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SOME PROPERTIES OF PICTURE FUZZY MULTIRELATIONS

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ABSTRACT. This paper investigates into the examination of picture fuzzy multirelations as an extension of picture fuzzy relations. We explore reflexivity, symmetry and transitivity of picture fuzzy multirelations over picture fuzzy multisets, and derive some associated properties.

1. INTRODUCTION

The concept of Fuzzy Relation (FR) was a generalisation of classical relation which was introduced by Zadeh [1]. The notion of Intuitionistic Fuzzy Sets (IFSs) put forward by Atanassov [2] served as the basis for the work of Bustince and Burillo [3] who introduced Intuitionistic Fuzzy Relation (IFR). Cuong and Kreinovich [4], introduced the notion of Picture Fuzzy Relations (PFRs) as a generalisation of fuzzy relations and intuitionistic fuzzy relations. Phong et al [5] examined some properties of composition of PFRs and proposed a new approach for medical diagnosis using composition of fuzzy relations. In [6], Dutta and Saikia studied equivalence picture fuzzy relation and some of its properties such as equivalence class, intersection and union of equivalence relations were obtained. Hasan et al [7] defined max-min composition and min-max composition for picture fuzzy relations and investigated some of their properties and also discussed an application of picture fuzzy relations in decision making. In [8], Hasan et al also defined picture fuzzy relation over picture fuzzy set, numerous properties related to picture fuzzy relation were established and some operations were discussed with examples.

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Yagar [9] initiated the idea of Fuzzy Multisets (FMs) as an extension of fuzzy sets. Shinoj and Sunil [10] introduced the concept of Intuitionistic Fuzzy Multisets (IFMSs) as an extension of IFSs and FSs. Sangodapo and Feng [11] introduced the notion of Picture Fuzzy Multisets (PFMSs) as a generalisation of the works in [9] and [10].

In this paper, we contribute to the work of Sangodapo and Kausar [12] on Picture Fuzzy MultiRelations (PFMRs). We discuss reflexivity, symmetry and transitivity of picture fuzzy multirelations over picture fuzzy multisets and obtained some properties associated with them. The paper is organised as follows; Section 2 is based on basic definitions, Section 3 discusses reflexivity, symmetric and transitivity of a picture fuzzy multirelation over a picture fuzzy multiset and some properties associated with them were obtained.

2. PRELIMINARY

Some basic definitions needed were stated from Zadeh [1], Cuong and Kreinovich [4], Sangodapo and Feng [11] and Sangodapo and Kausar [12].

Definition 1: Let Z be a nonempty set. A FR U on Z is a fuzzy set, defined as

$$U = \{\langle (r_1, r_2), \sigma_U(r_1, r_2)\rangle | (r_1, r_2) \in Z \times Z\}$$

where $\sigma_U : Z \times Z \longrightarrow [0,1]$ denotes the membership function of the fuzzy relation U, which assigns a degree of relatedness to each pair (r_1, r_2). The value of $\sigma_U(r_1, r_2)$ indicates to what extent r_1 is related to r_2 according to the fuzzy relation U.

Definition 2: Let X be a universe. A Picture Fuzzy Set (PFS) Z of X is an object of the form

$$Z = \{\langle r_1, \sigma_Z(r_1), \tau_Z(r_1), \eta_Z(r_1))| r_1 \in X\rangle\},$$

such that $\sigma_Z(r_1) \in [0,1]$ is called to as the degree of positive membership, $\tau_Z(r_1) \in [0,1]$ is called degree of neutral membership and $\eta_Z(r_1) \in [0,1]$ is called degree of negative membership of $r_1 \in X$ and for all $r_1 \in X$,

$$\sigma_Z(r_1) + \tau_Z(r_1) + \eta_Z(r_1) \leq 1$$

and the degree of refusal membership of $r_1 \in Z$ is $1 - (\sigma_Z(r_1) + \tau_Z(r_1) + \eta_Z(r_1))$.

Definition 3: Let Z_1 and Z_2 be two PFSs. Then, the inclusion, equality, union, intersection and complement are defined as follow:

- $Z_1 \subseteq Z_2$ if and only if for all $y \in X$, $\sigma_{Z_1}(y) \leq \sigma_{Z_2}(y)$, $\tau_{Z_1}(y) \leq \tau_{Z_2}(y)$ and $\eta_{Z_1}(y) \geq \eta_{Z_2}(y)$.

- $Z_1 = Z_2$ if and only if $Z_1 \subseteq Z_2$ and $Z_2 \subseteq Z_1$.
- $Z_1 \cup Z_2 = \{(y, \sigma_{Z_1}(y) \vee \sigma_{Z_2}(y), \tau_{Z_1}(y) \wedge \tau_{Z_2}(y)), \eta_{Z_1}(y) \wedge \eta_{Z_2}(y))| \, y \in X\}$.
- $Z_1 \cap Z_2 = \{(y, \sigma_{Z_1}(y) \wedge \sigma_{Z_2}(y), \tau_{Z_1}(y) \wedge \tau_{Z_2}(y)), \eta_{Z_1}(y) \vee \eta_{Z_2}(y))| \, y \in X\}$.
- $\overline{Z_1} = \{(y, \eta_{Z_1}(y), \tau_{Z_1}(y), \sigma_{Z_1}(y))| \, y \in X\}$.

Definition 4: Let Z_1 and Z_2 be nonempty sets. Then, a picture fuzzy relation (PFR) U is a PFS over $Z_1 \times Z_2$, defined as

$$U = \{\langle (r_1, r_2), \sigma_U(r_1, r_2), \tau_U(r_1, r_2), \eta_U(r_1, r_2) \rangle | (r_1, r_2) \in Z_1 \times Z_2\}$$

with $\sigma_U : Z_1 \times Z_2 \to [0,1], \; \tau_U : Z_1 \times Z_2 \to [0,1], \; \eta_U : Z_1 \times Z_2 \to [0,1]$, such that $0 \leq \sigma_U(r_1, r_2) + \tau_U(r_1, r_2) + \eta_U(r_1, r_2) \leq 1$ for every $(r_1, r_2) \in Z_1 \times Z_2$.

Definition 5: Let U be a PFR between Z_1 and Z_2. The inverse relation of U, U^{-1} between Z_2 and Z_1 is defined as

$$\sigma_{U^{-1}}(r_2, r_1) = \sigma_U(r_1, r_2), \tau_{U^{-1}}(r_2, r_1) = \tau_U(r_1, r_2), \eta_{U^{-1}}(r_2, r_1) = \eta_U(r_1, r_2),$$

$\forall \, (r_1, r_2) \in (Z_1 \times Z_2)$.

