

GLOBAL STABILITY IN THE DEGN-HARRISON MODEL

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Dedicated to the memory of Anthony Uyi Afuwape

ABSTRACT. The main goal of this paper is to investigate the global dynamics of the model proposed by Degn and Harrison to explain the oscillations in continuous bacterial cultures. We establish suitable conditions on the parameters ensuring the global asymptotic stability of the unique equilibrium. The technique of the invariant rectangles in the phase-space and the Lyapunov method are used. Our investigation confers a theoretical background for the numerical and experimental results in [6].

Keywords and phrases: Global stability; Degn-Harrison model; Attractive region; Isocline.

1. INTRODUCTION

The experimental and theoretical studies of ecological and chemical systems are an increasing object of research because of the importance of such systems in laboratory experiments and in industrial processes. In this paper we consider the model proposed by Degn and Harrison [4, 5, 7] for explaining the mechanism leading to respiratory oscillations in continuous cultures of *Klebsiella aerogenes*. From a mathematical point of view, the Degn-Harrison model yields the following differential equations for the temporal evolution of the system

$$\begin{cases} u' = a - u - \frac{uv}{1 + ku^2} \\ v' = b - \frac{uv}{1 + ku^2} \end{cases} \quad (1.1)$$

where $u(t)$ and $v(t)$ represent the dimensionless concentrations of oxygen and nutrient, respectively. The above model has been analyzed by Fairen and Velarde [6] using computer aided methods. For $k = 1$, ODE system (1.1) possesses striking similarity with regards

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to the occurrence of nonlinear terms in the Lengyel-Epstein system

$$\begin{cases} u' = a - u - \frac{4uv}{1+u^2} \\ v' = b(u - \frac{uv}{1+u^2}) \end{cases} \quad (1.2)$$

describing the chlorite-iodide-malonic acid (CIMA) chemical reaction [5, 7]. Both systems (1.1) and (1.2) have a unique equilibrium solution and produce oscillatory states but the Degn-Harrison model (1.1) presents a richer dynamics. We are mainly interested in the issue of the global stability of the constant solution (u^*, v^*) to (1.1). It is well known that different parameters values in a model can give rise to very different dynamics, such as equilibrium stability, oscillatory behavior or bifurcation. Thus, the property of global asymptotic stability is an important information to exclude other different dynamics such as the presence of periodic solutions. Through a systematic mathematical analysis, our investigation confers a theoretical back-ground for the numerical and experimental results found in [6]. We look at the phase portrait of nonlinear system (1.1), studying the mathematical properties of its isocline curves for all ranges of the parameters values. The existence of appropriate invariant rectangles, proved in Sections 3 and 4, becomes a fundamental tool for the understanding of the system dynamics. In particular for the existence of periodic solution proved in Corollary 3.1 and for the global stability of (u^*, v^*) obtained in Theorems 4.2 and 4.4. Our results can also contribute to a better investigation of the Degn-Harrison reaction-diffusion system

$$\begin{cases} \frac{\partial u}{\partial t} = d_1 \Delta u + a - u - \frac{uv}{1+ku^2} \\ \frac{\partial v}{\partial t} = d_2 \Delta v + b - \frac{uv}{1+ku^2} \end{cases} \quad x \in \Omega \quad (1.3)$$

in a bounded domain Ω in \mathbf{R}^n . Recently, in [8, 10], the authors analyze parabolic system (1.3) investigating the stability of the non constant steady-states, the Hopf bifurcation and the existence of Turing patterns. However they do not provide further informations on the corresponding ordinary differential equations system. Global attractivity results for the unique constant steady state of the reaction-diffusion Lengyel-Epstein model have been obtained by Yi, Wei and Shi[12] and Lisena[9]. They construct a suitable

Lyapunov function by using the monotonicity properties of the isocline curves. The mathematical method developed in [9, 12] cease to work for system (1.1) without deep modifications. The search of an invariant region surrounded (u^*, v^*) in which the isoclines are monotone permits to apply the Lyapunov methods when parameters a, b, k fall into certain ranges. Theorem 4.4 uses this arguments to demonstrate the global attractivity of (u^*, v^*) if $a > 2b$. The inserted Figures (obtained by applying MATHEMATICA) aim to strengthen our analytical analysis.

