{s^2}{u^2}Y(s,u) - Y(s,u) - s\frac{d}{ds}Y(s,u) - 4Y(s,u) &= 6\frac{u}{s} \\ -s\frac{d}{ds}Y(s,u) + Y(s,u)\left(\frac{s^2}{u^2} - 5\right) &= 6\frac{u}{s} \\ \frac{d}{ds}Y(s,u) - \frac{s^2 - 5u^2}{u^2 s}Y(s,u) &= -6\frac{u}{s^2} \end{aligned} \tag{x}$$

Now, solving the first order linear differential equation (x) for $Y(s,u)$ gives

Substituting $C = 0$ in (x)reduces it to

$$Y(s,u) = 6\left[\frac{u^3}{s^3} + 2\frac{u^5}{s^5}\right] \tag{xi}$$

Applying the inverse Shehu transform $\mathbb{S}^{-1}$ to both sides of (xi) yields

$$y(x) = 3x^2 + \frac{x^4}{2}. \tag{xii}$$

2.5 APPLICATION OF SHEHU TRANSFORM TO LINEAR VOLTERRA INTEGRAL EQUATIONS

Consider the general linear Volterra integral equation of the form

$$y(x) = f(x) + \lambda \int_0^x K(x,t)y(t)dt, \tag{2.5}$$

where, throughout our discussion in this work, $\lambda = 1$. Whenever $y(x)$ on the LHS of (4) is 0, we have Volterra integral equation of the first kind, otherwise it is the second kind.
The application of Shehu transform to (5) gives the result in convolution as follows:

$$\mathbb{S}\left\{y(x)\right\} = \mathbb{S}\left\{f(x)\right\} + \mathbb{S}\left\{\int_0^x K(x,t)y(t)dt\right\} \tag{2.6}$$

$$Y(s,u) = F(s,u) + \mathbb{S}\left\{y(x)\right\} * \mathbb{S}\left\{y(t)\right\} \tag{2.7}$$

The theorem stated in section 2.2 is therefore used to handle the resulting convolution in (7).

Example 1
Find the Shehu transform of $x - sinx = \int_0^x (x-4)y(t)dt$.
Solution

Taking the Shehu Transform of both sides, while using the convolution theorem, results in the following equations

$$\begin{aligned}
\frac{u^2}{s^2} - \frac{u^2}{s^2+u^2} &= \mathbb{S}\{x\}\mathbb{S}\{y(t)\} && \text{(i)}\\
\frac{u^2}{s^2} - \frac{u^2}{s^2+u^2} &= \frac{u^2}{s^2}Y(s,u)\\
Y(s,u) &= \frac{\frac{u^2}{s^2} - \frac{u^2}{s^2+u^2}}{\frac{u^2}{s^2}}\\
Y(s,u) &= 1 - \frac{s^2}{s^2+u^2}\\
Y(s,u) &= \frac{s^2+u^2-s^2}{s^2+u^2}\\
Y(s,u) &= \frac{u^2}{s^2+u^2} && \text{(ii)}
\end{aligned}$$

We then take the inverse Shehu transform of both sides of (ii) to get

$$\begin{aligned}
\mathbb{S}^{-1}\{Y(s,u)\} &= \mathbb{S}^{-1}\left\{\frac{u^2}{s^2+u^2}\right\}\\
y(x) &= \sin(x).
\end{aligned}$$

Example 2

Find the Shehu transform of

$$e^x + \sin(x) - \cos(x) = \int_0^x 2e^{x-t}y(t)dt. \quad \text{(iii)}$$

Solution

Taking the Shehu transform of both sides of (iii) and apply convolution theorem gives

$$\begin{aligned}
\frac{u}{s-u} + \frac{u^2}{s^2-u^2} - \frac{us}{s^2-u^2} &= 2\mathbb{S}\{e^x\}\mathbb{S}\{y(t)\}\\
\frac{u}{s-u} + \frac{u^2}{s^2-u^2} - \frac{us}{s^2-u^2} &= \frac{2u}{s-u}Y(s,u)\\
\frac{2u}{s-u}Y(s,u) &= \frac{u(s^2+u^2) + u^2(s-u) - us(s-u)}{(s-u)(s^2+u^2)}\\
2uY(s,u) &= \frac{us^2+u^3+u^2s-u^3-us^2+u^2s}{s^2+u^2}\\
2uY(s,u) &= \frac{2u^2s}{s^2+u^2}\\
Y(s,u) &= \frac{us}{s^2+u^2}. && \text{(iv)}
\end{aligned}$$