Definition 6: Let U and V be two PFRs between Z_1 and Z_2. Then,

- $U \leq V \Leftrightarrow (\sigma_U(r_1, r_2) \leq \sigma_V(r_1, r_2)), (\tau_U(r_1, r_2) \leq \tau_V(r_1, r_2))$ and $(\eta_U(r_1, r_2) \geq \eta_V(r_1, r_2))$
- $U \cup V = \{((r_1, r_2), \sigma_U(r_1, r_2) \vee \sigma_V(r_1, r_2), \tau_U(r_1, r_2) \wedge \tau_V(r_1, r_2), \eta_U(r_1, r_2) \wedge \eta_V(r_1, r_2))|(r_1, r_2) \in Z_1 \times Z_2\}$
- $U \cap V = \{((r_1, r_2), \sigma_U(r_1, r_2) \wedge \sigma_V(r_1, r_2), \tau_U(r_1, r_2) \wedge \tau_V(r_1, r_2), \eta_U(r_1, r_2) \vee \eta_V(r_1, r_2))|(r_1, r_2) \in Z_1 \times Z_2\}$
- $U^c = \{((r_1, r_2), \eta_U(r_1, r_2), \tau_U(r_1, r_2), \sigma_U(r_1, r_2))|(r_1, r_2) \in Z_1 \times Z_2\}$

for every $(r_1, r_2) \in (Z_1 \times Z_2)$.

Definition 7: Let Y be a nonempty set. A PFMS Z in Y is characterised by three functions namely positive membership count function *pmc*, neutral membership count function n_emc and negative membership count function *nmc* such that $pmc : Y \to W$, $n_emc : Y \to W$ and $nmc : Y \to W$, respectively, where W is the set of all crisp multisets drawn from $[0,1]$. Thus, for any $r \in Y$, *pmc* is the crisp multiset from $[0,1]$ whose positive membership sequence is defined by $(\sigma_Z^1(r), \sigma_Z^2(r), \cdots, \sigma_Z^n(r))$ such that $\sigma_Z^1(r) \geq \sigma_Z^2(r) \geq \cdots \geq \sigma_Z^n(r)$, n_emc is the crisp multiset from $[0,1]$ whose neutral membership sequence is defined by $(\tau_Z^1(r), \tau_Z^2(r), \cdots, \tau_Z^n(r))$ and *nmc* is the crisp multiset from $[0,1]$ whose negative membership sequence is defined by $(\eta_Z^1(r), \eta_Z^2(r), \cdots, \eta_Z^n(r))$, these can be either decreasing or increasing functions satisfying $0 \leq \sigma_Z^k(r) + \tau_Z^k(r) + \eta_Z^k(r) \leq 1 \; \forall r \in Y$, $k = 1, 2, \cdots, n$.

Thus, Z is represented by

$$Z = \{\langle r, \sigma_Z^k(r), \tau_Z^k(r), \eta_Z^k(r)\rangle | r \in Y\}$$

$k = 1,2,\cdots,n$.

Definition 8: Let

$$Z_1 = \{\langle r, \sigma_{Z_1}^k(r), \tau_{Z_1}^k(r)), \eta_{Z_1}^k(r)\rangle |\ r \in\ Y\}$$

and

$$Z_2 = \{\langle r, \sigma_{Z_2}^k(r)), \tau_{Z_2}^k(r), \eta_{Z_2}^k(r))\rangle |\ r \in\ Y\}$$

be two PFMSs drawn from Y. Then,

- $Z_1 \subseteq Z_2, \Leftrightarrow (\sigma_{Z_1}^k(r) \leq \sigma_{Z_2}^k(r)), (\tau_{Z_1}^k(r) \leq \tau_{Z_2}^k(r))$ and $(\eta_{Z_1}^k(r) \geq \eta_{Z_2}^k(r)); k = 1,2,\cdots,n,\ r \in Y$.
- $Z_1 = Z_2, \Leftrightarrow Z_1 \subseteq Z_2$ and $Z_2 \subseteq Z_1$.
- $Z_1 \cup Z_2 = \{(r, (\sigma_{Z_1}^k(r) \vee \sigma_{Z_2}^k(r)), (\tau_{Z_1}^k(r) \wedge \tau_{Z_2}^k(r)), (\eta_{Z_1}^k(r) \wedge \eta_{Z_2}^k(r)))|\ r \in Y\}$, $k = 1,2,\cdots,n$.
- $Z_1 \cap Z_2 = \{(r, (\sigma_{Z_1}^k(r) \wedge \sigma_{Z_2}^k(r))(\tau_{Z_1}^k(r) \wedge \tau_{Z_2}^k(r)),\ (\eta_{Z_1}^k(r) \vee \eta_{Z_2}^k(r)))|\ r \in Y\}$, $k = 1,2,\cdots,n$.
- $Z_1' = \{(r, \eta_{Z_1}^k(r), \tau_{Z_1}^k(r), \sigma_{Z_1}^k(e))|\ r \in\ Y\},\ k = 1,2,\cdots,n$.

Definition 9: Let Z be a nonempty set. Then, a picture fuzzy multirelation (PFMR) U on Z is PFMS defined by

$$U = \{\langle (r_1,r_2), \sigma_U^k(r_1,r_2), \tau_U^k(r_1,r_2), \eta_U^k(r_1,r_2)\rangle | (r_1,r_2) \in Z \times Z\}$$

where $k = 1,2,\cdots,\beta$ (β is the cardinality of the PFMS Z) $\sigma_Z^k(r), \tau_Z^k(r), \eta_Z^k(r) : Y \to W$, and W is the set of all crisp multisets drawn from $[0,1]$.

Definition 10: Let Y be a nonempty set and Z_1 and Z_2 be PFMS in Y with positive membership $\sigma_{Z_1}^k(r)$ and $\sigma_{Z_2}^k(r)$, neutral membership $\tau_{Z_1}^k(r)$ and $\tau_{Z_2}^k(r)$ and negative membership $\eta_{Z_1}^k(r)$ and $\eta_{Z_2}^k(r)$ such that

$$\sigma_{Z_1}^k(r), \sigma_{Z_2}^k(r), \tau_{Z_1}^k(r), \tau_{Z_2}^k(r), \eta_{Z_1}^k(r), \eta_{Z_2}^k(r) : Y \to W$$

and W is the set of all crisp multisets drawn from $[0,1]$. Then, the Cartesian product of Z_1 and Z_2, $Z_1 \times Z_2$ is the PFMS in $Y \times Y$ defined by

$$\sigma_{Z_1 \times Z_2}^k(r_1,r_2) = \bigwedge\{\sigma_{Z_1}^k(r_1), \sigma_{Z_2}^k(r_2)\},$$

$$\tau_{Z_1 \times Z_2}^k(r_1,r_2) = \bigwedge\{\tau_{Z_1}^k(r_1), \tau_{Z_2}^k(r_2)\}$$

and

$$\eta_{Z_1 \times Z_2}^k(r_1,r_2) = \bigvee\{\eta_{Z_1}^k(r_1), \eta_{Z_2}^k(r_2)\}$$

$\forall r_1, r_2 \in Y, k = 1,2,\cdots,\beta$ (β is the cardinality of the PFMS Z_1 and Z_2).