2. PRELIMINARY RESULTS

It is convenient to rewrite system (1.1) in the form

$$\begin{cases} u' = F(u, v), & t > 0 \\ v' = G(u, v) \end{cases} \quad (2.4)$$

where

$$F(u, v) = a - u - \phi_k(u) v, \quad G(u, v) = b - \phi_k(u) v \quad (2.5)$$

and

$$\phi_k(u) = \frac{u}{1 + k u^2}.$$

We consider positive solutions to the model subject to the initial condition

$$u(0) = u_0 > 0, \quad v(0) = v_0 > 0.$$

Assuming $a > b$, (2.1) has in \mathbf{R}_+^2 the equilibrium

$$(u^*, v^*), \quad u^* = a - b, \quad v^* = \frac{b}{\phi_k(u^*)}.$$

By the linearization technique one can easily obtain sufficient conditions for the stability of (u^*, v^*) . The Jacobian matrix evaluated at (u^*, v^*) has the form

$$J = \begin{pmatrix} -1 - \frac{b}{u^*} \frac{1 - k(u^*)^2}{1 + k(u^*)^2} & -\phi_k(u^*) \\ -\frac{b}{u^*} \frac{1 - k(u^*)^2}{1 + k(u^*)^2} & -\phi_k(u^*) \end{pmatrix}$$

Since

$$\det J = \phi_k(u^*) > 0,$$

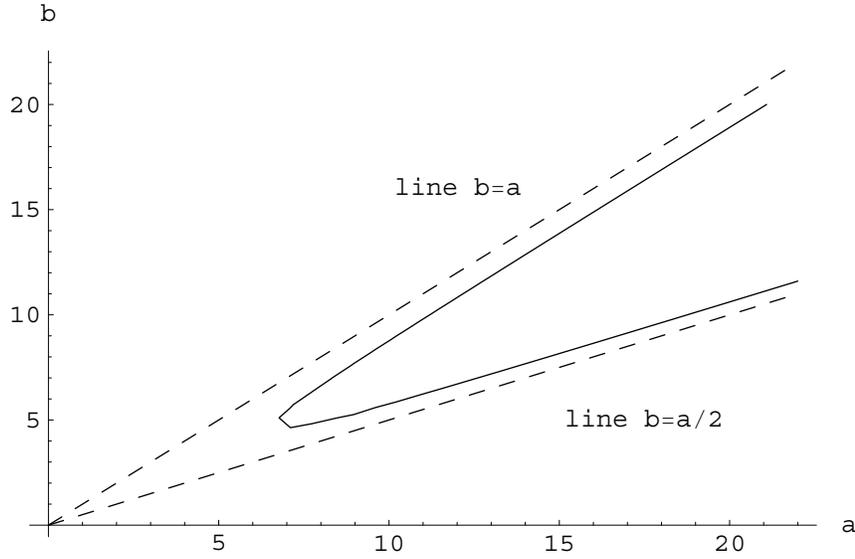


FIGURE 1. Curve $a + (a - b)^2(1 + a - 2b) = 0$, $b < a$

the eigenvalues of J have negative real parts if and only if

$$\operatorname{tr} J = -\frac{a + (a - b)^2(1 + k(a - 2b))}{(a - b)(1 + k(a - b)^2)} < 0.$$

Then (u^*, v^*) is (locally) asymptotically stable if

$$a + (a - b)^2(1 + k(a - 2b)) > 0, \quad (2.6)$$

it is instable if

$$a + (a - b)^2(1 - k(2b - a)) < 0.$$

Note that (2.3) is certainly verified when $2b \leq a$.

In Figure 1, the curve defined by the Cartesian equation

$$a + (a - b)^2(1 + k(a - 2b)) = 0, \quad a > b$$

is plotted in the plane ab , with fixed $k = 1$. It lies between the lines $b = a$ and $b = \frac{a}{2}$. In [10] it has been proved that (u^*, v^*) is a global attractor in \mathbf{R}_+^2 if $a^2k \leq 1$. Therefore, henceforth we suppose

$$a^2k > 1 \quad \text{that is} \quad \frac{1}{\sqrt{k}} \in]0, a[. \quad (2.7)$$

Further informations about the dynamics of system (2.1) can be derived from the detailed analysis of its isoclines. The plot of the v -isocline

$$\gamma_v : \quad v = \frac{b}{\phi_k(u)}$$

reflects the following properties:

$$i) \quad \lim_{u \rightarrow 0^+} \frac{b}{\phi_k(u)} = +\infty ;$$

$$ii) \quad \frac{b}{\phi_k(u)} \text{ is strictly decreasing in }]0, \frac{1}{\sqrt{k}}], \text{ is strictly increasing in }]\frac{1}{\sqrt{k}}, a];$$

iii) The minimum point has coordinates $(\frac{1}{\sqrt{k}}, 2b\sqrt{k})$.

The graph of the u -isocline

$$\gamma_u : \quad v = f_{a,k}(u), \quad f_{a,k}(u) = \frac{a-u}{\phi_k(u)}$$

is less simple because the monotonicity of $f_{a,k}$ is sensitive to different values of parameters a, k .

Theorem 2.1. *In interval $]0, a]$ the function $f_{a,k}$ has the following properties:*

$$i) \quad \lim_{u \rightarrow 0^+} f_{a,k}(u) = +\infty, \quad f_{a,k}(a) = 0 ;$$

ii) If

$$a^2k \leq 27 \tag{2.8}$$

$f_{a,k}(u)$ is strictly decreasing ;

iii) If $a^2k > 27$, $f_{a,k}(u)$ is strictly decreasing in $]0, \frac{1}{\sqrt{k}}]$ and in $[\frac{a}{2}, a]$.