We now take the inverse Shehu transform of both sides of (iv) to obtain $y(x)$ as

$$\mathbb{S}^{-1}\{Y(s,u)\} = \mathbb{S}^{-1}\left\{\frac{us}{s^2+u^2}\right\}.$$
$$y(x) = \cos(x).$$

Example 3
Find the Shehu transform of

$$x = \int_0^x (x-t+1)y(t)dt. \tag{v}$$

Solution
Taking the Shehu transform of both sides of (v), we have

$$\begin{aligned}
\frac{u^2}{s^2} &= \mathbb{S}\{x+1\}\mathbb{S}\{y(t)\}\\
\frac{u^2}{s^2} &= (\mathbb{S}\{x\}+\mathbb{S}\{1\})Y(s,u)\\
\frac{u^2}{s^2} &= \left(\frac{u^2}{s^2}+\frac{u}{s}\right)Y(s,u)\\
\frac{u^2}{s^2} &= \left(\frac{u^2+us}{s^2}\right)Y(s,u)\\
Y(s,u) &= \frac{u^2}{u^2+us}\\
Y(s,u) &= \frac{\frac{u^2}{u}}{\frac{u^2}{u}+\frac{us}{u}}\\
Y(s,u) &= \frac{u}{s+u}
\end{aligned} \tag{vi}$$

We then apply the inverse Shehu transform to both sides of (vi) to get

$$\mathbb{S}^{-1}\{Y(s,u)\} = \mathbb{S}^{-1}\left\{\frac{u}{s+u}\right\}.$$
$$y(x) = e^{-x}.$$

2.6 APPLICATION OF SHEHU TRANSFORM TO LINEAR VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS

Here, the Shehu tranform is applied to the general linear Volterra integro-differential equation of the form

$$y^{(n)}(x) = f(x) + \lambda \int_0^x K(x,t)y(t)dt, \tag{2.8}$$

where, λ is as given in the previous section, and $y^{(n)}(x)$ is interpreted as defined in section 2.3.1. The algorithm is as follows:

$$\mathbb{S}\left\{y^{(n)}(x)\right\} = \mathbb{S}\left\{f(x)\right\} + \mathbb{S}\left\{\int_0^x K(x,t)y(t)dt\right\} \tag{2.9}$$

$$\frac{s^n}{u^n}Y(s,u) - \sum_{i=0}^{n-1}\left(\frac{s}{u}\right)^{n-(i+1)} y^{(i)}(0) = F(s,u) + \mathbb{S}\left\{y(x)\right\} * \mathbb{S}\left\{y(t)\right\} \tag{2.10}$$

Example 1

Solve the first order integro-differential equation below using the Shehu transform method

$$y'(x) = 2 + x - \frac{1}{3!}x^3 + \int_0^x (x-t)y(t)dt, \quad y(0) = 1.$$

Solutions

Taking the Shehu transform of both sides and applying the convolution theorem to the equation

$$\mathbb{S}\{y'(x)\} = \mathbb{S}\{2 + x - \frac{1}{3!}x^3\} + \mathbb{S}\left\{\int_0^x (x-t)y(t)dt\right\}, \quad y(0) = 1.$$

we get

$$\begin{aligned}
\frac{s}{u}Y(s,u) - y(0) &= \frac{2u}{s} + \left(\frac{u}{s}\right)^2 - \left(\frac{u}{s}\right)^4 + \mathbb{S}\{x\}\mathbb{S}\{y(t)\} \\
\frac{s}{u}Y(s,u) &= 1 + \frac{2u}{s} + \left(\frac{u}{s}\right)^2 - \left(\frac{u}{s}\right)^4 + \left(\frac{u}{s}\right)^2 Y(s,u) \\
\left(\frac{s}{u} - \frac{u^2}{s^2}\right)Y(s,u) &= 1 + \frac{2u}{s} + \left(\frac{u}{s}\right)^2 - \left(\frac{u}{s}\right)^4 \\
\left(\frac{s^3 - u^3}{s^2 u}\right)Y(s,u) &= \frac{s^4 + 2us^3 + u^2s^2 - u^4}{s^4} \\
Y(s,u) &= \frac{us^4 + 2u^2s^3 + u^3s^2 - u^5}{s^2(s^3 - u^3)} \\
Y(s,u) &= \frac{u}{s-u} + \frac{u^2}{s^2}
\end{aligned}$$