Definition 11: Let U be a PFMS$(Y \times Y)$, $U \subseteq Z_1 \times Z_2$. Then, U is called a PFMR from Z_1 to Z_2 if for all $(r_1,r_2) \in Y \times Y$,

$$\sigma_U^k(r_1,r_2) \leq \sigma_{Z_1 \times Z_2}^k(r_1,r_2),\ \tau_U^k(r_1,r_2) \leq \tau_{Z_1 \times Z_2}^k(r_1,r_2),\ \eta_U^k(r_1,r_2) \geq \eta_{Z_1 \times Z_2}^k(r_1,r_2),$$

with $0 \leq \sigma_U^k(r_1,r_2) + \tau_U^k(r_1,r_2) + \eta_U^k(r_1,r_2) \leq 1$.
In particular, if $Z_1 = Z_2$, then U is called a PFMR on Z_1.
Definition 12: Let $U,V \in PFMR(Z_1 \times Z_2)$. Then, $U \subseteq V$ if for every $r_1,r_2 \in Y$ $(\sigma_U^k(r_1,r_2) \leq \sigma_V^k(r_1,r_2))$, $(\tau_U^k(r_1,r_2) \leq \tau_V^k(r_1,r_2))$ and $(\eta_U^k(r_1,r_2) \geq \eta_V^k(r_1,r_2))$; $k = 1,2,\cdots,n$. If $U \subseteq V$ and $V \subseteq U$, then $U = V$.
Definition 13: Let $U,V \in PFMR(Z_1 \times Z_2)$. Then, $U \cup V$ is a PFMR from Z_1 to Z_2 such that

$$\sigma_{U\cup V}^k(r_1,r_2) = \bigvee\{\sigma_U^k(r_1,r_2), \sigma_V^k(r_1,r_2)\},$$

$$\tau_{U\cup V}^k(r_1,r_2) = \bigwedge\{\tau_U^k(r_1,r_2), \tau_V^k(r_1,r_2)\}$$

and

$$\eta_{U\cup V}^k(r_1,r_2) = \bigwedge\{\eta_U^k(r_1,r_2), \eta_V^k(r_1,r_2)\}$$

$k = 1,2,\cdots,n$.
Definition 14: Let $U,V \in PFMR(Z_1 \times Z_2)$. Then, $U \cap V$ is a PFMR from Z_1 to Z_2 such that

$$\sigma_{U\cap V}^k(r_1,r_2) = \bigwedge\{\sigma_U^k(r_1,r_2), \sigma_V^k(r_1,r_2)\},$$

$$\tau_{U\cap V}^k(r_1,r_2) = \bigwedge\{\tau_U^k(r_1,r_2), \tau_V^k(r_1,r_2)\}$$

and

$$\eta_{U\cap V}^k(r_1,r_2) = \bigvee\{\eta_U^k(r_1,r_2), \eta_V^k(r_1,r_2)\}$$

$k = 1,2,\cdots,n$.
Definition 15: Let $U \in PFMR(Z_1 \times Z_2)$ and $V \in PFMR(Z_2 \times Z_3)$. Then, the composite relation $V \circ U$ is a PFMR between Z_1 and Z_3 defined by

$$V \circ U = \{\langle (r_1,r_3), \sigma_{V\circ U}^k(r_1,r_3), \tau_{V\circ U}^k(r_1,r_3), \eta_{V\circ U}^k(r_1,r_3)\rangle | (r_1,r_3) \in Z_1 \times Z_3\}$$

where $\forall\ (r_1,r_3) \in Z_1 \times Z_3$ and $\forall\ r_2 \in Z_2$, its positive membership, neutral membership and negative membership functions are defined by

$$\sigma_{V\circ U}^k(r_1,r_3) = \bigvee_{r_2\in V}\{\sigma_U^k(r_1,r_2) \wedge \sigma_V^k(r_2,r_3)\},$$

$$\tau_{V\circ U}^k(r_1,r_3) = \bigwedge_{r_2\in V}\{\tau_U^k(r_1,r_2) \wedge \tau_V^k(r_2,r_3)\}$$

and

$$\eta_{V\circ U}^k(r_1,r_3) = \bigwedge_{r_2\in V}\{\eta_U^k(r_1,r_2) \vee \eta_V^k(r_2,r_3)\},$$

respectively.

3. PICTURE FUZZY MULTIRELATIONS OVER PICTURE FUZZY MULTISETS

Definition 16: Let $U \in PFMR(Z \times Z)$. Then, R is called Reflexive if

$$\sigma_U^k(r,r) = 1,\ \tau_U^k(r,r) = 0,\ \text{and}\ \eta_U^k(r,r) = 0.$$

$k = 1, 2, \cdots, \beta$ (β is the cardinality of Z) for all $r \in Z$.

Proposition 1: Let $U \in PFMR(Z \times Z)$ be reflexive. Then,

1. U^{-1} is reflexive if and only if $U = U^{-1}$
2. $U \vee V$ is reflexive for every $V \in PFMR(Z \times Z)$.
3. $U \wedge V$ is reflexive if and only if $V \in PFMR(Z \times Z)$ is reflexive.

Proof:

1. Since U is reflexive, thus for every $r \in Z$;

$$\sigma_U^k(r,r) = 1,\ \tau_U^k(r,r) = 0,\ \text{and}\ \eta_U^k(r,r) = 0.$$

Now, suppose that U^{-1} is reflexive, then for every $r \in Z$,

$$\sigma_{U^{-1}}^k(r,r) = 1,\ \tau_{U^{-1}}^k(r,r) = 0,\ \text{and}\ \eta_{U^{-1}}^k(r,r) = 0.$$

By inverse definition, we have

$$\sigma_{U^{-1}}^k(r,r) = \sigma_U^k(r,r), \tau_{U^{-1}}^k(r,r) = \tau_U^k(r,r), \eta_{U^{-1}}^k(r,r) = \eta_U^k(r,r).$$

So, for all $r \in Z$,

$$\sigma_U^k(r,r) = 1;\ \tau_U^k(r,r) = 0 \text{ and } \eta_U^k(r,r) = 0.$$

Next, to show that $U = U^{-1}$,

$$\sigma_U^k(r_1,r_2) = \sigma_U^k(r_2,r_1);\ \tau_U^k(r_1,r_2) = \tau_U^k(r_2,r_1) \text{ and } \eta_U^k(r_1,r_2) = \eta_U^k(r_2,r_1).$$

Using the reflexivity of U and U^{-1} for all pairs (r_1, r_2) we have

$$\sigma_U^k(r_1,r_2) = \sigma_{U^{-1}}^k(r_2,r_1) = \sigma_U^k(r_2,r_1)$$

$$\tau_U^k(r_1,r_2) = \tau_{U^{-1}}^k(r_2,r_1) = \tau_U^k(r_2,r_1)$$

and

$$\eta_U^k(r_1,r_2) = \eta_{U^{-1}}^k(r_2,r_1) = \eta_U^k(r_2,r_1).$$

Hence, $U = U^{-1}$.

