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SOME NOTIONS ON CONVEXITY OF PICTURE FUZZY SETS

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ABSTRACT. The theory of picture fuzzy sets was formulated by Cuong and Kreinovich and is considered as one of the best effective tool in decision making problems. In this paper, the concept of picture fuzzy sets was studied. The notion of picture convex fuzzy sets was introduced and some of its properties are presented. Finally, we put forward the picture affine fuzzy sets and investigate some of its characteristics.

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1. INTRODUCTION

In 1965, Zadeh [1] introduced fuzzy sets (FS) which is most relevant in making decisions in an uncertain environment. He also initiated convexity of fuzzy sets as an extension of classical convex sets. In 1980, Lowen [2] extended Zadeh's work by introducing the concept of affine fuzzy sets in order to study convex fuzzy sets in greater details and investigated the properties of both convex and affine fuzzy sets using Euclidean points. After major contribution of Lowen to Zadeh's work on convex fuzzy sets, many researchers have also studied the concept, such as, Liu [3] 1985 established some properties of convex fuzzy sets. Yang [4] 1988, established the relationship among three types of convex fuzzy sets. Feiyue [5] 1991, extended Lowen's work by introducing affine hulls of fuzzy sets and established that the affine combination of a set of fuzzy points is an affine fuzzy set. This was also established for convex fuzzy sets. In 2002, Yang and Yang [6] were able to characterised convexity of

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fuzzy sets via semicontinuous conditions. Peng [7] 2010, proposed two new definitions of convex fuzzy sets, i.e., convex $(s,t]$-fuzzy sets via relation between fuzzy points and fuzzy subsets, and convex R-fuzzy sets via implication operators of fuzzy logic. In 2010, Cheng et al [8] put forward the concept of $(s,t]$-intuitionistic convex fuzzy sets via cut set of intuitionistic fuzzy sets and neighborhood relations between fuzzy point and intuitionistic fuzzy set. In 2017, Sangodapo and Ajayi [9] contributed to Lowen's work by establishing some properties of both convex and affine fuzzy sets using fuzzy points. Many researchers have also studied and developed various properties of convex fuzzy sets and affine fuzzy sets, see [3, 4, 10, 11, 12, 13, 14, 15].

In 1986, Atanassov [16] initiated the concept of intuitionistic fuzzy sets (IFSs) as one of the most generalisations of fuzzy sets in order to handle uncertainties more accurately. In 2012, Huang [17] generalised convex fuzzy sets to convex intuitionistic fuzzy sets and obtained equal characteristics in terms of cut sets. In 2018, Das and Mukhlalsah [18] also studied intuitionistic fuzy sets and developed intuitionistic concex fuzzy sets, established various theorems and illustrated with examples. In 2021, Sangodapo and Ajayi [19] extended the Huang's work to affine intuitionistic fuzzy sets and also established its characteristics.

Cuong and Kreinovich [20] in 2013, generalised Zadeh and Atanassov's works by putting forward the theory of PFSs. This theory is a new concept for computational intelligence problems to handle an important notion of neutrality degree that was lacking in IFSs theory. This degree of neutrality can be seen in a voting environment where voters may decide to: vote for, abstain, vote against and refuse of voting.

In 2014, Cuong [21] investigated some characteristics of PFSs, introduced distance measure and defined convex combination between two PFSs. Son [22] in 2016, introduced a generalised picture distance measure and applied it to establish an Hierarchical Picture Clustering. In 2017, Dutta and Ganju [23] studied decomposition theorems of PFSs, defined (α,γ,β)-cut of PFSs, and extension principle for PFSs and obtained their properties. PFSs has been extensively studied and applied, see [24, 23, 25, 26, 27, 28] for details.

In this paper, we continue the study of PFSs and introduce picture convex fuzzy sets (PCFSs) and picture affine fuzzy sets (PAFSs).

Finally, the characterisations of PCFSs and PAFSs were also established. The organisations of the paper is as follows. In section 2, we give necessary definitions and preliminary ideas of PFSs used throughout the paper. Section 3 introduces some properties of PCFSs and investigates its characterisations. In section 4, we introduce PAFSs and establish its characterisations. Section 5 concludes the paper and suggests direction for future research. In 1965, Zadeh [1] introduced fuzzy sets (FS) which is most relevant in making decisions in an uncertain environment. He also initiated convexity of fuzzy sets as an extension of classical convex sets. In 1980, Lowen [2] extended Zadeh's work by introducing the concept of affine fuzzy sets in order to study convex fuzzy sets in greater details and investigated the properties of both convex and affine fuzzy sets using Euclidean points. After major contribution of Lowen to Zadeh's work on convex fuzzy sets, many researchers have also studied the concept, such as, Liu [3] 1985 established some properties of convex fuzzy sets. Yang [4] 1988, established the relationship among three types of convex fuzzy sets. Feiyue [5] 1991, extended Lowen's work by introducing affine hulls of fuzzy sets and established that the affine combination of a set of fuzzy points is an affine fuzzy set. This was also established for convex fuzzy sets. In 2002, Yang and Yang [6] were able to characterised convexity of fuzzy sets via semicontinuous conditions. Peng [7] 2010, proposed two new definitions of convex fuzzy sets, i.e., convex $(s, t]$-fuzzy sets via relation between fuzzy points and fuzzy subsets, and convex R-fuzzy sets via implication operators of fuzzy logic. In 2010, Cheng et al [8] put forward the concept of $(s, t]$-intuitionistic convex fuzzy sets via cut set of intuitionistic fuzzy sets and neighborhood relations between fuzzy point and intuitionistic fuzzy set. In 2017, Sangodapo and Ajayi [9] contributed to Lowen's work by establishing some properties of both convex and affine fuzzy sets using fuzzy points. Many researchers have also studied and developed various properties of convex fuzzy sets and affine fuzzy sets, see [3, 4, 10, 11, 12, 13, 14, 15].

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2. PRELIMINARY

In this section, we state some basic definitions related to PFSs from Zadeh [1], Lowen [2], Atanasov [16], Huang [17], Das and Mukhlalsah [18], Sangodapo and Ajayi [19], Cuong and Kreinovich [20], Dogra and Pal [27].

Definition 1: Let X be a nonempty set. A FS A of X is an object of the form

$$A = \{\langle x, \sigma_A(x)\rangle | x \in X\}$$

with a membership function

$$\sigma_A : X \longrightarrow [0, 1]$$

where the function $\sigma_A(x)$ denotes the degree of membership of $x \in A$.