We now take the inverse Shehu transform of both sides

$$\mathbb{S}^{-1}\{Y(s,u)\} = \mathbb{S}^{-1}\left\{\frac{u}{s-u}\right\} + \mathbb{S}^{-1}\left\{\frac{u^2}{s^2}\right\}.$$

Thus, we have

$$y(x) = e^{-x} + x.$$

Example 2

Solve the third order integro-differential equation below using the Shehu transform method

$$y'''^2 + \int_0^x (x-t)y(t)dt, \quad y(0) = 5, \quad y'(0) = y''(0) = 1.$$

Solution
Taking the Shehu transform of both sides and applying the convolution theorem to the equation

$$\mathbb{S}\{y'''(x)\} = \mathbb{S}\{1 + x - 2x^2\} + \mathbb{S}\left\{\int_0^x (x-t)y(t)dt\right\},$$

we get

$$\frac{s^3}{u^3}Y(s,u) - \frac{s^2}{u^2}y(0) - \frac{s}{u}y'(0) - y''(0) = \frac{u}{s} + \frac{u^2}{s^2} - 4\left(\frac{u}{s}\right)^3 + \mathbb{S}\{x\}\mathbb{S}\{y(t)\}$$

$$\frac{s^3}{u^3}Y(s,u) - \frac{5s^2}{u^2} - \frac{s}{u} - 1 = \frac{u}{s} + \frac{u^2}{s^2} - 4\left(\frac{u}{s}\right)^3 + \frac{u^2}{s^2}Y(s,u)$$

$$\left(\frac{s^3}{u^3} - \frac{u^2}{s^2}\right)Y(s,u) = 1 + \frac{u}{s} + \frac{5s^2}{u^2} + \frac{u}{s} + \frac{u^2}{s^2} - 4\left(\frac{u}{s}\right)^3$$

$$\left(\frac{s^5 - u^5}{s^2u^3}\right)Y(s,u) = \frac{u^2s^3 + us^4 + 5s^5 + u^3s^2 + u^4s - 4u^5}{u^2s^3}$$

$$Y(s,u) = \frac{u^3s^3 + u^2s^4 + 5us^5 + u^4s^2 + u^5s - 4u^6}{s(s^5 - u^5)}$$

Resolving the above into partial fraction, and taking its inverse Shehu transform yields

$$\mathbb{S}^{-1}\{Y(s,u)\} = \mathbb{S}^{-1}\left\{\frac{u}{s-u}\right\} + \mathbb{S}^{-1}\left\{\frac{4u}{s}\right\}.$$

Thus, we have

$$y(x) = 4 + e^{-x}.$$

3. A REVIEW OF ADOMIAN DECOMPOSITION METHOD (ADM)

In this section, a brief review of George Adomian decomposition method is presented, as this method is combined with the Shehu transform in what follows.

Consider the initial value problem

$$Lu + Ru + N(u) = g(x), \tag{3.1}$$

together with the attached conditions

$$u(0) = \tau_0,\ u'(0) = \tau_1,\ ...,\ u^{(n-1)}(0) = \tau_{n-1}, \tag{3.2}$$

where L is the highest order linear operation, R is the remaining linear terms, N is the nonlinear term and $g(x)$ is the inhomogeneous source term.

In the ADM algorithm, L being a linear differential operator is expected to have an inverse operator L^{-1} which is the integral of equivalent order. Thus,

where

with

and

$$f(x) = L^{-1}g(x) \tag{3.3}$$

The other members of the series; $u_1(x)$, $u_2(x), \ldots$ are obtained through the recurrence formula

$$u_n(x) = -L^{-1}\left(Ru_{n-1}(x)\right) - L^{-1}\left(A_{n-1}(x)\right), \quad n = 1, 2, \ldots, \tag{3.4}$$

where $A_{n-1}(x)$ are the Adomian polynomials derived for the nonlinearity encountered in the given IVP. These are elegantly derived from the general formula

$$A_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n}N\left(\sum_{i=0}^{n}\lambda^i u_i\right)\right]_{\lambda=0}, \quad n = 0, 1, 2, \ldots \tag{3.5}$$

The Adomian polynomials thus developed are used in (17) to get various components of the series. The final solution is given by

That is

$$u(x) = \sum_{i=0}^{\infty} u_i(x). \tag{3.6}$$

The ADM would mostly give the exact solution whenever such exists in closed form. See ([1], [2], [3], [8].)