Conversely, suppose that $U = U^{-1}$. Since U is reflexive then for every $r \in Z$,

$$\sigma_U^k(r,r) = 1;\ \tau_U^k(r,r) = 0 \text{ and } \eta_U^k(r,r) = 0.$$

Since $U = U^{-1}$, we have for all $r_1, r_2 \in Z$

$$\sigma_U^k(r_1,r_2) = \sigma_U^k(r_2,r_1);\ \tau_U^k(r_1,r_2) = \tau_U^k(r_2,r_1) \text{ and } \eta_U^k(r_1,r_2) = \eta_U^k(r_2,r_1).$$

Thus, for the U^{-1}, we have

$$\sigma^k_{U^{-1}}(r,r) = \sigma^k_U(r,r) = 1$$

$$\tau^k_{U^{-1}}(r,r) = \tau^k_U(r,r) = 0$$

and

$$\eta^k_{U^{-1}}(r,r) = \eta^k_U(r,r) = 0.$$

Therefore, U^{-1} is reflexive.

2.

$$\sigma^k_{U\vee V}(r,r) = \sigma^k_U(r,r) \vee \sigma^k_V(r,r) = 1 \vee \sigma^k_V(r,r) = 1$$

$$\tau^k_{U\vee V}(r,r) = \tau^k_U(r,r) \wedge \tau^k_V(r,r) = 0 \wedge \tau^k_V(r,r) = 0$$

$$\eta^k_{U\vee V}(r,r) = \eta^k_U(r,r) \wedge \eta^k_V(r,r) = 0 \wedge \eta^k_V(r,r) = 0.$$

Hence, $U \vee V$ is reflexive.

3.

$$\sigma^k_{U\wedge V}(r,r) = \sigma^k_U(r,r) \wedge \sigma^k_V(r,r) = 1 \wedge \sigma^k_V(r,r) = 1$$

$$\tau^k_{U\wedge V}(r,r) = \tau^k_U(r,r) \wedge \tau^k_V(r,r) = 0 \wedge \tau^k_V(r,r) = 0$$

$$\eta^k_{U\wedge V}(r,r) = \eta^k_U(r,r) \vee \eta^k_V(r,r) = 0 \vee \eta^k_V(r,r) = 0.$$

Hence, $U \wedge V$ is reflexive if and only if V is reflexive.

Proposition 2: If U and V are reflexive PFMRs, then (i) $U \cup V$ and (ii) $U \cap V$ are reflexive PFMRs.

Proof: Since U and V are reflexive, thus for every $r \in Z$;

$$\sigma^k_U(r,r) = 1,\ \tau^k_U(r,r) = 0,\ \text{and}\ \eta^k_U(r,r) = 0.$$

$$\sigma^k_V(r,r) = 1,\ \tau^k_V(r,r) = 0,\ \text{and}\ \eta^k_V(r,r) = 0.$$

$k = 1, 2, \cdots, \beta$ (β is the cardinality of Z) for all $r \in Z$.

Now,

(i) $U \cup V$

$$\begin{aligned} \sigma^k_{U\cup V}(r,r) &= \sigma^k_U(r,r) \vee \sigma^k_V(r,r) \\ &= 1 \vee 1 \\ &= 1, \end{aligned}$$

$$\begin{aligned} \tau^k_{U\cup V}(r,r) &= \tau^k_U(r,r) \wedge \tau^k_V(r,r) \\ &= 0 \wedge 0 \\ &= 0, \end{aligned}$$

$$\begin{aligned} \eta^k_{U\cup V}(r,r) &= \eta^k_U(r,r) \wedge \eta^k_V(r,r) \\ &= 0 \wedge 0 \\ &= 0. \end{aligned}$$

Hence, $U \cup V$ is reflexive.
(ii) $U \cap V$

$$\begin{aligned} \sigma^k_{U\cap V}(r,r) &= \sigma^k_U(r,r) \wedge \sigma^k_V(r,r) \\ &= 1 \wedge 1 \\ &= 1, \end{aligned}$$

$$\begin{aligned} \tau^k_{U\cap V}(r,r) &= \tau^k_U(r,r) \wedge \tau^k_V(r,r) \\ &= 0 \wedge 0 \\ &= 0, \end{aligned}$$

$$\begin{aligned} \eta^k_{U\cap V}(r,r) &= \eta^k_U(r,r) \vee \eta^k_V(r,r) \\ &= 0 \vee 0 \\ &= 0. \end{aligned}$$

Hence, $U \cap V$ is reflexive.

Definition 17: Let $U \in PFMR(Z \times Z)$. Then, U is called Symmetric if

$\sigma^k_U(r_1,r_2) = \sigma^k_U(r_2,r_1)$, $\tau^k_U(r_1,r_2) = \tau^k_U(r_2,r_1)$, and $\eta^k_U(r_1,r_2) = \eta^k_U(r_2,r_1)$.

$i = 1,2,\cdots,\beta$ (β is the cardinality of Z) for all $r_1,r_2 \in Z$.

Proposition 3: If U is symmetric then U^{-1} is also symmetric.

Proof: U is symmetric means that

$\sigma^k_U(r_1,r_2) = \sigma^k_U(r_2,r_1)$, $\tau^k_U(r_1,r_2) = \tau^k_U(r_2,r_1)$, and $\eta^k_U(r_1,r_2) = \eta^k_U(r_2,r_1)$.

Also, since U^{-1} is an inverse relation, then

$\sigma^k_{U^{-1}}(r_1,r_2) = \sigma^k_U(r_2,r_1)$, $\tau^k_{U^{-1}}(r_1,r_2) = \tau^k_U(r_2,r_1)$, and $\eta^k_{U^{-1}}(r_1,r_2) = \eta^k_U(r_2,r_1)$.

$$\sigma^k_{U^{-1}}(r_1,r_2) = \sigma^k_U(r_2,r_1) = \sigma^k_U(r_1,r_2) = \sigma^k_{U^{-1}}(r_2,r_1),$$
$$\tau^k_{U^{-1}}(r_1,r_2) = \tau^k_U(r_2,r_1) = \tau^k_U(r_1,r_2) = \tau^k_{U^{-1}}(r_2,r_1)$$

and

$$\eta^k_{U^{-1}}(r_1,r_2) = \eta^k_U(r_2,r_1) = \eta^k_U(r_1,r_2) = \eta^k_{U^{-1}}(r_2,r_1).$$

Proposition 4: Let $U \in PFMR(Z \times Z)$. Then, U is symmetric if and only if $U = U^{-1}$.