Definition 2: Let Γ be a FS of X, the t-cut set of Γ, denoted by Γ_t is made up of members whose membership function is at least t. That is;

$$\Gamma_t = \{x \in X : \sigma(x) \geq t\}.$$

Definition 3: Let a nonempty set X be fixed. An IFS A of X is an object of the form

$$A = \{\langle x, \sigma_A(x), \tau_A(x)\rangle | x \in X\}$$

where the functions

$$\sigma_A : X \to [0,1] \text{ and } \tau_A : X \to [0,1]$$

are called the membership and non-membership degrees of the $x \in A$, respectively, and for every $x \in X$,

$$0 \leq \sigma_A(x) + \tau_A(x) \leq 1.$$

Definition 4: A picture fuzzy set A of X is an object of the form

$$A = \{(x, \sigma_A(x), \tau_A(x), \gamma_A(x)) | x \in X\},$$

where the functions

$$\sigma_A : X \to [0,1],\ \tau_A : X \to [0,1] \text{ and } \gamma_A : X \to [0,1]$$

are called the positive, neutral and negative membership degrees of $x \in A$, respectively, and $\sigma_A, \tau_A, \gamma_A$ satisfy

$$0 \leq \sigma_A(x) + \tau_A(x) + \gamma_A(x) \leq 1,\ \forall x \in X.$$

For each $x \in X$, $1 - (\sigma_A(x) - \tau_A(x) - \gamma_A(x))$ is called the refusal membership degree of $x \in A$.

Definition 5: Let A and B be two PFSs. Then, the inclusion, equality, union, intersection and complement are defined as follow:

- $A \subseteq B$ if and only if for all $x \in X$, $\sigma_A(x) \leq \sigma_B(x)$, $\tau_A(x) \leq \tau_B(x)$ and $\gamma_A(x) \geq \gamma_B(x)$.
- $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.
- $A \cup B = \{(x, \sigma_A(x) \vee \sigma_B(x), \tau_A(x) \wedge \tau_B(x)), \gamma_A(x) \wedge \gamma_B(x)) | x \in X\}$.
- $A \cap B = \{(x, \sigma_A(x) \wedge \sigma_B(x), \tau_A(x) \wedge \tau_B(x)), \gamma_A(x) \vee \gamma_B(x)) | x \in X\}$.
- $coA = \overline{A} = \{(x, \gamma_A(x), \tau_A(x), \sigma_A(x)) | x \in X\}$.

Definition 6: A FS A of X is called convex fuzzy set if for all $x, y \in X$ and $\lambda \in [0,1]$,

$$\sigma_A[\lambda x + (1-\lambda)y] \geq \sigma_A(x) \wedge \sigma_A(y)$$

Definition 7: A fuzzy set A is affine if for all $x, y \in E$, and $\lambda \in \mathbb{R}$

$$\sigma_A[\lambda x + (1-\lambda)y] \geq \sigma_A(x) \wedge \sigma_A(y)$$

Definition 8: Given an IFS A of X. Then, A is said to be intuitionistic convex fuzzy set if for all $x, y \in X$ and for all $\lambda \in [0, 1]$,

$$\sigma_A[\lambda x + (1-\lambda)y] \geq \sigma_A(x) \wedge \sigma_A(y)$$

and

$$\tau_A[\lambda x + (1-\lambda)y] \leq \tau_A(x) \vee \tau_A(y)$$

Definition 9: Let A be intuitionistic fuzzy set of E. Then, A is said to be intuitionistic affine fuzzy set if for all $x, y \in E$ and for all $\lambda \in \mathbb{R}$,

$$\sigma_A[\lambda x + (1-\lambda)y] \geq \sigma_A(x) \wedge \sigma_A(y)$$

and

$$\tau_A[\lambda x + (1-\lambda)y] \leq \tau_A(x) \vee \tau_A(y)$$

Definition 10: Let $A,\ B$ be PFSs on X and $\lambda \in [0, 1]$. Then, the convex combination of A and B is defined as

$$C_\lambda(A,\ B) = \{(x, \sigma_{C_\lambda}(x), \tau_{C_\lambda}(x), \gamma_{C_\lambda}(x))|x \in X\}$$

where

$$\sigma_{C_\lambda}(x) = \lambda\sigma_A(x) + (1-\lambda)\sigma_B(x),$$
$$\tau_{C_\lambda}(x) = \lambda\tau_A(x) + (1-\lambda)\tau_B(x),$$
$$\gamma_{C_\lambda}(x) = \lambda\gamma_A(x) + (1-\lambda)\gamma_B(x)$$

$\forall\ x \in X$.

Remark 1: Definition 10 can be rewritten as a convex combination of two points of a PFS A on X. That is;

$$PCC(A) = \{((x, y), (\sigma; \lambda)(x, y), (\tau, ; \lambda)(x, y), (\gamma; \lambda)(x, y))|x, y \in E\}$$

for a PFS A and $\lambda \in [0, 1]$ where

$$(\sigma; \lambda)(x, y) = (1-\lambda)\sigma_A(x) + \lambda\sigma_A(y)$$

$$(\tau; \lambda)(x, y) = (1-\lambda)\tau_A(x) + \lambda\tau_A(y)$$

and

$$(\gamma; \lambda)(x, y) = (1-\lambda)\gamma_A(x) + \lambda\gamma_A(y)$$

$\forall\ x, y \in E$.

Definition 11: Let $A = \{(x, \sigma_A, \tau_A, \gamma_A)|x \in X\}$ be PFS over the universe X. Then, (r, s, t)-cut of A is the crisp set in A, denoted by $C_{r,s,t}(A)$ and is defined by

$$C_{r,s,t}(A) = \{x \in X|\sigma_A(x) \geq r,\ \tau_A(x) \geq s, \gamma_A(x) \leq t\}$$

$r, s, t \in [0, 1]$ with the condition $0 \leq r + s + t \leq 1$

Zadeh [1] and Brown [29] established that convexity of fuzzy set and cut set of fuzzy set are equivalent.
Lemma 1: A fuzzy set A is convex if and only if its cut sets are convex.
Huang [**?**] established the equivalence of intuitionistic fuzzy set and its cut sets
Lemma 2: An intuitionistic fuzzy set A is convex if and only if the cut sets of intuitionistic fuzzy set A are convex.
Das and Mukhlalsah [18] also obtained the relationship between intuitionistic convex fuzzy set and its cut sets.
Lemma 3: An intuitionistic fuzzy set is intuitionistic convex fuzzy set if its every cut set is convex set.

3. PICTURE CONVEX FUZZY SETS

This section introduces picture convex fuzzy sets and establishes some of its properties. From Remark 1, we define picture convex fuzzy sets. Hereafter, we denote $[0,1] = \mathbb{I}$, and assume that X is an n-dimensional real Euclidean space $\mathbb{R}^n$ denoted by E for concreteness.