4. SOLUTION OF NONLINEAR VOLTERRA INTEGRO-DIFFERENTIAL EQUATION (NVIDE) BY SHEHU TRANSFORM ADOMIAN DECOMPOSITION METHOD (STADM)

A method of solving NVIDEs is proposed in this section, where the Shehu transform is implemented and ADM is employed to decompose any nonlinearity encountered in a manner that reduces the computational works.

Consider the nonlinear Volterra integro-differential equation

$$y^{(n)}(x) = f(x) + \lambda\int_0^x K(x-t)N\left(y(t)\right)dt, \tag{4.1}$$

As usual, we shall apply Shehu transform to both sides of (21)

$$\mathbb{S}\left\{y^{(n)}(x)\right\} = \mathbb{S}\left\{f(x)\right\} + \mathbb{S}\left\{\int_0^x K(x-t)N\left(y(t)\right)dt\right\} \tag{4.2}$$

$$\frac{s^n}{u^n}Y(s,u) - \sum_{i=0}^{n-1}\left(\frac{s}{u}\right)^{n-(i+1)} y^{(i)}(0) = F(s,u) + \mathbb{S}\left\{K(x-t)\right\} * \mathbb{S}\left\{N(y(t))\right\} \tag{4.3}$$

For us to be able to handle the nonlinear aspect of (23) above, that is $N(y(x))$. The desired Adomian polynomials shall be obtained using the formula stated in (18), i.e.

$$N(y(x)) = \sum_{n=0}^{\infty} A_n(x). \tag{4.4}$$

Substitute (24) into (23) to have

$$\frac{s^n}{u^n}Y(s,u) - \sum_{i=0}^{n-1}\left(\frac{s}{u}\right)^{n-(i+1)} y^{(i)}(0) = F(s,u) + \mathbb{S}\left\{K(x-t)\right\} * \mathbb{S}\left\{\sum_{n=0}^{\infty} A_n(x)\right\}. \tag{4.5}$$

Thus, the recurrence relation expected in Adomian decomposition method now takes the form

$$\begin{aligned} \mathbb{S}\left\{y_0(x)\right\} &= \frac{u^n}{s^n}\left(\sum_{i=0}^{n-1}\left(\frac{s}{u}\right)^{n-(i+1)} y^{(i)}(0) + F(s,u)\right), \\ \mathbb{S}\left\{y_{k+1}(x)\right\} &= \frac{u^n}{s^n}\mathbb{S}\left\{\sum_{k=0}^{\infty} A_k(x)\right\}, \quad k \geq 0. \end{aligned} \tag{4.6}$$

For the results in (26) to be valid, the function $y(x)$ must be piecewise continuous and of exponential order.
The application of inverse Shehu transform to the first part of (26) produces $y_0(x)$. Thus, derivation of $A_0(x)$ is no longer a problem, and would lead to obtaining all other members of $y_{k+1}(x)$, $k \geq 0$ derivable from the second part of (26).
Examples on the algorithm discussed in this section, that is, the implementation of Shehu transform combined with Adomian decomposition method shall be presented in the sequel.

4.1 NUMERICAL EXPERIMENTS ON THE APPLICATION OF STADM

Selected problems are considered for the application of the proposed method in this section.

Problem 1 [1]
Solve the first order nonlinear integro-differential equation below using

STADM

$$y'(x) = \frac{17}{4} + \frac{9}{2}x - 2x^2 - 3e^x - \frac{1}{4}e^{2x} + \int_0^x (x-t)y^2(t)dt, \qquad y(0) = 3. \text{ (i)}$$

