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SOME GEOMETRIC PROPERTIES OF A FAMILY WALKER METRIC ON AN EIGHT-DIMENSIONAL MANIFOLDS

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ABSTRACT. A pseudo-Riemannian manifold which admits a field of parallel null r-planes, with $r \leq \frac{n}{2}$ is a Walker manifold. A. G. Walker in [18] investigated the canonical forms of the metrics and came out with some interesting results. Of special interest are the even-dimensional Walker manifolds ($n = 2m$) with fields of parallel null planes of half dimension ($r = m$). In this paper, we consider a particular eight-dimensional Walker manifold, derive and investigate some geometric properties of the curvature tensors of the manifold. We give some theorems for the metric to be Einstein and locally conformally flat.

Keywords and phrases: Einstein manifolds, locally conformally flats; Walker metrics.
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1. Introduction

The study of the curvature properties of a given class of pseudo-Riemannian manifolds is important to our knowledge of these spaces. They are use to exemplify some of the main differences between the geometry of Riemannian manifolds and the geometry of pseudo-Riemannian manifolds and thereby illustrate phenomena in pseudo-Riemannian geometry that are quite different from those which occur in Riemannian geometry.

Walker n-manifold is a pseudo-Riemannian manifold which admit a non-trivial parallel null r-plane field with $r \leq n$. Walker n-manifold is applicable in physics. Lorentzian Walker manifolds have been studied extensively in physics since they constitute the background metric of the

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pp-wave models. A *pp*-wave spacetime admits a covariantly constant null vector field U [1].

The theory of Walker manifolds is outlined in [2]. The authors treated hypersurfaces with nilpotent shape operators, locally conformally flat metrics with nilpotent Ricci operator, degenerate pseudo-Riemannian homogeneous structures, para-Kaehler structures, and 2-step nilpotent Lie groups with degenerate center. The curvature properties of a large class of 4-dimensional Walker metrics are treated in [3] and several interesting examples are given (see [4, 5, 9, 10, 11, 12] and references therein). The Walker 6-dimensional Walker metrics was also considered in ([6, 7]).

There are few studies in the class of 8-dimensional Walker metrics manifolds. Some examples in this direction may be found in [8, 13] and references therein. In [8], the geometric properties of some curvature tensors of an 8-dimensional Walker manifold are investigated, theorems for the metric to be Einstein, locally conformally flat and for the 8-dimensional manifold to admit a Kähler structure are given. We want to extent this study to a canonical form for a non-strict eight dimensional walker manifold. We derive the $(0,4)$-curvature tensor, the Ricci tensor, Weyl tensor and study some of the properties associated with a class of non-strict 8-dimensional Walker manifold. We investigate the Einstein property and establish a theorem for the metric to be locally conformally flat.

The paper is organized as follows. In Section 2, we will give the canonical form of Walker metrics and some theorems associated with it. One specific family of Walker metric of 8-dimensional manifolds is investigated in Section 3. We find the form of the defining functions that makes those metrics similar to Einstein and locally conformally flat metrics.

2. The canonical form of Walker metrics

Let M be a pseudo-Riemannian manifold of signature (m,m). We suppose given a splitting of the tangent bundle in the form $TM = \mathcal{D}_1 \oplus \mathcal{D}_2$ where $\mathcal{D}_1$ and $\mathcal{D}_2$ are smooth subbundles which are called distributions. This define two complementary projection π_1 and π_2 of TM onto $\mathcal{D}_1$ and $\mathcal{D}_2$. We say that $\mathcal{D}_1$ is parallel distribution if $\nabla\pi_1 = 0$. Equivalently this means that if X_1 is any smooth vector field taking values in $\mathcal{D}_1$, then ∇X_1 again takes values in $\mathcal{D}_1$. If M is Riemannian, we can take $\mathcal{D}_2 = \mathcal{D}_1^{\perp}$ to be the orthogonal complement of $\mathcal{D}_1$ and in that case $\mathcal{D}_2$ is again parallel. In the pseudo-Riemannian setting, $\mathcal{D}_1 \cap \mathcal{D}_2$ need not be trivial. We say that $\mathcal{D}_1$ is a null parallel distribution if it is parallel and the metric restricted to $\mathcal{D}_1$ vanish identically.

Walker [18] studied pseudo-Riemannian manifolds (M, g) with a parallel field of null planes $\mathcal{D}$ and derived a canonical form. Motivated by this seminal work, one says that a pseudo-Riemannian manifold M which admits a null parallel (i.e., degenerate) distribution $\mathcal{D}$ is a Walker manifold.