Definition 12: Let A be a PFS on E. Then, A is called a picture convex fuzzy set, (PCFS) if for all $x, y \in E$ and $\lambda \in \mathbb{I}$,

$$\sigma_A[(1-\lambda)x + \lambda y] \geq \sigma_A(x) \wedge \sigma_A(y),$$

$$\tau_A[(1-\lambda)x + \lambda y] \geq \tau_A(x) \wedge \tau_A(y)$$

and

$$\gamma_A[(1-\lambda)x + \lambda y] \leq \gamma_A(x) \vee \gamma_A(y)$$

Remark 1 can be extended to finitely many points of PFS A as follows:
Definition 13: Let A be a PFS on E, $x_1, \cdots, x_n \in A$. Then, the picture convex combination of A, denoted by $PCC(A)$ is a point x with $x = \sum\limits_{i=1}^{n} \lambda_i x_i$ where

$$x = (\sigma_x, \tau_x, \gamma_x),\ \lambda_i = (\lambda_{i\sigma}, \lambda_{i\tau}, \lambda_{i\gamma}),\ x_i = (\sigma_A(x_i), \tau_A(x_i), \gamma_A(x_i)),$$

$$\sigma_x = \sum_{i=1}^{n} \lambda_{i\sigma}\sigma_A(x_i),\ \tau_x = \sum_{i=1}^{n} \lambda_{i\tau}\tau_A(x_i),\ \gamma_x = \sum_{i=1}^{n} \lambda_{i\gamma}\gamma_A(x_i)$$

$\sum\limits_{i=1}^{n}(\lambda_{i\sigma} + \lambda_{i\tau} + \lambda_{i\gamma}) = 1$ and $(\lambda_{i\sigma} + \lambda_{i\tau} + \lambda_{i\gamma}) \in \mathbb{I}$ for each i.
Definition 12 can be defined in terms of **Definition 13** as;

Definition 14: Let A be a PFS on E and $x_1, \cdots, x_n$ be points in A. Then, A is called a PCFS if for all $x_i \in A,\ i = 1, \cdots, n$ and $\lambda \in \mathbb{I}$

$$\sigma_A\{\sum_{i=1}^{n} \lambda_i x_i\} \geq \sigma_A(x_1) \wedge \cdots \wedge \sigma_A(x_n),$$

$$\tau_A\{\sum_{i=1}^{n} \lambda_i a_i\} \geq \tau_A(x_1) \wedge \cdots \wedge \tau_A(x_n),$$

and

$$\gamma_A\{\sum_{i=1}^{n} \lambda_i a_i\} \leq \gamma_A(x_1) \vee \cdots \vee \gamma_A(x_n)$$

and $\sum_{i=1}^{d} \lambda_i = 1$. Where λ_i and x_i are as defined above.

Definition 15: Let $A = \{(x, \sigma_A(x), \tau_A(x), \gamma_A(x))|x \in E\}$ be PFS. Then, the picture convex hull of A, denoted by $Pch(A)$ is defined as

$$Pch(A) := \{x = \sum_{i=1}^{n} \lambda_i x_i : x_i \in A; n \geq 1, \lambda_i \in \mathbb{I}, \sum_{i=1}^{n} \lambda_i = 1\}$$

where x, λ_i and x_i are as defined in **Definition 13** and the positive, neutral and negative membership degrees are respectively defined as follow:

$$\sigma_{Pch(A)}(x) = \wedge\{\sigma_B(x)|\sigma_B \geq \sigma_A(x)\},$$

$$\tau_{Pch(A)}(x)(x) = \wedge\{\tau_B(x)|\tau_B \geq \tau_A(x)\}$$

$$\gamma_{Pch(A)}(x)(x) = \vee\{\gamma_B(x)|\gamma_B \leq \gamma_A(x)\},$$

where B is a PFS of the form $B = \{(x, \sigma_B(x), \tau_B(x), \gamma_B(x))|x \in E\}$.

Proposition 1: A PFS $A = \{(x, \sigma_A(x), \tau_A(x), \gamma_A(x))|x \in E\}$ is PCFS if and only if $C_{r,s,t}(A)$ is a PCFS.

Proof: Suppose that A is a PCFS. Let $x, y \in C_{r,s,t}(A)$, then $\sigma_A(x) \geq r$, $\sigma_A(y) \geq r$, $\tau_A(x) \geq s$, $\tau_A(y) \geq s$ and $\gamma_A(x) \leq t$, $\gamma_A(y) \leq t$ thus $\sigma_A(x) \wedge \sigma_A(y) \geq r$, $\tau_A(x) \wedge \tau_A(y) \geq s$ and $\gamma_A(x) \vee \gamma_A(y) \leq t$. Since A is PCFS, then

$$\sigma_A[(1-\lambda)x + \lambda y] \geq \sigma_A(x) \wedge \sigma_A(y) \geq r,$$

$$\tau_A[(1-\lambda)x + \lambda y] \geq \tau_A(x) \wedge \tau_A(y) \geq s,$$

$$\gamma_A[(1-\lambda)x + \lambda y] \leq \tau_A(x) \vee \tau_A(y) \leq t$$

for all $\lambda \in \mathbb{I}$. Thus, $(1-\lambda)x + \lambda y \in C_{r,s,t}(A)$. Hence, $C_{r,s,t}(A)$ is PCFS.

Conversely, suppose that $C_{r,s,t}(A)$ is PCFS for all $r, s, t \in \mathbb{I}$, let

$x, y \in E,\ \lambda \in \mathbb{I}$. Let $r = \sigma_A(x) \wedge \sigma_A(y)$, $s = \tau_A(x) \wedge \tau_A(y)$, and $t = \gamma_A(x) \vee \gamma_A(y)$. Now

$$\sigma_A(x) \geq \sigma_A(x) \wedge \sigma_A(y) = r,\ \sigma_A(y) \geq \sigma_A(x) \wedge \sigma_A(y) = r,$$

$$\tau_A(x) \geq \tau_A(y) \wedge \tau_A(y) = s,\ \tau_A(y) \geq \tau_A(x) \wedge \tau_A(y) = s,$$

$$\gamma_A(x) \leq \gamma_A(y) \vee \gamma_A(y) = t,\ \gamma_A(y) \leq \gamma_A(x) \vee \gamma_A(y) = t$$

which imply that $x, y \in C_{r,s,t}(A)$. Since $C_{r,s,t}(A)$ is PCFS, we have $(1-\lambda)x + \lambda y \in C_{r,s,t}(A)$. Thus,

$$\sigma_A[(1-\lambda)x + \lambda y] \geq r = \sigma_A(x) \wedge \sigma_A(y),$$

$$\tau_A[(1-\lambda)x + \lambda y] \geq s = \tau_A(x) \wedge \tau_A(y)$$

and

$$\gamma_A[(1-\lambda)x + \lambda y] \leq t = \gamma_A(x) \vee \gamma_A(y)$$

Therefore, A is a PCFS.