Solution

Taking the Shehu transform of both sides of (i) and applying the convolution theorem, that is

$$\mathbb{S}\left\{y'(x)\right\} = \mathbb{S}\left\{\frac{17}{4} + \frac{9}{2}x - 2x^2 - 3e^x - \frac{1}{4}e^{2x} + \int_0^x (x-t)y^2(t)dt\right\},$$

we get

$$\begin{aligned}
\frac{s}{u}Y(s,u) - y(0) &= \frac{17u}{4s} + \frac{9u^2}{2s^2} - \frac{4u^3}{s^3} - \frac{3u}{s-u} - \frac{u}{4(s-2u)} \\
&\quad + \mathbb{S}\{x\}\mathbb{S}\{y^2(t)\} \\
\frac{s}{u}Y(s,u) &= 3 + \frac{17u}{4s} + \frac{9u^2}{2s^2} - \frac{4u^3}{s^3} - \frac{3u}{s-u} - \frac{u}{4(s-2u)} \\
&\quad + \frac{u^2}{s^2}\mathbb{S}\{y^2(t)\} \\
Y(s,u) &= \frac{3u}{s} + \frac{17u^2}{4s^2} + \frac{9u^3}{2s^3} - \frac{4u^4}{s^4} - \frac{3u^2}{s(s-u)} - \frac{u^2}{4s(s-2u)} \\
&\quad + \frac{u^3}{s^3}\mathbb{S}\{y^2(t)\}.
\end{aligned}$$

So that

$$Y_0(s,u) = \frac{3u}{s} + \frac{17u^2}{4s^2} + \frac{9u^3}{2s^3} - \frac{4u^4}{s^4} - \frac{3u^2}{s(s-u)} - \frac{u^2}{4s(s-2u)}$$

We then take the inverse Shehu transform of $Y_0(s,u)$, to get

$$y_0(x) = \frac{49}{8} - 3e^x - \frac{e^{2x}}{8} + \frac{17}{4}x + \frac{9}{4}x^2 - \frac{2}{3}x^3.$$

Now, we compute $y_1(x)$ using the recurrence relation:

$$\mathbb{S}\{y_{k+1}(x)\} = \frac{u^3}{s^3}\mathbb{S}\{A_k(x)\}, \qquad k \geq 0,$$

where $A_k(x)$ is the set of Adomian polynomials corresponding to the nonlinearity $y^2(x)$.

For the generation of the required Adomian polynomials see ([1], [2].)

Recall that the Adomian polynomials for $F(y(x)) = y^2(x)$ are:

$$\begin{aligned}
A_0 &= y_0^2, & A_1 &= 2y_0y_1, \\
A_2 &= 2y_0y_2 + y_1^2, & A_3 &= 2y_0y_3 + 2y_1y_2, \quad \text{etc.}
\end{aligned}$$

Now for $k = 0$, we obtain $y_1(x)$:

$$
\begin{aligned}
A_0 &= y_0^2 \\
A_0 &= \left(\frac{49}{8} - 3e^x - \frac{e^{2x}}{8} + \frac{17x}{4} + \frac{9x^2}{4} - \frac{2x^3}{3}\right)^2 \\
A_0 &= \frac{2401}{64} - \frac{147e^x}{4} + \frac{239e^{2x}}{32} + \frac{3e^{3x}}{4} + \frac{e^{4x}}{64} + \frac{833x}{16} - \frac{51e^x x}{2} \\
&\quad - \frac{17}{16}e^{2x}x + \frac{365x^2}{8} - \frac{27e^x x^2}{2} - \frac{9}{16}e^{2x}x^2 + \frac{263x^3}{24} + 4e^x x^3
\end{aligned}
$$

Taking Shehu transform of A_0, we have

$$
\begin{aligned}
\mathbb{S}\{A_0(x)\} = {} & \frac{u}{64(-4u+s)} + \frac{3u}{4(-3u+s)} + \frac{u^4}{(-2u+s)^4} - \frac{9u^3}{8(-2u+s)^3} \\
& - \frac{17u^2}{16(-2u+s)^2} + \frac{239u}{32(-2u+s)} + \frac{24u^4}{(-u+s)^4} - \frac{27u^3}{(-u+s)^3} \\
& - \frac{51u^2}{2(-u+s)^2} - \frac{147u}{4(-u+s)} + \frac{320u^7}{s^7} - \frac{360u^6}{s^6} - \frac{29u^5}{2s^5} \\
& + \frac{263u^4}{4s^4} + \frac{365u^3}{4s^3} + \frac{833u^2}{16s^2} + \frac{2401u}{64s}.
\end{aligned}
$$