In interval $]\frac{1}{\sqrt{k}}, \frac{a}{2}[$ there is a local minimum point u_1 , a local maximum point u_2 .

Moreover $\sqrt[3]{\frac{a}{k}}$ is a saddle point and $u_1 < \sqrt[3]{\frac{a}{k}} < u_2$.

Proof. Since $f_{a,k}(u) = (\frac{a}{u} - 1)(1 + ku^2)$, $i)$ is obvious.

To prove $ii)$ and $iii)$ observe that

$$f'_{a,k}(u) = -\frac{a}{u^2} + ka - 2ku = -\frac{a}{u^2} + k(a - 2u) = a(k - \frac{1}{u^2}) - 2ku \tag{2.9}$$

so that $f_{a,k}(u)$ is strictly decreasing in $]0, \frac{1}{\sqrt{k}}]$ and in $[\frac{a}{2}, a]$. To study the sign of $f'_{a,k}(u)$ in $]\frac{1}{\sqrt{k}}, \frac{a}{2}[$, consider $f''_{a,k}(u) = \frac{2a}{u^3} - 2k$ and note that assumption (2.4) implies $\frac{1}{\sqrt{k}} < \sqrt[3]{\frac{a}{k}}$. We deduce that $f''_{a,k}(u) > 0$ in $]\frac{1}{\sqrt{k}}, \sqrt[3]{\frac{a}{k}}[$, $f''_{a,k}(u) < 0$ in $]\sqrt[3]{\frac{a}{k}}, a[$, $\sqrt[3]{\frac{a}{k}}$ is a maximum point for $f'_{a,k}(u)$ and $f'_{a,k}(\sqrt[3]{\frac{a}{k}}) = ka - 3\sqrt[3]{ak^2}$. Under assumption (2.5)

$$f'_{a,k}(u) \leq f'_{a,k}(\sqrt[3]{\frac{a}{k}}) \leq 0$$

and $ii)$ easily follows.

If $a^2k > 27$, $\frac{1}{\sqrt{k}} < \sqrt[3]{\frac{a}{k}} < \frac{a}{2}$. A comparison between the curves $v = \frac{a}{u^2}$ and $v = k(a - 2u)$ completes the proof of $iii)$.

□

Figure 2.(a)-(d), illustrates nullclines γ_u and γ_v for different values of parameters a, b, k .

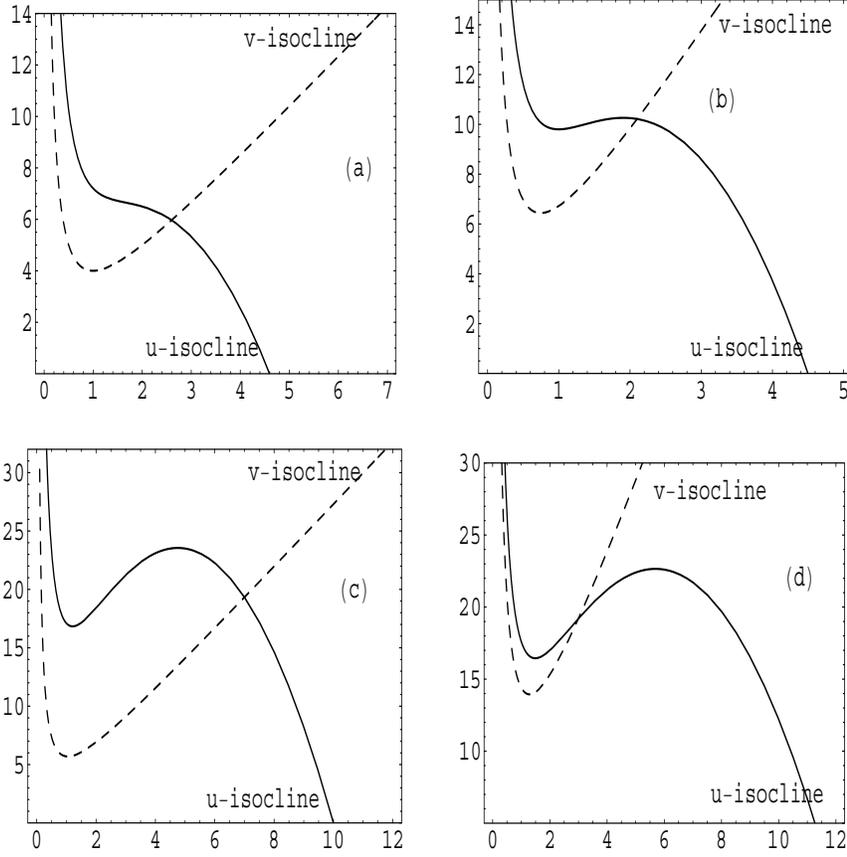


FIGURE 2. Isoclines: (a) $a=4.6, b=2, k=1$ (b) $a=4.5, b=2.4, k=1.8$ (c) $a=10, b=3, k=0.9$ (d) $a=12, b=9, k=0.6$

3. INVARIANT REGION

In this section we will show that ODE system (2.1) has an invariant region

$$\mathcal{R} = [\bar{u}, a] \times [2b\sqrt{k}, f_{a,k}(\bar{u})], \quad \bar{u} = \frac{bu^*}{a(1+ka^2)}$$

in the phase plane which actually attracts all solutions, regardless of the initial values u_0 and v_0 .