It is known that Walker metrics have served as a powerful tool for constructing interesting indefinite metrics which exhibit various aspects of geometric properties not given by any positive definite metrics. Among these, the significant Walker manifolds are the examples of the non-symmetric and non-homogeneous Osserman manifolds [2].

Canonical forms were known previously for parallel non-degenerate distributions. In this case, the metric tensor, in matrix notation, expresses in canonical form as

$$(g_{ij}) = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \tag{1}$$

where A is a symmetric $r \times r$ matrix whose coefficients are functions of $(u_1, \ldots, u_r)$ and B is a symmetric $(n-r) \times (n-r)$ matrix whose coefficients are functions of $(u_{r+1}, \ldots, u_n)$. Here n is the dimension of M and r is the dimension of the distribution $\mathcal{D}$. We will refer [2] for the proof of the following theorems.

[2] A canonical form for an n dimensional pseudo-Riemannian manifold (M, g) admitting a parallel field of null r dimensional planes $\mathcal{D}$ is given by the metric tensor in matrix form as

$$(g_{ij}) = \begin{pmatrix} 0 & 0 & \mathrm{Id}_r \\ 0 & A & H \\ \mathrm{Id}_r & {}^tH & B \end{pmatrix}, \tag{2}$$

where Id_r is the $r \times r$ identity matrix and A, B, H are matrices whose coefficients are functions of the coordinates satisfying the following:

(1) A and B are symmetric matrices of order $(n-2r) \times (n-2r)$ and $r \times r$ respectively. H is a matrix of order $(n-2r) \times r$ and ${}^tH$ stands for the transpose of H.
(2) A and H are independent of the coordinates $(u_1, \ldots, u_r)$.

Furthermore, the null parallel r-plane $\mathcal{D}$ is locally generated by the coordinate vector fields $\{\partial_{u_1}, \ldots, \partial_{u_r}\}$.

[2] A canonical form for an m dimensional pseudo-Riemannian manifold (M, g) admitting a strictly parallel field of null r dimensional planes $\mathcal{D}$ is given by the metric tensor as in Theorem 2, where B is independent of the coordinates $(u_1, \ldots, u_r)$.

3. On 8-dimensional Walker metrics

A neutral g on an 8-manifold M is said to be a Walker metric if there exists a 4-dimensional null distribution $\mathcal{D}$ on M which is parallel with respect to g. From Walker theorem [18], there is a system of coordinates $(u_1, \ldots, u_8)$ with respect to which g takes the local canonical form

$$(g_{ij}) = \begin{pmatrix} 0 & I_4 \\ I_4 & B \end{pmatrix}, \tag{3}$$

where I_4 is the 4×4 identity matrix and B is an 4×4 symmetric matrix whose coefficients are the functions of $(u_1, \ldots, u_8)$. Note that g is of neutral signature $(4, 4)$ and that the parallel null 4-plane $\mathcal{D}$ is spanned locally by $\{\partial_1, \ldots, \partial_4\}$, where $\partial_i = \frac{\partial}{\partial_i}, i = 1, 2, 3, 4$.

In this paper, we consider the specific Walker metrics on 8-dimensional M with B of the form

$$B = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & b \end{pmatrix} \tag{4}$$

where a, b are smooth functions of the coordinates $(u_1, \ldots, u_8)$. We will denote $a_i = \frac{\partial a(u_1,\ldots,u_8)}{\partial u_i}$ and $b_i = \frac{\partial b(u_1,\ldots,u_8)}{\partial u_i}$.

Our purpose is to systematically study the Walker metrics by focusing on their curvature properties. The main purpose is to study conditions for the eight dimensional Walker manifolds which admit a field of parallel null 4-planes to be Einsteinian.