Proposition 2: Let $A = \{(x, \sigma_A(x), \tau_A(x), \gamma_A(x)) | x \in E\}$ and $B = \{(x, \sigma_B(x), \tau_B(x), \gamma_B(x)) | x \in E\}$ be PCFSs. Then, $D = A \cap B$ is a PCFS.

Proof: Let $x, y \in E$, take $\lambda \in \mathbb{I}$. Then,

$$\sigma_D[(1-\lambda)x + \lambda y] = \sigma_A[(1-\lambda)x + \lambda y] \cap \sigma_B[(1-\lambda)x + \lambda y],$$

$$\tau_D[(1-\lambda)x + \lambda y] = \tau_A[(1-\lambda)x + \lambda y] \cap \tau_B[(1-\lambda)x + \lambda y]$$

and

$$\gamma_D[(1-\lambda)x + \lambda y] = \gamma_A[(1-\lambda)x + \lambda y] \cup \gamma_B[(1-\lambda)x + \lambda y]$$

Since A and B are PCFSs,

$$\sigma_A[(1-\lambda)x+\lambda y] \geq \sigma_A(x) \wedge \sigma_A(y),\ \sigma_B[(1-\lambda)x+\lambda y] \geq \sigma_B(x) \wedge \sigma_B(y),$$

$$\tau_A[(1-\lambda)x+\lambda y] \geq \tau_A(x) \wedge \tau_A(y),\ \tau_B[(1-\lambda)x+\lambda y] \geq \tau_B(x) \wedge \tau_b(y)$$

and

$$\gamma_A[(1-\lambda)x+\lambda y] \leq \gamma_A(x) \vee \gamma_A(y),\ \gamma_B[(1-\lambda)x+\lambda y] \leq \gamma_B(x) \vee \gamma_b(y).$$

Thus,

$$\begin{aligned}
\sigma_D[(1-\lambda)x + \lambda y] &= [\sigma_A(x) \wedge \sigma_A(y)] \cap [\sigma_B(x) \wedge \sigma_B(y)] \\
&\geq [\sigma_A(x) \cap \sigma_B(x)] \wedge [\sigma_A(y) \cap \sigma_B(y)] \\
&\geq [\sigma_A \cap \sigma_B](x) \wedge [\sigma_A \cap \sigma_B](y) \\
&\geq [\sigma_{A\cap B}](x) \wedge [\sigma_{A\cap B}](y) \\
&= \sigma_D(x) \wedge \sigma_D(y).
\end{aligned}$$

,

$$\begin{aligned} \tau_D[\lambda x + ((1-\lambda)x + \lambda y] &= [\tau_A(x) \wedge \tau_A(y)] \cap [\tau_B(x) \wedge \tau_B(y)] \\ &\geq [\tau_A(x) \cap \tau_B(x)] \wedge [\tau_A(y) \cap \tau_B(y)] \\ &\geq [\tau_A \cap \tau_B](x) \wedge [\tau_A \cap \tau_B](y) \\ &\geq [\tau_{A\cap B}](x) \wedge [\tau_{A\cap B}](y) \\ &= \tau_D(x) \wedge \tau_D(y). \end{aligned}$$

and

$$\begin{aligned} \gamma_D[(1-\lambda)x + \lambda y] &= [\gamma_A(x) \vee \gamma_A(y)] \cup [\gamma_B(x) \vee \gamma_B(y)] \\ &\leq [\gamma_A(x) \cup \gamma_B(x)] \vee [\gamma_A(y) \cup \gamma_B(y)] \\ &\leq [\gamma_A \cup \gamma_B](x) \vee [\gamma_A \cup \gamma_B](y) \\ &\leq [\gamma_{A\cup B}](x) \vee [\gamma_{A\cup B}](y) \\ &= \gamma_D(x) \vee \gamma_D(y). \end{aligned}$$

Corollary 1: Given any arbitrary family $\{A_i, i = 1, 2, \cdots,\}$ of PCFSs in E, then their intersection is also a PCFS.
Proof: Let $\{A_i\}_i$ be an arbitrary family of PCFS of E. Then, the intersection $B = \bigwedge_i A_i$ is PCFS. If

$\sigma_B[(1-\lambda)x+\lambda y] \in B,\ \tau_B[(1-\lambda)x+\lambda y] \in B$ and $\eta_B[(1-\lambda)x+\lambda y] \in B$,

then they belong also to every A_i, since A_i is PCFS. It implies that

$\sigma_B[(1-\lambda)x+\lambda y] \in B,\ \tau_B[(1-\lambda)x+\lambda y] \in B$ and $\eta_B[(1-\lambda)x+\lambda y] \in B$

belong to every A_i, and consequently to their intersection, i.e., to B.
Proposition 3: A PFS A is a PCFS if and only if for all finite family $r_i \in E$ and $\lambda_i \in \mathbb{I},\ i = 1, 2, \cdots n$ such that $\sum_{i=1}^{n} \lambda_i = 1$, we have

$$\sigma_A\left(\sum_{i=1}^{n} \lambda_i r_i\right) \geq \bigwedge_{i=1}^{n} \sigma_A(r_i),\ \tau_A\left(\sum_{i=1}^{n} \lambda_i r_i\right) \geq \bigwedge_{i=1}^{n} \tau_A(r_i) \text{ and } \gamma_A\left(\sum_{i=1}^{n} \lambda_i r_i\right) \leq \bigvee_{i=1}^{n} \gamma_A(r_i), \quad (1).$$

Proof: Assume that A contains all finite family $r_i \in E$ and $\lambda_i \in \mathbb{I},\ i = 1, 2, \cdots n$ such that $\sum_{i=1}^{n} \lambda_i = 1$, then this holds for two points $a,\ b \in E$ where $a = \sum_{i=1}^{p} \lambda_i r_i,\ a = (\sigma_a, \tau_a, \gamma_a)$,

$\lambda_i = (\lambda_{i\sigma_1}, \lambda_{i\tau_1}, \lambda_{i\gamma_1})$ and $r_i = (\sigma_A(r_i), \tau_A(r_i), \gamma_A(r_i)),\ \sigma_a = \sum_{i=1}^{p} \lambda_{i\sigma_1}\sigma_A(r_i),\ \tau_a =$