Theorem 3.1. *Let $(u(t), v(t))$ be any positive solution of (2.1) with initial conditions $(u_0, v_0) \in \mathcal{R}$. Then*

$$(u(t), v(t)) \in \mathcal{R}, \quad t > 0.$$

Proof. First we check that

$$\bar{u} < \frac{1}{\sqrt{k}}.$$

Since $b < a$ we are allowed to write $b = ta$, $0 < t < 1$, so that

$$bu^* = t(1-t)a^2 \leq \frac{a^2}{4}, \quad t \in]0, 1[.$$

Consequently

$$\bar{u} \leq \frac{a}{4(1+ka^2)} = \frac{\phi_k(a)}{4} \leq \frac{1}{8\sqrt{k}} < \frac{1}{\sqrt{k}}. \quad (3.10)$$

By Theorem 2.1, $f_{a,k}(u)$ and $\frac{b}{\phi_k(u)}$ are strictly decreasing in $]0, \bar{u}]$ (for each k).

As second step let us verify that

$$\frac{b}{\phi_k(a)} < f_{a,k}(\bar{u}). \quad (3.11)$$

Observe that, using (3.1),

$$f_{a,k}(\bar{u}) = \left(\frac{a}{\bar{u}} - 1\right)(1 + k\bar{u}^2) > (4(1 + ka^2) - 1)(1 + k\bar{u}^2) > 4ka^2 + 3$$

and

$$\frac{b}{\phi_k(a)} = \frac{b}{a}(1 + ka^2) < 1 + ka^2.$$

Hence (3.2) easily follows. Previous estimates proves, in particular, that (u^*, v^*) lies in the interior of \mathcal{R} . Taking into account our investigation in Section 2, we can state that, on the boundary of \mathcal{R} , the vector field $(F(u, v), G(u, v))$, defined by (2.2), does not point outwards. Indeed

$$F(\bar{u}, v) \geq 0 \quad \text{and} \quad F(a, v) \leq 0 \quad \text{for} \quad 2b\sqrt{k} \leq v \leq f_{a,k}(\bar{u}),$$

$$G(u, 2b\sqrt{k}) \geq 0 \quad \text{and} \quad G(u, f_{a,k}(\bar{u})) \leq 0 \quad \text{for} \quad \bar{u} \leq u \leq a.$$

Hence rectangle \mathcal{R} is an invariant region and the proof is complete.

□

Theorem 3.2. *Let $(u(t), v(t))$ be any solution of (2.1). Then there is a constant $T > 0$, which may depend on u_0 and v_0 , such that*

$$(u(t), v(t)) \in \mathcal{R} \quad \text{for all } t > T.$$

Proof. It is obvious that

$$u'(t) \leq a - u(t).$$

Since all positive solutions of the ODE

$$x'(t) = a - x(t)$$

tends to a as $t \rightarrow +\infty$, by the comparison theorem,

$$u(t) \leq a \quad \text{for } t \text{ sufficiently large.}$$

Analogously, since $\phi_k(u) \leq \frac{1}{2\sqrt{k}}$,

$$v'(t) \geq b - \frac{v(t)}{2\sqrt{k}}.$$

Taking into account that all solutions of the differential equation

$$y'(t) = b - \frac{y(t)}{2\sqrt{k}}$$

approach $2b\sqrt{k}$ as $t \rightarrow +\infty$, we get

$$v(t) \geq 2b\sqrt{k} \quad \text{for } t \text{ sufficiently large.}$$

Previous argument proves that

$$\mathcal{Q} =]0, a[\times [2b\sqrt{k}, +\infty[$$

is an invariant region. Note that

$$\frac{b}{\phi_k(u)} < f_{a,k}(u) \quad \text{if } u < u^*,$$

so that we can divide the region $\mathcal{Q} \setminus \mathcal{R}$ in three parts:

- I) $(u, v) \in \mathcal{Q} \setminus \mathcal{R}, \quad 0 < u < \bar{u}, \quad v \leq \frac{b}{\phi_k(u)};$
- II) $(u, v) \in \mathcal{Q} \setminus \mathcal{R}, \quad v \geq f_{a,k}(u);$
- III) $(u, v) \in \mathcal{Q} \setminus \mathcal{R}, \quad 0 < u < \bar{u}, \quad \frac{b}{\phi_k(u)} < v < f_{a,k}(u).$