Then

which yields

$$
\begin{aligned}
Y_1(s,u) = \frac{u^3}{s^3} \quad \times \quad & \left(\frac{u}{64(-4u+s)} + \frac{3u}{4(-3u+s)} + \frac{u^4}{(-2u+s)^4}\right. \\
& - \frac{9u^3}{8(-2u+s)^3} - \frac{17u^2}{16(-2u+s)^2} + \frac{239u}{32(-2u+s)} + \frac{24u^4}{(-u+s)^4} \\
& - \frac{27u^3}{(-u+s)^3} - \frac{51u^2}{2(-u+s)^2} - \frac{147u}{4(-u+s)} + \frac{320u^7}{s^7} - \frac{360u^6}{s^6} \\
& \left. - \frac{29u^5}{2s^5} + \frac{263u^4}{4s^4} + \frac{365u^3}{4s^3} + \frac{833u^2}{16s^2} + \frac{2401u}{64s}\right).
\end{aligned}
$$

$$
\begin{aligned}
Y_1(s,u) = {} & \frac{2401u^4}{64s^4} + \frac{u^4}{64s^3(s-4u)} + \frac{3u^4}{4s^3(s-3u)} + \frac{239u^4}{32s^3(s-2u)} \\
& - \frac{147u^4}{4s^3(s-u)} + \frac{833u^5}{16s^5} - \frac{17u^5}{16s^3(s-2u)^2} - \frac{51u^5}{2s^3(s-u)^2} \\
& + \frac{365u^6}{4s^6} - \frac{9u^6}{8s^3(s-2u)^3} - \frac{27u^6}{s^3(s-u)^3} + \frac{263u^7}{4s^7} + \frac{u^7}{s^3(s-2u)^4}
\end{aligned}
$$

Taking the Inverse Shehu transform of both sides of $Y_1(s,u)$ above, we have

$$y_1(x) = \frac{3x^3}{2} + \frac{x^4}{4} + \frac{x^5}{15} - \frac{7x^6}{120} - \frac{11x^7}{630} - \frac{x^8}{160} + \frac{7x^9}{2592} + \frac{373x^{10}}{604800}$$

Now taking the approximate solution as the sum of $y_0(x)$ and $y_1(x)$, we have

$$y(x) = 3 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + ...$$
$$y(x) = 2 + \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + ...\right)$$

that results into

$$y(x) = 2 + e^x$$

Problem 2 [1]
Solve the third order nonlinear integro-differential equation below using STADM transform method

$$y'''(x) = \frac{3}{2}e^x - \frac{1}{2}e^{3x} + \int_0^x (x-t)y^3(t)dt, \qquad y(0) = y'(0) = y''(0) = 1. \tag{ii}$$

Solution
Taking the Shehu transform of both sides of (i) and applying the convolution theorem, that is

$$\mathbb{S}\left\{y'''(x)\right\} = \mathbb{S}\left\{\frac{3}{2}e^x - \frac{1}{2}e^{3x} + \int_0^x (x-t)y^3(t)dt\right\},$$

we get

$$\begin{aligned}
\frac{s^3}{u^3}Y(s,u) - \frac{s^2}{u^2}y(0) - \frac{s}{u}y'(0) - y''(0) &= \frac{3u}{2(s-u)} - \frac{u}{2(s-3u)} \\
&\quad + \mathbb{S}\{e^x\}\mathbb{S}\{y^3(t)\} \\
\frac{s^3}{u^3}Y(s,u) - \frac{s^2}{u^2} - \frac{s}{u} - 1 &= \frac{3u}{2(s-u)} - \frac{u}{2(s-3u)} \\
&\quad + \frac{u}{s-u}\mathbb{S}\{y^3(t)\} \\
\frac{s^3}{u^3}Y(s,u) &= 1 + \frac{s}{u} + \frac{s^2}{u^2} + \frac{3u}{2(s-u)} - \frac{u}{2(s-3u)} \\
&\quad + \frac{u}{s-u}\mathbb{S}\{y^3(t)\} \\
Y(s,u) &= \frac{u^3}{s^3} + \frac{u^2}{s^2} + \frac{u}{s} + \frac{3u^4}{2s^3(s-u)} - \frac{u^4}{2s^3(s-3u)} \\
&\quad + \frac{u^4}{s^3(s-u)}\mathbb{S}\{y^3(t)\}
\end{aligned}$$

So that, $Y_0(s,u)$ is given by

$$Y_0(s,u) = \frac{u^3}{s^3} + \frac{u^2}{s^2} + \frac{u}{s} + \frac{3u^4}{2s^3(s-u)} - \frac{u^4}{2s^3(s-3u)}.$$