$\sum_{i=1}^{p} \lambda_{i\tau_1}\tau_A(r_i)$,
$\gamma_a = \sum_{i=1}^{p} \lambda_{i\gamma_1}\gamma_A(r_i)$, $\sum_{i=1}^{p}(\lambda_{i\sigma_1}+\lambda_{i\tau_1}+\lambda_{i\gamma_1}) = 1$ and $(\lambda_{i\sigma_1}+\lambda_{i\tau_1}+\lambda_{i\gamma_1}) \in \mathbb{I}$ for all i and
$b = \sum_{i=1}^{p} \phi_i r_i$, $b = (\sigma_b, \tau_b, \gamma_b)$, $\phi_i = (\phi_{i\sigma_2}, \phi_{i\tau_2}, \phi_{i\gamma_2})$ and $r_i = (\sigma_A(r_i), \tau_A(r_i), \gamma_A)$,
$\sigma_b = \sum_{i=1}^{p} \phi_{i\sigma_2}\sigma_A(r_i)$, $\tau_b = \sum_{i=1}^{p} \phi_{i\tau_2}\tau_A(r_i)$, $\gamma_b = \sum_{i=1}^{p} \phi_{i\gamma_2}\gamma_A(r_i)$, $\sum_{i=1}^{p}(\phi_{i\sigma_2}+ \phi_{i\tau_2} + \phi_{i\gamma_2}) = 1$ and
$(\phi_{i\sigma_2} + \phi_{i\tau_2} + \phi_{i\gamma_2}) \in \mathbb{I}$ for all i.
Thus,

$$\sigma_A((1-\delta)a + \delta b) \geq \sigma_A(a) \wedge \sigma_A(b),$$
$$\tau_A((1-\delta)a + \delta b) \geq \tau_A(a) \wedge \tau_A(b)$$

and

$$\gamma_A((1-\delta)a + \delta b) \leq \gamma_A(a) \vee \gamma_A(b)$$

The positive, neutral and negative membership degrees of a and b are given by

$$\begin{aligned}\sigma_A &= (1-\delta)\sum_{i=1}^{p} \lambda_{i\sigma_1}\sigma_A(r_i) + \delta \sum_{i=1}^{p} \phi_{i\sigma_2}\sigma_A(r_i) \\ &= \sum_{i=1}^{p} ((1-\delta)\lambda_{i\sigma_1} + \delta\phi_{i\sigma_2})\,\sigma_A(r_i)\end{aligned}$$

$$\begin{aligned}\tau_A &= (1-\delta)\sum_{i=1}^{p} \lambda_{i\tau_1}\tau_A(r_i) + \delta \sum_{i=1}^{p} \phi_{i\tau_2}\tau_A(r_i) \\ &= \sum_{i=1}^{p} ((1-\delta)\lambda_{i\tau_1} + \delta\phi_{i\tau_2})\,\tau_A(r_i)\end{aligned}$$

and

$$\begin{aligned}\gamma_A &= (1-\delta)\sum_{i=1}^{p} \lambda_{i\gamma_1}\gamma_A(r_i) + \delta \sum_{i=1}^{p} \phi_{i\gamma_2}\gamma_A(r_i) \\ &= \sum_{i=1}^{p} ((1-\delta)\lambda_{i\gamma_1} + \delta\phi_{i\gamma_2})\,\gamma_A(r_i)\end{aligned}$$

respectively with

$$\sum_{i=1}^{p} [(1-\delta)\lambda_{i\sigma_1} + \delta\phi_{i\sigma_2}] + \sum_{i=1}^{p} [(1-\delta)\lambda_{i\tau_1} + \delta\phi_{i\tau_2}] + \sum_{i=1}^{p} [(1-\delta)\lambda_{i\gamma_1} + \delta\phi_{i\gamma_2}]$$

$$\begin{aligned}
&= \left(\sum_{i=1}^{p}(1-\delta)\lambda_{i\sigma_1} + \sum_{i=1}^{p}(1-\delta)\lambda_{i\tau_1}\lambda_{i\tau_1} + \sum_{i=1}^{p}(1-\delta)\lambda_{i\gamma_1}\lambda_{i\gamma_1}\right) \\
&+ \left(\sum_{i=1}^{p}\delta\phi_{i\sigma_2} + \sum_{i=1}^{p}\delta\phi_{i\tau_2}\sum_{i=1}^{p}\delta\phi_{i\gamma_2}\right) \\
&= (1-\delta)\left(\sum_{i=1}^{p}\lambda_{i\sigma_1} + \sum_{i=1}^{p}\lambda_{i\tau_1} + \sum_{i=1}^{p}\lambda_{i\gamma_1}\right) + \delta\left(\sum_{i=1}^{p}\phi_{i\sigma_2} + \sum_{i=1}^{p}\phi_{i\tau_2} + \sum_{i=1}^{p}\phi_{i\gamma_2}\right) \\
&= 1-\delta+\delta \\
&= 1.
\end{aligned}$$

and since

$$(\lambda_{i\sigma} + \lambda_{i\tau} + \lambda_{i\gamma}) \in \mathbb{I},\ (\phi_{i\sigma} + \phi_{i\tau} + \phi_{i\gamma}) \in \mathbb{I}$$

then

$$(((1-\delta)\lambda_{i\sigma_1} + \delta)\phi_{i\sigma_2}) + ((1-\delta)\lambda_{i\tau_1} + \delta\phi_{i\tau_2})) + ((1-\delta)\lambda_{i\gamma_1} + \delta\phi_{i\gamma_2}) \in \mathbb{I}$$

such that

$$\sigma_A[(1-\delta)a + \lambda b] \geq \sigma_A(a) \wedge \sigma_A(b),$$

$$\tau_A[(1-\lambda)a + \lambda b] \geq \tau_A(a) \wedge \tau_A(b)$$

and

$$\gamma_A[(1-\lambda)a + \lambda b] \leq \gamma_A(a) \vee \gamma_A(b).$$

The convexity of A means that A is closed under convex combinations of two points. Therefore, A is a PCFS.
Conversely, suppose that A is PFS and $n \in \mathbb{N}$. We shall use induction on n. For $n = 1$, the assertion is trivial. For $(n-1)$ to $n,\ n \geq 2$, let $r_1, \cdots, r_n \in A$ and $\lambda_i \in \mathbb{I},\ i = 1, 2, \cdots n$ with $\sum_{i=1}^{n}\lambda_i = 1$. Assume $\lambda_i \in \{0, 1\}$ and $\delta_i := \dfrac{\lambda_i}{1-\lambda_n},\ i = 1, \cdots, n-1$, thus $\delta_i \in \mathbb{I}$ and $\sum_{i=1}^{n}\delta_i = 1$. By the induction hypothesis, $\sum_{i=1}^{n-1}\delta_i r_i \in A$. Put

$$\sum_{i=1}^{n}\lambda_i r_i = (1-\lambda_n)\sum_{i=1}^{n-1}\delta_i r_i + \lambda_n r_n$$