The trajectories starting in region I) have both components strictly increasing then they go into \mathcal{R} , at some t , otherwise they get into region III) after intersecting isocline γ_v . Similarly, the trajectories starting in region II) have both components strictly decreasing then, after intersecting the line $v = f_{a,k}(\bar{u})$, they go into \mathcal{R} or they intersect isocline γ_u and enter into region III). Considering the direction of the vector field on isoclines γ_u and γ_v , we deduce that the trajectories starting in region III) cannot leave this region till

they go inside \mathcal{R} , for some t . \square

Corollary 3.1. *Suppose*

$$a + (a - b)^2(1 - k(2b - a)) < 0. \quad (3.12)$$

Then model (2.1) admits at least a periodic solution $(\overset{\circ}{u}(t), \overset{\circ}{v}(t))$ whose orbit lies in \mathcal{R} . The integral average of its first component satisfies

$$m[\overset{\circ}{u}] = u^*.$$

Moreover

$$m[\phi_k(\overset{\circ}{u})\overset{\circ}{v}] = b, \quad m\left[\frac{1}{\overset{\circ}{v}}\right] = \frac{m[\phi_k(\overset{\circ}{u})]}{b}.$$

Proof. Under (3.4) the equilibrium (u^*, v^*) is instable in the attractive invariant rectangle \mathcal{R} . Thus, from the Poincaré-Bendixson theorem (see [2, 3]), it follows that some orbits as limit cycles have to exist in \mathcal{R} . Denoting by $(\overset{\circ}{u}(t), \overset{\circ}{v}(t))$ one of such periodic solutions with period τ , the integration of both sides of (2.1) over $[0, \tau]$ leads to

$$0 = m[(\overset{\circ}{u})'] = a - m[\overset{\circ}{u}] - m[\phi_k(\overset{\circ}{u})\overset{\circ}{v}], \quad 0 = m[(\overset{\circ}{v})'] = b - m[\phi_k(\overset{\circ}{u})\overset{\circ}{v}],$$

thus

$$m[\overset{\circ}{u}] = a - b = u^*.$$

Moreover, after dividing both sides of equation $(\overset{\circ}{v})' = b - \phi_k(\overset{\circ}{u})\overset{\circ}{v}$ by $\overset{\circ}{v}(t)$, one gets

$$0 = m\left[\frac{(\overset{\circ}{v})'}{\overset{\circ}{v}}\right] = m\left[\frac{b}{\overset{\circ}{v}}\right] - m[\phi_k(\overset{\circ}{u})]$$

concluding the proof. \square

In Figure 3 the orbit of the periodic solution located in [6] ($a = 19.4$, $b = 11$, $k = 0.5$) is plotted in the phase plane. Inside the trajectories of two solutions starting close to (u^*, v^*) . In Figure 4, the orbit passing through (4.6, 42.9) is given for the model with parameters values $a = 10$, $b = 6$, $k = 1.5$. The numerical simulation shows that it is the only limit cycle.

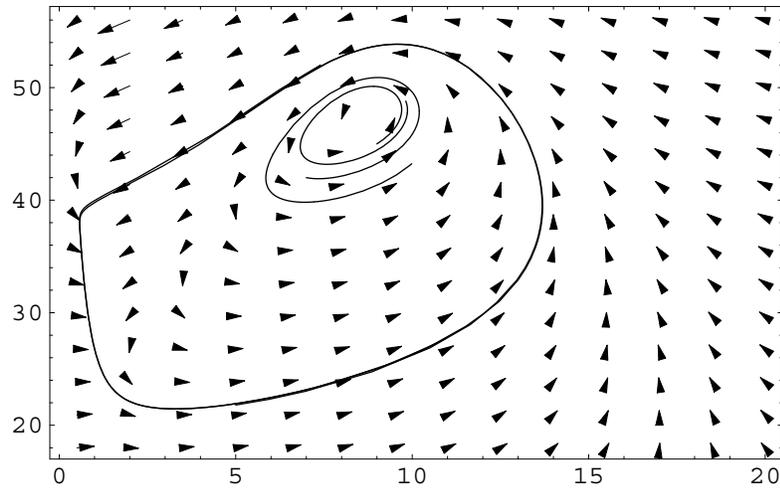


FIGURE 3. $a = 19.4$, $b = 11$, $k = 0.5$. The periodic solution found in [6] and the vector field generated by (2.1)

4. GLOBAL ATTRACTIVITY

In the next theorem a certain range of parameters is determined for which no periodic solution may exist. Obviously we are excluding the constant solution (u^*, v^*) .