Proof: Suppose that U is symmetric, then

$\sigma^k_U(r_1,r_2) = \sigma^k_U(r_2,r_1)$, $\tau^k_U(r_1,r_2) = \tau^k_U(r_2,r_1)$, and $\eta^k_U(r_1,r_2) = \eta^k_U(r_2,r_1)$.

Since U^{-1} is an inverse relation,

$\sigma^k_{U^{-1}}(r_1,r_2) = \sigma^k_U(r_2,r_1)$, $\tau^k_{U^{-1}}(r_1,r_2) = \tau^k_U(r_2,r_1)$, and $\eta^k_{U^{-1}}(r_1,r_2) = \eta^k_U(r_2,r_1)$.

Now,

$$\sigma^k_{U^{-1}}(r_1,r_2) = \sigma^k_U(r_2,r_1) = \sigma^k_U(r_1,r_2),$$
$$\tau^k_{U^{-1}}(r_1,r_2) = \tau^k_U(r_2,r_1) = \tau^k_U(r_1,r_2)$$

and

$$\eta^k_{U^{-1}}(r_1,r_2) = \eta^k_U(r_2,r_1) = \eta^k_U(r_1,r_2).$$

Therefore, $U = U^{-1}$.
Conversely, suppose that $U = U^{-1}$, then

$$\sigma^k_U(r_1,r_2) = \sigma^k_{U^{-1}}(r_1,r_2) = \sigma^k_U(r_2,r_1),$$

$$\tau^k_U(r_1,r_2) = \tau^k_{U^{-1}}(r_1,r_2) = \tau^k_U(r_2,r_1)$$

and

$$\eta^k_U(r_1,r_2) = \eta^k_{U^{-1}}(r_1,r_2) = \eta^k_U(r_2,r_1).$$

Therefore, U is symmetric.
Proposition 5: If U and V are symmetric PFMRs, then $U \cap V$ and $U \cup V$ are symmetric PFMRs.
Proof: If U and V are symmetric PFMRs, then

$\sigma^k_U(r_1,r_2) = \sigma^k_U(r_2,r_1)$, $\tau^k_U(r_1,r_2) = \tau^k_U(r_2,r_1)$, and $\eta^k_U(r_1,r_2) = \eta^k_U(r_2,r_1)$.

and

$\sigma^k_V(r_1,r_2) = \sigma^k_V(r_2,r_1)$, $\tau^k_V(r_1,r_2) = \tau^k_V(r_2,r_1)$, and $\eta^k_V(r_1,r_2) = \eta^k_V(r_2,r_1)$.

Now,
• $\sigma^k_{U\wedge V}(r_1,r_2) = \sigma^k_U(r_1,r_2) \wedge \sigma^k_V(r_1,r_2) = \sigma^k_U(r_2,r_1) \wedge \sigma^k_V(r_2,r_1) = \sigma^k_{U\wedge V}(r_2,r_1)$, $\tau^k_{U\wedge V}(r_1,r_2) = \tau^k_U(r_1,r_2) \wedge \tau^k_V(r_1,r_2) = \tau^k_U(r_2,r_1) \wedge \tau^k_V(r_2,r_1) = \tau^k_{U\wedge V}(r_2,r_1)$ and $\eta^k_{U\wedge V}(r_1,r_2) = \eta^k_U(r_1,r_2) \vee \eta^k_V(r_1,r_2) = \eta^k_U(r_2,r_1) \vee \eta^k_V(r_2,r_1) = \eta^k_{U\wedge V}(r_2,r_1)$.
Hence, $U \cap V$ is symmetric PFMR.

• $\sigma^k_{U\vee V}(r_1,r_2) = \sigma^k_U(r_1,r_2) \vee \sigma^k_V(r_1,r_2) = \sigma^k_U(r_2,r_1) \vee \sigma^k_V(r_2,r_1) = \sigma^k_{U\vee V}(r_2,r_1)$, $\tau^k_{U\vee V}(r_1,r_2) = \tau^k_U(x,y) \wedge \tau^k_V(r_1,r_2) = \tau^k_U(r_2,r_1) \wedge \tau^k_V(r_2,r_1) = \tau^k_{U\vee V}(r_2,r_1)$ and $\eta^k_{U\vee V}(r_1,r_2) = \eta^k_U(r_1,r_2) \wedge \eta^k_V(r_1,r_2) = \eta^k_U(r_2,r_1) \wedge \eta^k_V(r_2,r_1) = \eta^k_{U\vee V}(r_2,r_1)$.
Hence, $U \cup V$ is symmetric PFMR.
Remark 1: Note that; $U \circ V$ is not symmetric in general because,

$$\begin{aligned} \sigma^k_{U\circ V}(r_1,r_3) &= \bigvee_{r_2}\{\sigma^k_V(r_1,r_2) \wedge (\sigma^k_U(r_2,r_3)\} \\ &= \bigvee_{r_2}\{\sigma^k_V(r_2,r_1) \wedge (\sigma^k_U(r_3,r_2)\} \\ &\neq \sigma^k_{U\circ V}(r_3,r_1), \end{aligned}$$

$$\begin{aligned}
\tau^k_{U\circ V}(r_1,r_3) &= \bigwedge_{r_2}\{\tau^k_V(r_1,r_2)\wedge(\tau^k_U(r_2,r_3)\} \\
&= \bigwedge_{r_2}\{\tau^k_V(r_2,r_1)\wedge(\tau^k_U(r_3,r_2)\} \\
&\neq \tau^k_{U\circ V}(r_3,r_1)
\end{aligned}$$

and

$$\begin{aligned}
\eta^k_{U\circ V}(r_1,r_3) &= \bigwedge_{r_2}\{\eta^k_V(r_1,r_2)\vee(\eta^k_U(r_2,r_3)\} \\
&= \bigwedge_{r_2}\{\eta^k_V(r_2,r_1)\vee(\eta^k_U(r_3,r_2)\} \\
&\neq \eta^k_{U\circ V}(r_3,r_1)
\end{aligned}$$