We then take the inverse Shehu transform of $Y_0(s,u)$, to get

$$y_0(x) = -\frac{13}{27} + \frac{3e^x}{2} - \frac{e^{3x}}{54} - \frac{4x}{9} - \frac{x^2}{6}.$$

Now, we compute $y_1(x)$ using the recurrence relation

$$\mathbb{S}\{y_{k+1}(x)\} = \frac{u^4}{s^3(s-u)}\mathbb{S}\{A_k(x)\}, \qquad k \geq 0.$$

Recall that the Adomian polynomials for $F(y(x)) = y^3(x)$ are:

$$\begin{aligned} A_0 &= y_0^3, & A_1 &= 3y_1y_0^2, \\ A_2 &= 3(y_2y_0^2 + y_1^0), & A_3 &= 3y_3y_0^2 + 6y_2y_1y_0 + y_1^3, \quad \text{etc.} \end{aligned}$$

Now for $k = 0$, we obtain $y_1(x)$:

$$
\begin{aligned}
A_0 &= y_0^3\\
A_0 &= \left(-\frac{13}{27} + \frac{3e^x}{2} - \frac{e^{3x}}{54} - \frac{4x}{9} - \frac{x^2}{6}\right)^3\\
A_0 &= -\frac{2197}{19683} + \frac{169e^x}{162} - \frac{13e^{2x}}{4} + \frac{176471e^{3x}}{52488} + \frac{13e^{4x}}{162} - \frac{e^{5x}}{8}\\
&- \frac{13e^{6x}}{26244} + \frac{e^{7x}}{648} - \frac{e^{9x}}{157464} - \frac{676x}{2187} + \frac{52e^x x}{27} - 3e^{2x}x - \frac{52e^{3x}x}{2187}\\
&+ \frac{2}{27}e^{4x}x - \frac{e^{6x}x}{2187} - \frac{65x^2}{162} + \frac{29e^x x^2}{18} - \frac{9}{8}e^{2x}x^2 - \frac{29e^{3x}x^2}{1458}\\
&+ \frac{1}{36}e^{4x}x^2 - \frac{e^{6x}x^2}{5832} - \frac{220x^3}{729} + \frac{2e^x x^3}{3} - \frac{2}{243}e^{3x}x^3 - \frac{5x^4}{36}
\end{aligned}
$$

Taking Shehu transform of A_0, we have:

$$
\begin{aligned}
\mathbb{S}\{A_0(x)\} = &-\frac{u}{157464(-9u+s)} + \frac{u}{648(-7u+s)} - \frac{u^3}{2916(-6u+s)^3}\\
&- \frac{u^2}{2187(-6u+s)^2} - \frac{13u}{26244(-6u+s)} - \frac{u}{8(-5u+s)}\\
&+ \frac{u^3}{18(-4u+s)^3} + \frac{2u^2}{27(-4u+s)^2} + \frac{13u}{162(-4u+s)}\\
&- \frac{u^5}{27(-3u+s)^5} - \frac{4u^4}{81(-3u+s)^4} - \frac{29u^3}{729(-3u+s)^3}\\
&- \frac{52u^2}{2187(-3u+s)^2} + \frac{176471u}{52488(-3u+s)} - \frac{9u^3}{4(-2u+s)^3}\\
&- \frac{3u^2}{(-2u+s)^2} - \frac{13u}{4(-2u+s)} + \frac{3u^5}{(-u+s)^5} + \frac{4u^4}{(-u+s)^4}\\
&+ \frac{29u^3}{9(-u+s)^3} + \frac{52u^2}{27(-u+s)^2} + \frac{169u}{162(-u+s)} - \frac{10u^7}{3s^7} - \frac{40u^6}{9s^6}\\
&- \frac{10u^5}{3s^5} - \frac{440u^4}{243s^4} - \frac{65u^3}{81s^3} - \frac{676u^2}{2187s^2} - \frac{2197u}{19683s}
\end{aligned}
$$