Then, we have

$$\begin{aligned}\sigma_A\left(\sum_{i=1}^{n}\lambda_i r_i\right) &\geq \sigma_A(\sum_{i=1}^{n-1}\delta_i r_i)\wedge\sigma_A(r_n)\\ &\geq \sigma_A(\sum_{i=1}^{n-1}(\frac{\lambda_i}{1-\lambda_n})(r_i))\wedge\sigma_A(r_n)\\ &\geq \bigwedge_{i}^{n}\sigma_A(r_i),\end{aligned}$$

$$\begin{aligned}\tau_A\left(\sum_{i=1}^{n}\lambda_i r_i\right) &\geq \tau_A(\sum_{i=1}^{n-1}\delta_i r_i)\wedge\tau_A(r_n)\\ &\geq \tau_A(\sum_{i=1}^{n-1}(\frac{\lambda_i}{1-\lambda_n})(r_i))\wedge\tau_A(r_n)\\ &\geq \bigwedge_{i}^{n}\tau_A(r_i)\end{aligned}$$

and

$$\begin{aligned}\gamma_A\left(\sum_{i=1}^{n}\lambda_i r_i\right) &\leq \gamma_A(\sum_{i=1}^{n-1}\delta_i r_i)\vee\gamma_A(r_n)\\ &\leq \gamma_A(\sum_{i=1}^{n-1}(\frac{\lambda_i}{1-\lambda_n})(r_i))\vee\gamma_A(r_n)\\ &\leq \bigvee_{i}^{n}\gamma_A(r_i)\end{aligned}$$

Proposition 4: For a PCFS A on E, $Pch(A)$ consists of all the convex combinations of the points of A.

Proof: The points of A belong to $Pch(A)$, so all their convex combinations belong to $Pch(A)$ by **Proposition 3**.

Conversely, let $a,\ b\in\mathbb{E}$ such that $a=\sum\limits_{i=1}^{m}\lambda_i r_i,\ a=(\sigma_a,\tau_a,\gamma_a)$,

$\lambda_i=(\lambda_{i\sigma_1},\lambda_{i\tau_1},\lambda_{i\gamma_1})$ and $r_i=(\sigma_A(r_i),\tau_A(r_i),\gamma_A(r_i))$, $\sigma_a=\sum\limits_{i=1}^{m}\lambda_{i\sigma_1}\sigma_A(r_i)$, $\tau_a=\sum\limits_{i=1}^{m}\lambda_{i\tau_1}\tau_A(r_i)$,

$\gamma_a=\sum\limits_{i=1}^{m}\lambda_{i\gamma_1}\gamma_A(r_i),\ \sum\limits_{i=1}^{m}(\lambda_{i\sigma_1}+\lambda_{i\tau_1}+\lambda_{i\gamma_1})=1$ and $(\lambda_{i\sigma_1}+\lambda_{i\tau_1}+\lambda_{i\gamma_1})\in$

$\mathbb{I}$ for all i and $r_i \in A$ and $b = \sum_{j=1}^{n} \beta_j s_j$, $b = (\sigma_b, \tau_b, \gamma_b)$, $\beta_j = (\beta_{j\sigma_2}, \beta_{j\tau_2}, \beta_{j\gamma_2})$ and $s_j = (\sigma_A(s_j), \tau_A(s_j), \gamma_A(s_j))$,
$\sigma_b = \sum_{j=1}^{n} \beta_{j\sigma_2}\sigma_A(s_j)$, $\tau_{b_2} = \sum_{j=1}^{n} \beta_{j\tau_2}\tau_A(s_j)$, $\gamma_{b_2} = \sum_{j=1}^{n} \beta_{j\gamma_2}\gamma_A(s_j)$, $\sum_{j=1}^{n}(\beta_{j\sigma_2}+ \beta_{j\tau_2} + \beta_{j\gamma_2}) = 1$ and
$(\phi_{i\sigma_2} + \phi_{i\tau_2} + \phi_{i\gamma_2}) \in \mathbb{I}$ for all i and $s_i \in A$. Thus,
$((1-\delta)\lambda_{i\sigma_1} + \delta\phi_{j\sigma_2}) + ((1-\delta)\lambda_{i\tau_1} + \delta\phi_{j\tau_2}) + ((1-\delta)\lambda_{i\gamma_1} + \delta\phi_{j\gamma_2})) \in \mathbb{I}$
Now,

$$\begin{aligned} \sigma_A(a) \wedge \sigma_A(b) &\geq \sigma_A(\sum_{i=1}^{m} \lambda_i r_i) \wedge \sigma_A(\sum_{j=1}^{n} \phi_j s_j) \\ &\geq \sigma_A(r_1) \wedge \cdots \wedge \sigma_A(r_m) \wedge \sigma_A(s_{1)} \wedge \cdots \wedge \sigma_A(s_n) \end{aligned}$$

$$\begin{aligned} \tau_A(a) \wedge \tau_A(b) &\geq \tau_A(\sum_{i=1}^{m} \lambda_i r_i) \wedge \tau_A(\sum_{j=1}^{n} \phi_j a_j) \\ &\geq \tau_A(r_1) \wedge \cdots \wedge \tau_A(r_m) \wedge \tau_A(s_{1)} \wedge \cdots \wedge \tau_A(s_n) \end{aligned}$$

and

$$\begin{aligned} \gamma_A(a) \vee \gamma_A(b) &\leq \gamma_A(\sum_{i=1}^{m} \lambda_i r_i) \vee \gamma_A(\sum_{j=1}^{n} \phi_j a_j) \\ &\leq \gamma_A(r_1) \vee \cdots \vee \gamma_A(r_m) \vee \gamma_A(s_{1)} \vee \cdots \vee \gamma_A(s_n) \end{aligned}$$

Hence, $\lambda a + \phi b$ is another convex combination of points of A. Therefore, the set of convex combinations of points of A is itself a convex set that contains A which must coincide with the $Pch(A)$.
Example 1: Let $X = \{1, 2, 3, 4, 5\}$ and
$A = \{(1, 0.1, 0.2, 0.7), (2, 0.2, 0.5, 0.3), (3, 0.2, 0.4, 0.4), (4, 0.3, 0, 5, 0.2)\}$
be a PFS. To show that A is a PCFS. Take $\lambda = \frac{1}{2}, x = 1, y = 3$. Then,

$$\begin{aligned} \sigma_A\left[(1-\lambda)x + \lambda y\right] &= \sigma_A(\frac{1}{2} \times 1 + \frac{1}{2} \times 3) \\ &= \sigma_A(2) \\ &= 0.2 \end{aligned}$$

Also,

$$\begin{aligned} \sigma_A(x) \wedge \sigma_A(y) &= \sigma_A(1) \wedge \sigma_A(3) \\ &= 0.1 \wedge 0.2 \\ &= 0.1 \end{aligned}$$

Since $0.2 > 0.1$, it implies that $\sigma_A\left[(1-\lambda)x+\lambda y\right] > \sigma_A(x) \wedge \sigma_A(y)$.

$$\begin{aligned}\tau_A\left[(1-\lambda)x+\lambda y\right] &= \tau_A(\frac{1}{2}\times 1+\frac{1}{2}\times 3)\\ &= \tau_A(2)\\ &= 0.5\end{aligned}$$