Example 1: Let

$$Z_1=\{1,4\},\ Z_2=\{2,5\} \text{ and } Z_3=\{3,6\}.$$

Define relation on U as

$$\sigma_U(1,2)=0.7,\ \tau_U(1,2)=0.2 \text{ and } \eta_U(1,2)=0.1$$

and

$$\sigma_U(4,5)=0.8,\ \tau_U(4,5)=0.1 \text{ and } \eta_U(4,5)=0.1$$

Define relation on V as

$$\sigma_V(2,3)=0.6,\ \tau_V(2,3)=0.3 \text{ and } \eta_V(2,3)=0.1$$

and

$$\sigma_V(5,6)=0.5,\ \tau_V(5,6)=0.4 \text{ and } \eta_V(5,6)=0.1$$

Let $P=U\circ V$. For $r_1=1$ and $r_3=3$ we have

$$\sigma_P(1,3)=\bigvee(\sigma_U(1,2)\wedge\sigma_V(2,3))=0.7\wedge 0.6=0.6,$$

$$\tau_P(1,3)=\bigwedge(\tau_U(1,2)\wedge\tau_V(2,3))=0.2\wedge 0.3=0.2$$

and

$$\eta_P(1,3)=\bigwedge(\eta_U(1,2)\vee\eta_V(2,3))=0.1\vee 0.1=0.1.$$

For $r_1=3$ and $r_3=1$, let $P'=V\circ U$. Since there is no direct path from 3 to 1 via U and V, there is a need to check for possible compositions. Suppose that a possible set for U and V defined backward,

$$\sigma_{P'}(3,1)=0 \text{ (if no composition exists)},$$

$$\tau_{P'}(3,1)=1 \text{ (if no composition exists)}$$

and

$$\eta_{P'}(3,1)=0 \text{ (if no composition exists)}.$$

Thus, $\sigma_P(1,3) \neq \sigma_{P'}(3,1)$, $\tau_P(1,3) \neq \tau_{P'(3,1)}$ and $\eta_P(1,3) \neq \eta_{P'(3,1)}$. Similarly,

$$\sigma_P(4,6) = \bigvee(\sigma_U(4,5) \wedge \sigma_V(5,6)) = 0.8 \wedge 0.5 = 0.5,$$

$$\tau_P(4,6) = \bigwedge(\tau_U(4,5) \wedge \tau_V(5,6)) = 0.1 \wedge 0.4 = 0.1$$

and

$$\eta_P(4,6) = \bigwedge(\eta_U(4,5) \vee \eta_V(5,6)) = 0.1 \vee 0.1 = 0.1.$$

Also, there is no direct path from 6 to 4 via U and V,

$$\sigma_{P'(6,4)} = 0 \text{ (if no composition exists)},$$

$$\tau_{P'(6,4)} = 1 \text{ (if no composition exists)}$$

and

$$\eta_{P'(6,4)} = 0 \text{ (if no composition exists)}.$$

Thus, $\sigma_P(4,6) \neq \sigma_{P'(6,4)}$, $\tau_P(4,6) \neq \tau_{P'(6,4)}$ and $\eta_P(4,6) \neq \eta_{P'(6,4)}$. Hence, $U \circ V \neq V \circ U$. Therefore, composition of PFMRs is not true in general.

Proposition 6: Given $U \in PFMR(Z_1 \times Z_2)$ and $V \in PFMR(Z_2 \times Z_3)$. Then, $U \circ V$ is symmetric if and only if $U \circ V = V \circ U$, for symmetric relations U and V.

Proof:

$$\begin{aligned}
\sigma^k_{U\circ V}(r_1,r_3) &= \bigvee_{r_2}\{\sigma^k_V(r_1,r_2) \wedge \sigma^k_U(r_2,r_3)\} \\
&= \bigvee_{r_2}\{\sigma^k_V(r_2,r_1) \wedge (\sigma^k_U(r_3,r_2)\} \\
&= \bigvee_{r_2}\{\sigma^k_U(r_3,r_2) \wedge (\sigma^k_V(r_2,r_1)\} \\
&= \sigma^k_{U\circ V}(r_3,r_1),
\end{aligned}$$

$$\begin{aligned}
\tau^k_{U\circ V}(r_1,r_3) &= \bigwedge_{r_2}\{\tau^k_V(r_1,r_2) \wedge \tau^k_U(r_2,r_3)\} \\
&= \bigwedge_{r_2}\{\tau^k_V(r_2,r_1) \wedge (\tau^k_U(r_3,r_2)\} \\
&= \bigwedge_{r_2}\{\tau^k_U(r_3,r_2) \wedge (\tau^k_V(r_2,r_1)\} \\
&= \tau^k_{U\circ V}(r_3,r_1)
\end{aligned}$$

and

$$\begin{aligned}\eta^k_{U\circ V}(r_1,r_3) &= \bigwedge_{r_2}\{\eta^k_V(r_1,r_2)\vee\eta^k_U(r_2,r_3)\}\\ &= \bigwedge_{r_2}\{\eta^k_V(r_2,r_1)\vee(\eta^k_U(r_3,r_2)\}\\ &= \bigwedge_{r_2}\{\eta^k_U(r_3,r_2)\vee(\eta^k_V(r_2,r_1)\}\\ &= \eta^k_{U\circ V}(r_3,r_1)\end{aligned}$$

for every $(r_1,r_3)\in Z_1\times Z_3$ and for every $r_2\in Z_2$.

Definition 18: Let $U\in PFMR(Z\times Z)$. Then, U is called Transitive if for every triplet (r_1,r_2,r_3) in $Z\times Z\times Z$ whenever (r_1,r_2) and $(r_2,r_3)\in U$ with certain degrees of relatedness $\sigma^k_U(r_1,r_2)$ and $\sigma^k_U(r_2,r_3)$; $\tau^k_U(r_1,r_2)$ and $\tau^k_U(r_2,r_3)$; $\eta^k_U(r_1,r_2)$ and $\eta^k_U(r_2,r_3)$ then $(r_1,r_3)\in U$ with a degree of relatedness $\sigma^k_U(r_1,r_3)\geq min\{\sigma^k_U(r_1,r_2),\sigma^k_U(r_2,r_3)\}$; $\tau^k_U(r_1,r_3)\leq max\{\tau^k_U(r_1,r_2),\tau^k_U(r_2,r_3)\}$; $\eta^k_U(r_1,r_3)\leq max\{\eta^k_U(r_1,r_2),\eta^k_U(r_2,r_3)\}$, respectively.

Proposition 7: Let U be a transitive relation. Then U^{-1} is transitive if and only if $U=U^{-1}$.