which yields

$$
\begin{aligned}
Y_1(s,u) = {} & \frac{u^4}{s^3(s-u)} \quad \times \quad \Bigg(-\frac{u}{157464(-9u+s)} + \frac{u}{648(-7u+s)} \\
& - \frac{u^3}{2916(-6u+s)^3} - \frac{u^2}{2187(-6u+s)^2} - \frac{13u}{26244(-6u+s)} \\
& - \frac{u}{8(-5u+s)} + \frac{u^3}{18(-4u+s)^3} + \frac{2u^2}{27(-4u+s)^2} \\
& + \frac{13u}{162(-4u+s)} - \frac{u^5}{27(-3u+s)^5} - \frac{4u^4}{81(-3u+s)^4} \\
& - \frac{29u^3}{729(-3u+s)^3} - \frac{52u^2}{2187(-3u+s)^2} + \frac{176471u}{52488(-3u+s)} \\
& - \frac{9u^3}{4(-2u+s)^3} - \frac{3u^2}{(-2u+s)^2} - \frac{13u}{4(-2u+s)} + \frac{3u^5}{(-u+s)^5} \\
& + \frac{4u^4}{(-u+s)^4} + \frac{29u^3}{9(-u+s)^3} + \frac{52u^2}{27(-u+s)^2} + \frac{169u}{162(-u+s)} \\
& - \frac{10u^7}{3s^7} - \frac{40u^6}{9s^6} - \frac{10u^5}{3s^5} - \frac{440u^4}{243s^4} - \frac{65u^3}{81s^3} - \frac{676u^2}{2187s^2} - \frac{2197u}{19683s} \Bigg)
\end{aligned}
$$

$$
\begin{aligned}
Y_1(s,u) = {} & \frac{25606049146299857u}{629974412928000s} - \frac{u}{918330048(s-9u)} + \frac{u}{1333584(s-7u)} \\
& - \frac{23u}{88573500(s-6u)} - \frac{u}{4000(s-5u)} + \frac{53u}{248832(s-4u)} \\
& + \frac{353545u}{5668704(s-3u)} - \frac{19u}{32(s-2u)} - \frac{8664787u}{216000(s-u)} \\
& + \frac{21187613209157u^2}{999959385600s^2} - \frac{19u^2}{94478400(s-6u)^2} + \frac{u^2}{13824(s-4u)^2} \\
& + \frac{103u^2}{944784(s-3u)^2} + \frac{21u^2}{64(s-2u)^2} + \frac{3175u^2}{162(s-u)^2} \\
& + \frac{12255086041u^3}{793618560s^3} - \frac{u^3}{3149280(s-6u)^3} + \frac{u^3}{3456(s-4u)^3} \\
& - \frac{53u^3}{157464(s-3u)^3} - \frac{9u^3}{32(s-2u)^3} - \frac{371u^3}{27(s-u)^3} + \frac{278416u^4}{19683s^4} \\
& + \frac{u^4}{8748(s-3u)^4} + \frac{83u^4}{9(s-u)^4} + \frac{30691u^5}{2187s^5} - \frac{u^5}{1458(s-3u)^5} \\
& - \frac{5u^5}{(s-u)^5} + \frac{3335u^6}{243s^6} + \frac{3u^6}{(s-u)^6} + \frac{3140u^7}{243s^7} + \frac{100u^8}{9s^8} + \frac{70u^9}{9s^9} + \frac{10u^{10}}{3s^{10}}
\end{aligned}
$$

Taking the Inverse Shehu transform of the above, we have $y_1(x)$ to be:

$$\begin{aligned} y_1(x) = &\frac{x^4}{24} + \frac{x^5}{30} + \frac{13x^6}{720} + \frac{x^7}{126} + \frac{59x^8}{20160} + \frac{319x^9}{362880} + \frac{137x^{10}}{725760} \\ &+ \frac{1546841x^{11}}{117573120000} - \frac{37171x^{12}}{7838208000} - \frac{151x^{13}}{195955200} - \frac{79x^{14}}{174182400} \\ &+ \frac{x^{15}}{145152000} \end{aligned}$$

Now taking the approximate solution based on $y_0(x)$ and $y_1(x)$, we have

which results to

$$y(x) = e^x.$$

5. DISCUSSION OF RESULTS AND CONCLUSION

In Section 2.5, Shehu transform has been applied to linear Volterra integral equations, while in 2.6 it was applied to linear integro-differential equations. The analytical results obtained for problems considered in the two cases are similar to those obtained in [1] where the author used Laplace transform.

The major result in this paper is presented Section 4. The algorithm developed in that section has been validated by two nonlinear integro-differential equation problems.

A new analytical approach to the solution of nonlinear integro-differential equation named Shehu Transform Adomian Decomposition Method has been presented. The results obtained from selected problems in the literature show the reliability of the proposed method.

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