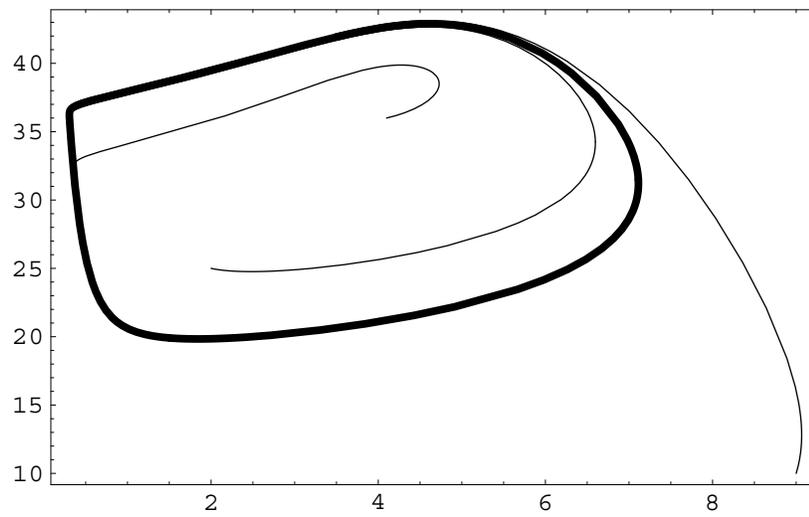


FIGURE 4. $a = 10$, $b = 6$, $k = 1.5$. The equilibrium is unstable and the plotted trajectories tends to the orbit of the periodic solution through $(4.6, 42.9)$

Theorem 4.1. *Let us suppose that*

$$3\sqrt[3]{a k^2} - a k + 1 > 0, \quad (4.13)$$

then, for any $b < a$, system (2.1) admits no (non constant) periodic solutions.

Proof. Rewrite system (2.1) in the form

$$\begin{cases} u' = \phi_k(u)(f_{a,k}(u) - v), & t > 0 \\ v' = \phi_k(u)\left(\frac{b}{\phi_k(u)} - v\right). \end{cases} \quad (4.14)$$

By the Dulac's criterion, if $\frac{\partial}{\partial u}(f_{a,k}(u) - v) + \frac{\partial}{\partial v}\left(\frac{b}{\phi_k(u)} - v\right)$ does not change sign in \mathcal{R} , then (4.2) has no closed orbits lying entirely in \mathcal{R} (see [3, 11, 1]). Obviously

$$\frac{\partial}{\partial u}(f_{a,k}(u) - v) + \frac{\partial}{\partial v}\left(\frac{b}{\phi_k(u)} - v\right) = f'_{a,k}(u) - 1.$$

In Theorem 2.1 we saw that

$$f'_{a,k}(u) = -\frac{a}{u^2} + k a - 2k u$$

and $f'_{a,k}(u) \leq 0$ for $a^2 k \leq 27$. Moreover, if $a^2 k > 27$, it is enough to prove

$$f'_{a,k}\left(\sqrt[3]{\frac{a}{k}}\right) - 1 < 0$$

to provide inequality $f'_{a,k}(u) - 1 < 0$. The equality

$$f'_{a,k}\left(\sqrt[3]{\frac{a}{k}}\right) - 1 = -3\sqrt[3]{a k^2} + k a - 1$$

concludes the proof. \square

Next Theorems 4.2 and 4.4 are the main results of this paper.

Theorem 4.2. *Assume that condition (4.1) is satisfied, then (u^*, v^*) is globally asymptotically stable in \mathbf{R}_+^2 .*

Proof. Firstly let us prove that, under condition (4.1), (u^*, v^*) is asymptotically stable. By the results in Section 2, it is enough to demonstrate that inequality (2.3) holds. We may assume

$$\frac{a}{2} < b < a$$

because the validity of (2.3) is evident for $b \leq \frac{a}{2}$. Introduce the function

$$g(x) = \frac{a}{(a-x)^2} - (2x-a)k + 1. \quad (4.15)$$

We are going to prove that, under assumption (4.1),

$$g(x) > 0, \quad \frac{a}{2} < x < a. \quad (4.16)$$

It is easy to check that

$$g\left(\frac{a}{2}\right) = \frac{4}{a} + 1 > 0, \quad \lim_{x \rightarrow a^-} g(x) = +\infty,$$

and

$$g'(x) = \frac{2a}{(a-x)^3} - 2k$$

is strictly positive if and only if $x > a - \sqrt[3]{\frac{a}{k}}$. When $a^2k \leq 8$, it turns out $a - \sqrt[3]{\frac{a}{k}} \leq \frac{a}{2}$ so that $g(x)$ is strictly increasing in interval $]\frac{a}{2}, a[$. As a consequence

$$g(x) > g\left(\frac{a}{2}\right) > 0, \quad \frac{a}{2} < x < a.$$

Let us consider the case $a^2k > 8$. The point $a - \sqrt[3]{\frac{a}{k}}$ is a minimum for $g(x)$,

$$g\left(a - \sqrt[3]{\frac{a}{k}}\right) = 3\sqrt[3]{ak^2} - ak + 1$$

therefore, using (4.1),

$$g(x) \geq g\left(a - \sqrt[3]{\frac{a}{k}}\right) > 0, \quad \frac{a}{2} < x < a.$$