Proof: Suppose that U^{-1} is transitive, for every $r_1,r_2,r_3\in Z_1\times Z_2\times Z_3$,

$$\sigma^k_{U^{-1}}(r_1,r_3)\geq\bigvee_{r_2\in Z_2}\left(\sigma^k_{U^{-1}}(r_1,r_2)\wedge\sigma^k_{U^{-1}}(r_2,r_3)\right),$$

$$\tau^k_{U^{-1}}(r_1,r_3)\leq\bigwedge_{r_2\in Z_2}\left(\tau^k_{U^{-1}}(r_1,r_2)\vee\tau^k_{U^{-1}}(r_2,r_3)\right)$$

and

$$\eta^k_{U^{-1}}(r_1,r_3)\leq\bigwedge_{r_2\in Z_2}\left(\eta^k_{U^{-1}}(r_1,r_2)\vee\eta^k_{U^{-1}}(r_2,r_3)\right)$$

By inverse definition,

$$\sigma^k_{U^{-1}}(r_1,r_3)=\sigma^k_U(r_3,r_1),\ \tau^k_{U^{-1}}(r_1,r_3)=\eta^k_U(r_3,r_1),\eta^k_{U^{-1}}(r_1,r_3)=\eta^k_U(r_3,r_1)$$

Substitute these into the transitivity conditions for U^{-1}, we have

$$\sigma^k_U(r_3,r_1)\geq\bigvee_{r_2\in Z_2}\left(\sigma^k_U(r_2,r_1)\wedge\sigma^k_U(r_3,r_2)\right),$$

$$\tau^k_U(r_3,r_1)\leq\bigwedge_{r_2\in Z_2}\left(\tau^k_U(r_2,r_1)\vee\tau^k_U(r_3,r_2)\right)$$

and

$$\eta^k_U(r_3,r_1)\leq\bigwedge_{r_2\in Z_2}\left(\eta^k_U(r_2,r_1)\vee\eta^k_U(r_3,r_2)\right)$$

Since U is transitive, it means that U is symmetric. Thus,
$\sigma_U^k(r_1,r_2)=\sigma_U^k(r_2,r_1)$, $\tau_U^k(r_1,r_2)=\tau_U^k(r_2,r_1)$ and $\eta_U^k(r_1,r_2)=\eta_U^k(r_2,r_1)$
Hence, $U=U^{-1}$.
Conversely, suppose that $U=U^{-1}$. Thus,
$\sigma_U^k(r_1,r_2)=\sigma_U^k(r_2,r_1)$, $\tau_U^k(r_1,r_2)=\tau_U^k(r_2,r_1)$ and $\eta_U^k(r_1,r_2)=\eta_U^k(r_2,r_1)$
Since U is transitive, for every $r_1,r_2,r_3\in Z_1\times Z_2\times Z_3$,

$$\sigma_U^k(r_1,r_3)\geq\bigvee_{r_2\in Z_2}\left(\sigma_U^k(r_1,r_2)\wedge\sigma_U^k(r_2,r_3)\right),$$

$$\tau_U^k(r_1,r_3)\leq\bigwedge_{r_2\in Z_2}\left(\tau_U^k(r_1,r_2)\vee\tau_U^k(r_2,r_3)\right)$$

and

$$\eta_U^k(r_1,r_3)\leq\bigwedge_{r_2\in Z_2}\left(\eta_U^k(r_1,r_2)\vee\eta_U^k(r_2,r_3)\right)$$

Since $U=U^{-1}$, it means that transitivity conditions for U can be written in terms of U^{-1}. So,

$$\begin{aligned}\sigma_{U^{-1}}^k(r_1,r_3) &= \sigma_U^k(r_3,r_1)\\ &\geq \bigvee_{r_2\in Z_2}\left(\sigma_U^k(r_3,r_2)\wedge\sigma_U^k(r_2,r_1)\right)\\ &= \bigvee_{r_2\in Z_2}\left(\sigma_{U^{-1}}^k(r_1,r_2)\wedge\sigma_{U^{-1}}^k(r_2,r_3)\right)\end{aligned}$$

$$\begin{aligned}\tau_{U^{-1}}^k(r_1,r_3) &= \tau_U^k(r_3,r_1)\\ &\leq \bigwedge_{r_2\in Z_2}\left(\tau_U^k(r_3,r_2)\vee\tau_U^k(r_2,r_1)\right)\\ &= \bigwedge_{r_2\in Z_2}\left(\tau_{U^{-1}}^k(r_1,r_2)\vee\tau_{U^{-1}}^k(r_2,r_3)\right)\end{aligned}$$

$$\begin{aligned}\eta_{U^{-1}}^k(r_1,r_3) &= \eta_U^k(r_3,r_1)\\ &\leq \bigwedge_{r_2\in Z_2}\left(\eta_U^k(r_3,r_2)\vee\eta_U^k(r_2,r_1)\right)\\ &= \bigwedge_{r_2\in Z_2}\left(\eta_{U^{-1}}^k(r_1,r_2)\vee\eta_{U^{-1}}^k(r_2,r_3)\right)\end{aligned}$$

Hence, U^{-1} is transitive.

Proposition 8:

Proof: First to show that $U \cap V$ is transitive for transitive U and V

$$\sigma^k_{U\cap V}(r_1,r_2) = min\{\sigma^k_U(r_1,r_2), \sigma^k_U(r_1,r_2)\},$$
$$\tau^k_{U\cap V}(r_1,r_2) = max\{\tau^k_U(r_1,r_2), \tau^k_U(r_1,r_2)\}$$

and

$$\eta^k_{U\cap V}(r_1,r_2) = max\{\eta^k_U(r_1,r_2), \eta^k_U(r_1,r_2)\}.$$

For $\sigma^k_{U\cap V}(r_1,r_3)$,

$$\sigma^k_{U\cap V}(r_1,r_3) = min\{\sigma^k_U(r_1,r_2), \sigma^k_U(r_2,r_3)\}.$$

Since U and V are transitive, we have,

$$\sigma^k_U(r_1,r_3) \geq min\left(\sigma^k_U(r_1,r_2), \sigma^k_U(r_2,r_3)\right)$$

and

$$\sigma^k_V(r_1,r_3) \geq min\left(\sigma^k_V(r_1,r_2), \sigma^k_V(r_2,r_3)\right)$$

Thus,

$$min\left(\sigma^k_U(r_1,r_3), \sigma^k_V(r_1,r_3)\right) \geq min\left[min\left(\sigma^k_U(r_1,r_2), \sigma^k_U(r_2,r_3)\right), min\left(\sigma^k_V(r_1,r_2), \sigma^k_V(r_2,r_3)\right)\right]$$

since

$$min\left(\sigma^k_{U\cap V}(r_1,r_2), \sigma^k_{U\cap V}(r_2,r_3)\right) = min\left[min\left(\sigma^k_U(r_1,r_2), \sigma^k_V(r_1,r_2)\right), min\left(\sigma^k_U(r_2,r_3), \sigma^k_V(r_2,r_3)\right)\right]$$

we have

$$\sigma^k_{U\cap V}(r_1,r_3) \geq min\left(\sigma^k_{U\cap V}(r_1,r_2), \sigma^k_{U\cap V}(r_2,r_3)\right)$$