Also,

$$\begin{aligned}\tau_A(x)\wedge\tau_A(y) &= \tau_A(1)\wedge\tau_A(3)\\ &= 0.2\wedge 0.4\\ &= 0.2\end{aligned}$$

Since $0.5 > 0.2$, it implies that $\tau_A\left[(1-\lambda)x+\lambda y\right] > \tau_A(x) \wedge \tau_A(y)$.

$$\begin{aligned}\eta_A\left[(1-\lambda)x+\lambda y\right] &= \eta_A(\frac{1}{2}\times 1+\frac{1}{2}\times 3)\\ &= \eta_A(2)\\ &= 0.3\end{aligned}$$

Also,

$$\begin{aligned}\eta_A(x)\wedge\eta_A(y) &= \eta_A(1)\wedge\eta_A(3)\\ &= 0.7\wedge 0.4\\ &= 0.7\end{aligned}$$

Since $0.3 < 0.7$, it implies that $\eta_A\left[(1-\lambda)x+\lambda y\right] < \eta_A(x) \wedge \eta_A(y)$.
Hence, A is a PCFS.

3.1 PICTURE AFFINE FUZZY SETS

Affine set and its related notions have already been defined and studied in each of classical sets, fuzzy sets and intuitionistic fuzzy sets. In a similar way, we define picture affine fuzzy sets and investigate some of its characteristics.

Definition 16: Let A be PFSs on E and $\lambda \in \mathbb{R}$. Then, the picture affine combination of A, denoted by $PAC(A)$ is defined as

$$PAC(A) = \{((x,y),(\sigma;\lambda)(x,y),(\tau;\lambda)(x,y),(\gamma;\lambda)(x,y))|x,y\in E\}$$

where

$$(\sigma;\lambda)(x,y) = (1-\lambda)\sigma_A(x)+\lambda\sigma_A(y)$$
$$(\tau;\lambda)(x,y) = (1-\lambda)(x)+\lambda\tau_A(y)$$

and

$$(\gamma;\lambda)(x,y) = (1-\lambda)\gamma_A(x)+\lambda\gamma_A(y)$$

$\forall\ x, y \in E$.

Definition 17: Let A be a PFS on E. Then, A is called a picture affine fuzzy set, (PAFS) if for all $x, y \in E$ and $\lambda \in \mathbb{R}$,

$$\sigma_A[(1-\lambda)x + \lambda y] \geq \sigma_A(x) \wedge \sigma_A(y)$$
$$\tau_A[(1-\lambda)x + \lambda y] \geq \tau_A(x) \wedge \tau_A(y)$$

and

$$\gamma_A[(1-\lambda)x + \lambda y] \leq \gamma_A(x) \vee \gamma_A(y)$$

Remark 2: In **Definition 12**, the membership degrees of the line segment for the values taken from the unit interval, $[0, 1]$ are embedded in the membership degrees for the values taken from the whole real numbers, $\mathbb{R}$. Also, **Definition 12** takes its value only from the unit interval $[0, 1]$ while **Definition 17** takes any value in $\mathbb{R}$. It means that for convexity, there is a restriction on the values it can take but affinity has no restriction on the scalar, because the unit interval is a segment of $\mathbb{R}$. Hence, **Definition 12** is embedded in **Definition 17** but the converse is not true.

Definition 17 can be redefined for finitely many points of PFS A as follows;

Definition 18: Let A be a PFS on E, $x_1, \cdots, x_n \in A$. Then, the $PAC(A)$ is a point x with $x = \sum_{i=1}^{n} \lambda_i x_i$ where

$$x = (\sigma_x, \tau_x, \gamma_x),\ \lambda_i = (\lambda_{i\sigma}, \lambda_{i\tau}, \lambda_{i\gamma}),\ x_i = (\sigma_A(x_i), \tau_A(x_i), \gamma_A(x_i)),$$

$$\sigma_x = \sum_{i=1}^{n} \lambda_{i\sigma}\sigma_A(x_i),\ \tau_x = \sum_{i=1}^{n} \lambda_{i\tau}\tau_A(x_i),\ \gamma_x = \sum_{i=1}^{n} \lambda_{i\gamma}\gamma_A(x_i)$$

$\sum_{i=1}^{n}(\lambda_{i\sigma} + \lambda_{i\tau} + \lambda_{i\gamma}) = 1$ and $\lambda_{i\sigma} + \lambda_{i\tau} + \lambda_{i\gamma} \in \mathbb{R}$ for each i.

Definition 17 can also be defined in terms of **Definition 18**.

Definition 19: Let A be a PFS on E and $x_1, \cdots, x_n$ be points in A. Then, A is called a PAFS if for all $x_i \in A$, $i = 1, \cdots, n$ and $\lambda \in \mathbb{R}$

$$\sigma_A\{\sum_{i=1}^{n} \lambda_i x_i\} \geq \sigma_A(x_1) \wedge \cdots \wedge \sigma_A(x_n),$$

$$\tau_A\{\sum_{i=1}^{n} \lambda_i a_i\} \geq \tau_A(x_1) \wedge \cdots \wedge \tau_A(x_n),$$

and

$$\gamma_A\{\sum_{i=1}^{n} \lambda_i a_i\} \leq \gamma_A(x_1) \vee \cdots \vee \gamma_A(x_n)$$

and $\sum_{i=1}^{d} \lambda_i = 1$. Where λ_i and x_i are as defined above.

Definition 20: Let $A = \{(x, \sigma_A(x), \tau_A(x), \gamma_A(x))|x \in E\}$ be PFS. Then, the picture affine hull of A, denoted by $Pah(A)$ is defined as

$$Pah(A) := \{x = \sum_{i=1}^{n} \lambda_i x_i : x_i \in A; n \geq 1, \lambda_i \in \mathbb{R}, \sum_{i=1}^{n} \lambda_i = 1\}$$

where x, λ_i and x_i are as defined in Definition **??** and the positive, neutral and negative membership degrees are defined as follow:

$$\sigma_{Pch(A)}(x)(x) = \wedge\{\sigma_B(x)|\sigma_B \geq \sigma_A(x)\},$$
$$\tau_{Pch(A)}(x)(x) = \wedge\{\tau_B(x)|\tau_B \geq \tau_A(x)\}$$
$$\gamma_{Pch(A)}(x)(x) = \vee\{\gamma_B(x)|\gamma_B \leq \gamma_A(x)\},$$

respectively, where B is a PFS of the form $B = \{(x, \sigma_B(x), \tau_B(x), \gamma_B(x))|x \in E\}$.