and (4.4) is proved. It implies that $a + (a-b)^2(1 - (2b-a)k) > 0$ for each $b \in]\frac{a}{2}, a[$. From the first step of the proof, (u^*, v^*) is asymptotically stable in \mathcal{R} , by Theorems 3.1 and 3.2 \mathcal{R} is invariant and attractive, by Theorem 4.1 no periodic solutions exist in \mathbf{R}_+^2 . Applying the Poincaré-Bendixson theorem, for any positive solution $(u(t), v(t))$ to system (2.1), we get

$$\lim_{t \rightarrow +\infty} |u(t) - u^*| = 0 = \lim_{t \rightarrow +\infty} |v(t) - v^*|$$

and the global stability is proved. \square

Arguing as in the previous theorem, we can obtain a new criterion of global stability for the equilibrium of the Lengyel-Epstein system.

Theorem 4.3. *Let*

$$a - 3\sqrt[3]{a} < b, \quad (4.17)$$

then $(\alpha, 1 + \alpha^2)$, $\alpha = \frac{a}{5}$, is globally asymptotically stable as equilibrium of system (1.2).

Proof. The rectangle $\mathcal{R} =]0, a[\times]0, 1 + a^2[$ is an invariant, attractive region for model (1.2) (see [9]) and $(\alpha, 1 + \alpha^2)$ is the unique constant solution of (1.2). It is locally asymptotically stable under the condition $3\alpha^2 - 5 < (\alpha b)$ [9]. Notice that

$$a - 3\sqrt[3]{a} \geq \frac{3\alpha^2 - 5}{\alpha} = \frac{3a^2 - 125}{5a}.$$

System (1.2) can be written in the form

$$\begin{cases} u' = \phi(u)(f_a(u) - 4v), & t > 0 \\ v' = \phi(u)(b(1 + u^2) - bv), \end{cases}$$

where $\phi(u) = \frac{u}{1+u^2}$, $f_a(u) = \frac{a-u}{\phi(u)}$. By the Dulac's criterion, if

$$\frac{\partial}{\partial u}(f_a(u) - 4v) + \frac{\partial}{\partial v}(b(1 + u^2) - bv) < 0 \quad \text{in } \mathcal{R}$$

then (1.2) has no nonconstant periodic solution in \mathcal{R} . It is obvious that

$$\frac{\partial}{\partial u}(f_a(u) - 4v) + \frac{\partial}{\partial v}(b(1 + u^2) - bv) = f'_a(u) - b = -\frac{a}{u^2} + a - 2u - b.$$

Let us prove that

$$f'_a(u) < b, \quad u \in]0, a[.$$

It is easy to see that $f'_a(u)$ attains its maximum value at $u = \sqrt[3]{a}$ and

$$f'_a(\sqrt[3]{a}) = a - 3\sqrt[3]{a}.$$

Thus, under (4.5),

$$f'_a(u) \leq f'_a(\sqrt[3]{a}) < b.$$

An application of the Poincaré-Bendixson theorem gives the statement. \square

Theorem 4.4. *Suppose that $3\sqrt[3]{a k^2} - a k + 1 \leq 0$ and*

$$b \leq \frac{a}{4} \left(\frac{4 + a^2 k}{1 + a^2 k} \right). \quad (4.18)$$

Then (u^, v^*) is globally attractive in \mathbf{R}_+^2 .*

Proof. From the arguments in Theorem 4.1, we deduce that our first assumption ensures that $a^2 k > 27$. This inequality guarantees that $\frac{4+a^2 k}{1+a^2 k} < \frac{4}{3}$. Consequently, using (4.6)

$$b < \frac{a}{3} \quad \text{and} \quad u^* = a - b > \frac{2}{3}a .$$

Notice that

$$f_{a,k}\left(\frac{a}{2}\right) = 1 + k \frac{a^2}{4}$$

so that inequality (4.6) gives

$$\frac{b}{\phi_k(a)} \leq f_{a,k}\left(\frac{a}{2}\right) .$$

Let

$$\mathcal{T} = \left[\frac{a}{2}, a\right] \times \left[2b\sqrt{k}, f_{a,k}\left(\frac{a}{2}\right)\right] \subset \mathcal{R} .$$

Previous arguments ensures that, on the boundary of \mathcal{T} , the vector field (F, G) , defined by (2.2), does not point outwards, so that \mathcal{T} is an invariant region. In rectangle \mathcal{T} isocline γ_u is strictly decreasing, isocline γ_v is strictly increasing and they intersect in $(u^*, v^*) \in \mathcal{T}$. Taking advantage of such monotone properties, we are able to employ the Lyapunov method. Once again rewrite system (2.1) in the form

$$\begin{cases} u' = \phi_k(u)[(f_{a,k}(u) - f_{a,k}(u^*)) - (v - v^*)] \\ v' = \phi_k(u) \left[\left(\frac{b}{\phi_k(u)} - \frac{b}{\phi_k(u^*)} \right) - (v - v^*) \right] . \end{cases} \quad (4.19)$$