For $\tau^k_{U\cap V}(r_1,r_3)$,
$\tau^k_{U\cap V}(r_1,r_3) = max\{\tau^k_U(r_1,r_2), \tau^k_U(r_2,r_3)\}$. Since U and V are transitive, we have,

$$\tau^k_U(r_1,r_3) \leq max\left(\tau^k_U(r_1,r_2), \tau^k_U(r_2,r_3)\right)$$

and

$$\tau^k_V(r_1,r_3) \leq max\left(\tau^k_V(r_1,r_2), \tau^k_V(r_2,r_3)\right)$$

Thus,

$$max\left(\tau^k_U(r_1,r_3), \tau^k_V(r_1,r_3)\right) \leq max\left[max\left(\tau^k_U(r_1,r_2), \tau^k_U(r_2,r_3)\right), max\left(\tau^k_V(r_1,r_2), \tau^k_V(r_2,r_3)\right)\right]$$

since

$$max\left(\tau^k_{U\cap V}(r_1,r_2), \tau^k_{U\cap V}(r_2,r_3)\right) = max\left[max\left(\tau^k_U(r_1,r_2), \tau^k_V(r_1,r_2)\right), max\left(\tau^k_U(r_2,r_3), \tau^k_V(r_2,r_3)\right)\right]$$

we have

$$\tau^k_{U\cap V}(r_1,r_3) \le max\left(\tau^k_{U\cap V}(r_1,r_2),\tau^k_{U\cap V}(r_2,r_3)\right)$$

For $\eta^k_{U\cap V}(r_1,r_3)$,
$\eta^k_{U\cap V}(r_1,r_3) = max\{\eta^k_U(r_1,r_2),\eta^k_U(r_2,r_3)\}$. Since U and V are transitive, we have,

$$\eta^k_U(r_1,r_3) \le max\left(\eta^k_U(r_1,r_2),\eta^k_U(r_2,r_3)\right)$$

and

$$\eta^k_V(r_1,r_3) \le max\left(\eta^k_V(r_1,r_2),\eta^k_V(r_2,r_3)\right)$$

Thus,

$$max\left(\eta^k_U(r_1,r_3),\eta^k_V(r_1,r_3)\right) \le max\left[max\left(\eta^k_U(r_1,r_2),\eta^k_U(r_2,r_3)\right),max\left(\eta^k_V(r_1,r_2),\eta^k_V(r_2,r_3)\right)\right]$$

since

$$max\left(\eta^k_{U\cap V}(r_1,r_2),\eta^k_{U\cap V}(r_2,r_3)\right) = max\left[max\left(\eta^k_U(r_1,r_2),\eta^k_V(r_1,r_2)\right),max\left(\eta^k_U(r_2,r_3),\eta^k_V(r_2,r_3)\right)\right]$$

we have

$$\eta^k_{U\cap V}(r_1,r_3) \le max\left(\eta^k_{U\cap V}(r_1,r_2),\eta^k_{U\cap V}(r_2,r_3)\right)$$

Next to show that $U\cup V$ not transitive for transitive U and V. This will be done by counter example.
Let $Z = \{1,2,3\}$.
Define relation U as

$$\sigma_U(1,2) = 0.9,\ \tau_U(1,2) = 0.1 \text{ and } \eta_U(1,2) = 0.0,$$
$$\sigma_U(2,3) = 0.8,\ \tau_U(2,3) = 0.1 \text{ and } \eta_U(2,3) = 0.1$$
$$\sigma_U(1,3) = 0.7,\ \tau_U(1,3) = 0.2 \text{ and } \eta_U(1,3) = 0.1$$

Also, define relation V as

$$\sigma_V(1,2) = 0.7,\ \tau_V(1,2) = 0.2 \text{ and } \eta_V(1,2) = 0.1,$$
$$\sigma_V(2,3) = 0.6,\ \tau_V(2,3) = 0.3 \text{ and } \eta_V(2,3) = 0.1$$
$$\sigma_V(1,3) = 0.5,\ \tau_V(1,3) = 0.4 \text{ and } \eta_V(1,3) = 0.1$$

For U,

$$\sigma_U(1,3) = 0.7 \ge min\{0.9,0.8\} = 0.8,$$
$$\tau_U(1,3) = 0.2 \le max\{0.1,0.1\} = 0.1$$

and

$$\eta_U(1,3) = 0.2 \le max\{0.0,0.1\} = 0.1.$$

For V,

$$\sigma_V(1,3) = 0.5 \ge min\{0.7,0.6\} = 0.6,$$

$$\tau_V(1,3) = 0.4 \leq max\{0.2, 0.3\} = 0.3$$

and

$$\eta_U(1,3) = 0.1 \leq max\{0.1, 0.1\} = 0.1.$$

Now, for $U \cup V$,

$$\sigma_{U\cup V}(1,2) = max\{0.9, 0.7\} = 0.9,$$

$$\tau_{U\cup V}(1,2) = min\{0.1, 0.2\} = 0.1$$

and

$$\eta_{U\cup V}(1,2) = min\{0.0, 0.1\} = 0.0$$

$$\sigma_{U\cup V}(2,3) = max\{0.8, 0.6\} = 0.8,$$

$$\tau_{U\cup V}(2,3) = min\{0.1, 0.3\} = 0.1$$

and

$$\eta_{U\cup V}(2,3) = min\{0.1, 0.1\} = 0.1$$

$$\sigma_{U\cup V}(1,3) = max\{0.7, 0.5\} = 0.7,$$

$$\tau_{U\cup V}(1,3) = min\{0.2, 0.4\} = 0.2$$

and

$$\eta_{U\cup V}(1,3) = min\{0.1, 0.1\} = 0.1$$

So, checking transitivity for $U \cup V$;

$\sigma_{U\cup V}(1,3) = 0.7 \neq min\{\sigma_{U\cup V}(1,2), \sigma_{U\cup V}(2,3)\} = min\{0.9, 0.8\} = 0.8.$

Since $0.7 < 0.8$, i.e

$$\sigma_{U\cup V}(1,3) \neq min\{\sigma_{U\cup V}(1,2), \sigma_{U\cup V}(2,3)\}$$

which implies that $\sigma_{U\cup V}(1,3)$ does not satisfy the transitivity condition. Similarly $\tau_{U\cup V}(1,3)$ and $\tau_{U\cup V}(1,3)$ satisfy not the transitivity condition.
Therefore, $U \cup V$ not transitive.
Remark 2: A PFMR R is said to be Picture Fuzzy Multi Equivalence Relation (PFMER) if it satisfies reflexive, symmetric and transitive.

4. CONCLUDING REMARKS

In this paper, it has been established that the Picture Fuzzy MultiRelations (PFMR) are extension of the Picture Fuzzy Relations (PFR). Reflexivity, symmetric and transitivity of picture fuzzy multirelations over picture fuzzy multisets were introduced and some properties associated with them were obtained.

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