Definition 21: Let $A = \{(x, \sigma_A(x), \tau_A(x), \gamma_A(x))|x \in E\}$ be PFS and $r, s, t \in \mathbb{R},\ r + s + t \leq 1$. Then, the (r, s, t)-level set of A, $(P, A)_{(r,s,t)}$ is defined as

$$(P, A)_{(r,s,t)} = \{x \in \mathbb{E}|\sigma_A(x) \geq r,\ \tau_A(x) \geq s, \gamma_A(x) \leq t\}$$

Lemma 1: A FS A is an affine fuzzy set if and only if Γ_t is an affine set.

Proof: Suppose that A is an affine fuzzy set. Let $x, y \in \Gamma_t$, then $\sigma_A(x) \geq t$ and $\sigma_A(y) \geq t$, thus $\sigma_A(x) \wedge \sigma_A(y) \geq t$. Since A is n affine fuzzy set, then

$$\sigma_A\left[(1-\lambda)x + \lambda y\right] \geq \sigma_A(x) \wedge \sigma_A(y) \geq t \cap t = t$$

for each $\lambda \in \mathbb{R}$. Thus, $(1-\lambda)x + \lambda y \in \Gamma_t$. Hence, Γ_t is an affine set. Conversely, suppose that Γ_t is an affine set, for all $\lambda \in \mathbb{R}$. Let $t = \sigma_A(x) \wedge \sigma_A(y)$. From first part, we obtained $\sigma_A(x) \geq t$ and $\sigma_A(y) \geq t$ $x, y \in \Gamma_t$. Thus, by affinity of Γ_t, we have $(1-\lambda)x+\lambda y \in \Gamma_t$. Hence, $\sigma_A\left[(1-\lambda)x + \lambda y\right] \geq t = \sigma_A(x) \wedge \sigma_A(y)$. Therefore, A is an affine fuzzy set.

Proposition 5: A PFS $A = \{(x, \sigma_A(x), \tau_A(x), \gamma_A(x))|x \in E\}$ is PAFS if and only if $(P, A)_{(r,s,t)}$ is a PAFS.

Proof: The proof is similar to the proof of **Proposition 1**.

Proposition 6: Let $A = \{(x, \sigma_A(x), \tau_A(x), \gamma_A(x))|x \in E\}$ and $B = \{(x, \sigma_B(x), \tau_B(x), \gamma_B(x))|x \in E\}$ be PAFSs. Then, $D = A \cap B$ is a PAFS.

The proof is similar to the proof of **Proposition 2**.

Corollary 2: Given any arbitrary family $\{A_i, i = 1, 2, \cdots, \}$ of PAFSs in E, then their intersection is also a PAFS.

The proof is similar to the proof of **Corollary 1**.

Proposition 7: A PFS A is a PAFS if and only if for all finite family $r_i \in E$ and $\lambda_i \in \mathbb{R},\ i = 1, 2, \cdots n$ such that $\sum_{i=1}^{n} \lambda_i = 1$, we have

$$\sigma_A\left(\sum_{i=1}^{n} \lambda_i r_i\right) \geq \bigwedge_{i=1}^{n} \sigma_A(r_i),\ \tau_A\left(\sum_{i=1}^{n} \lambda_i r_i\right) \geq \bigwedge_{i=1}^{n} \tau_A(r_i),\ \gamma_A\left(\sum_{i=1}^{n} \lambda_i r_i\right) \leq \bigvee_{i=1}^{n} \gamma_A(r_i), \tag{1}$$

The proof is similar to the proof of **Proposition 3**.

Proposition 8: For a PAFS A on E, $A(A)$ consists of all the affine combinations of the points of A.

The proof is similar to the proof of **Proposition 4**.

Example 2: Let $X = \{1, 2, 3, 4, 5\}$ and

$A = \{(1, 0.1, 0.2, 0.7), (2, 0.2, 0.5, 0.3), (3, 0.2, 0.4, 0.4), (4, 0.3, 0, 5, 0.2)\}$

be a PFS. To show that A is a PCFS. Take $\lambda = \frac{3}{2}, x = 1, y = 3$. Then,

$$\begin{aligned} \sigma_A\left[(1-\lambda)x + \lambda y\right] &= \sigma_A(-\frac{1}{2} \times 1 + \frac{3}{2} \times 3) \\ &= \sigma_A(4) \\ &= 0.3 \end{aligned}$$

Also,

$$\begin{aligned} \sigma_A(x) \wedge \sigma_A(y) &= \sigma_A(1) \wedge \sigma_A(3) \\ &= 0.1 \wedge 0.2 \\ &= 0.1 \end{aligned}$$

Since $0.3 > 0.1$, it implies that $\sigma_A\left[(1-\lambda)x + \lambda y\right] > \sigma_A(x) \wedge \sigma_A(y)$.

$$\begin{aligned} \tau_A\left[(1-\lambda)x + \lambda y\right] &= \tau_A(-\frac{1}{2} \times 1 + \frac{3}{2} \times 3) \\ &= \tau_A(4) \\ &= 0.5 \end{aligned}$$

Also,

$$\begin{aligned} \tau_A(x) \wedge \tau_A(y) &= \tau_A(1) \wedge \tau_A(3) \\ &= 0.2 \wedge 0.4 \\ &= 0.2 \end{aligned}$$

Since $0.5 > 0.2$, it implies that $\tau_A\left[(1-\lambda)x + \lambda y\right] > \tau_A(x) \wedge \tau_A(y)$.

$$\begin{aligned}\eta_A\left[(1-\lambda)x+\lambda y\right] &= \eta_A(-\frac{1}{2}\times 1+\frac{3}{2}\times 3)\\ &= \eta_A(4)\\ &= 0.2\end{aligned}$$

Also,

$$\begin{aligned}\eta_A(x)\wedge\eta_A(y) &= \eta_A(1)\wedge\eta_A(3)\\ &= 0.7\wedge 0.4\\ &= 0.7\end{aligned}$$

Since $0.2 < 0.7$, it implies that $\eta_A\left[(1-\lambda)x+\lambda y\right] < \eta_A(x)\wedge\eta_A(y)$. Hence, A is a PAFS.

4. CONCLUDING REMARKS

Picture fuzzy sets is a new area of research developed as a generalisation of both fuzzy sets and intuitionistic fuzzy sets. In this paper, the concept of convexity of picture fuzzy sets has been introduced and some of its properties are obtained. Also, picture affine fuzzy sets has been initiated and some of its characteristics are established. An example is given for each of PCFS and PAFS. For future research, picture convex fuzzy set can be extended to $(s,t]$-picture convex fuzzy sets and some of its properties via cut set of picture fuzzy sets and relations between fuzzy point and picture fuzzy set will be investigated. Also, the characterisations of $(s,t]$-picture convex fuzzy sets to optimisation will be explored. The theory can be applied to solve related problems in operations research, optimisation problem, decision making problems, etc.

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