Consider the following Lyapunov function

$$V(u, v) = \int_{u^*}^u \left(\frac{b}{\phi_k(s)} - \frac{b}{\phi_k(u^*)} \right) ds + \frac{(v - v^*)^2}{2} .$$

Let $(u(t), v(t))$ be a solution of (4.7) in \mathcal{T} for $t \geq 0$ and put $V(t) = V(u(t), v(t))$. Then

$$\begin{aligned} V'(t) &= \left(\frac{b}{\phi_k(u)} - \frac{b}{\phi_k(u^*)} \right) u' + (v - v^*) v' \\ &= \phi_k(u) \left[\left(\frac{b}{\phi_k(u)} - \frac{b}{\phi_k(u^*)} \right) (f_{a,k}(u) - f_{a,k}(u^*)) - (v - v^*)^2 \right] . \end{aligned}$$

Since

$$\left(\frac{b}{\phi_k(u)} - \frac{b}{\phi_k(u^*)} \right) < 0, \quad (f_{a,k}(u) - f_{a,k}(u^*)) > 0 \quad \text{for} \quad u < u^*$$

$$\left(\frac{b}{\phi_k(u)} - \frac{b}{\phi_k(u^*)} \right) > 0, (f_{a,k}(u) - f_{a,k}(u^*)) < 0 \quad \text{for } u > u^*,$$

it turns out that the derivative of $V(u, v)$ along trajectories of (4.7) on \mathcal{T} is negative and $V'(t) = 0$ only on the equilibrium solution (u^*, v^*) . Thus, for all initial values $(u_0, v_0) \in \mathcal{T}$, the solution of (4.7) converges, as $t \rightarrow +\infty$, to (u^*, v^*) . At this point we can state that no periodic solution can exist in \mathcal{R} . Indeed by contradiction, assume $(\overset{\circ}{u}(t), \overset{\circ}{v}(t))$ be a periodic solution in \mathcal{R} . Corollary 3.1 yields $m[\overset{\circ}{u}] = u^*$ hence, a time value $\bar{t} > 0$ there exists such that

$$(\overset{\circ}{u}(\bar{t}), \overset{\circ}{v}(\bar{t})) \in \mathcal{T}.$$

Obviously this last property cannot hold. Arguing as in Theorem 4.2, we conclude that (u^*, v^*) is globally attractive in R_+^2 . \square

The next Figure shows in rectangle \mathcal{T} the equilibrium (u^*, v^*) and the direction of the vectorial field generated by (2.1), taking $a = 7.6, b = 2, k = 0.6$. For the same parameters values, in Figure 6, some trajectories starting in \mathcal{T} are plotted.

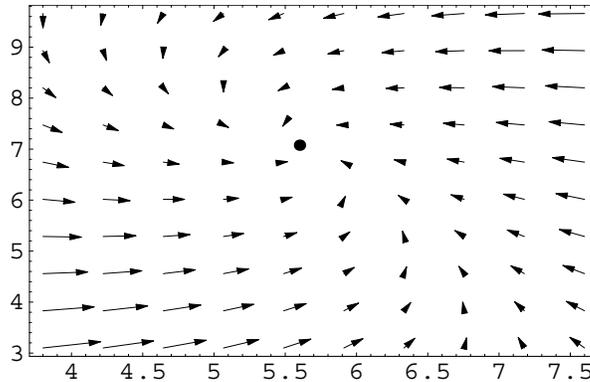


FIGURE 5. $a = 7.6, b = 2, k = 0.6$ The vector field in \mathcal{T} . The filled point denotes (u^*, v^*) .

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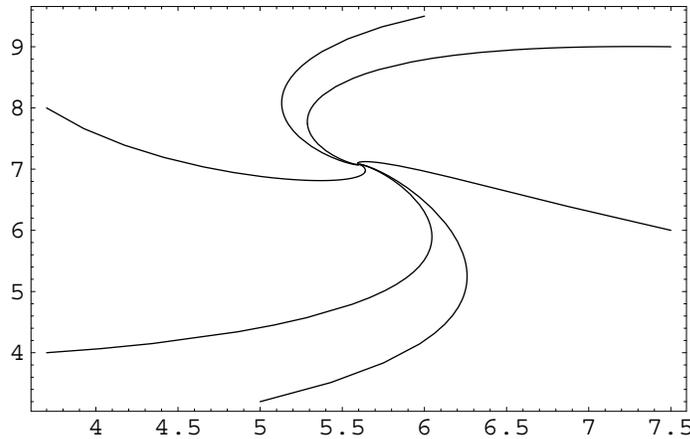


FIGURE 6. $a = 7.6$, $b = 2$, $k = 0.6$ In rectangle \mathcal{T} the equilibrium is globally attractive.

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