From the following formula:

$$\Gamma_{ij}^k = \sum_{l=1}^{8} \frac{1}{2} g^{kl}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}),$$

we have:

$$\begin{aligned}
\Gamma_{ij}^1 &= \frac{1}{2}g^{11}(\partial_i g_{j1} + \partial_j g_{i1} - \partial_1 g_{ij}) + \frac{1}{2}g^{15}(\partial_i g_{j5} + \partial_j g_{i5} - \partial_5 g_{ij});\\
\Gamma_{ij}^2 &= \frac{1}{2}g^{22}(\partial_i g_{j2} + \partial_j g_{i2} - \partial_2 g_{ij}) + \frac{1}{2}g^{26}(\partial_i g_{j6} + \partial_j g_{i6} - \partial_6 g_{ij});\\
\Gamma_{ij}^3 &= \frac{1}{2}g^{33}(\partial_i g_{j3} + \partial_j g_{i3} - \partial_3 g_{ij}) + \frac{1}{2}g^{37}(\partial_i g_{j7} + \partial_j g_{i7} - \partial_7 g_{ij});\\
\Gamma_{ij}^4 &= \frac{1}{2}g^{44}(\partial_i g_{j4} + \partial_j g_{i4} - \partial_4 g_{ij}) + \frac{1}{2}g^{48}(\partial_i g_{j8} + \partial_j g_{i8} - \partial_8 g_{ij});\\
\Gamma_{ij}^5 &= \frac{1}{2}g^{51}(\partial_i g_{j1} + \partial_j g_{i1} - \partial_1 g_{ij}); \quad \Gamma_{ij}^6 = \frac{1}{2}g^{62}(\partial_i g_{j2} + \partial_j g_{i2} - \partial_2 g_{ij});\\
\Gamma_{ij}^7 &= \frac{1}{2}g^{73}(\partial_i g_{j3} + \partial_j g_{i3} - \partial_3 g_{ij}); \quad \Gamma_{84}^8 = \frac{1}{2}g^{84}(\partial_i g_{j4} + \partial_j g_{i4} - \partial_4 g_{ij});
\end{aligned}$$

where g_{ij} and g^{ij} are the coefficients of the matrix (3). Thus, the non-zero Christoffel symbols Γ_{ij}^k of the Levi-Civita connection of the Walker metric (3) and (4) are given by

$$\begin{aligned}
\Gamma_{ij}^1 &= \frac{-a}{2}\Big(\partial_i g_{j1} + \partial_j g_{i1} - \partial_1 g_{ij}\Big) + \frac{1}{2}\Big(\partial_i g_{j5} + \partial_j g_{i5} - \partial_5 g_{ij}\Big);\\
\Gamma_{ij}^2 &= \frac{-a}{2}\Big(\partial_i g_{j2} + \partial_j g_{i2} - \partial_2 g_{ij}\Big) + \frac{1}{2}\Big(\partial_i g_{j6} + \partial_j g_{i6} - \partial_6 g_{ij}\Big),\\
\Gamma_{ij}^3 &= \frac{-b}{2}\Big(\partial_i g_{j3} + \partial_j g_{i3} - \partial_3 g_{ij}\Big) + \frac{1}{2}\Big(\partial_i g_{j7} + \partial_j g_{i7} - \partial_7 g_{ij}\Big),\\
\Gamma_{ij}^4 &= \frac{-b}{2}\Big(\partial_i g_{j4} + \partial_j g_{i4} - \partial_4 g_{ij}\Big) + \frac{1}{2}\Big(\partial_i g_{j8} + \partial_j g_{i8} - \partial_8 g_{ij}\Big),\\
\Gamma_{ij}^5 &= \frac{1}{2}\Big(\partial_i g_{j1} + \partial_j g_{i1} - \partial_1 g_{ij}\Big), \quad \Gamma_{ij}^6 = \frac{1}{2}\Big(\partial_i g_{j2} + \partial_j g_{i2} - \partial_2 g_{ij}\Big),\\
\Gamma_{ij}^7 &= \frac{1}{2}\Big(\partial_i g_{j3} + \partial_j g_{i3} - \partial_3 g_{ij}\Big), \quad \Gamma_{ij}^8 = \frac{1}{2}\Big(\partial_i g_{j4} + \partial_j g_{i4} - \partial_4 g_{ij}\Big).
\end{aligned}$$

From now on we shall let $\partial_i a = a_i$ and $\partial_i b = b_i$ for all $i = 1,2,3,4,5,6,7,8$ so that we have the following christofell symbols.

The non-zero Christoffel symbols Γ_{ij}^k of the Levi-Civita connection of the Walker metric (3) and (4) are given by

$$\begin{aligned}
\Gamma_{55}^1 &= \frac{aa_1}{2} + \frac{a_5}{2};\ \Gamma_{55}^2 = \frac{aa_2}{2} - \frac{a_6}{2};\ \Gamma_{55}^3 = \frac{ba_3}{2} - \frac{a_7}{2};\ \Gamma_{55}^4 = \frac{ba_4}{2} - \frac{a_8}{2};\\
\Gamma_{55}^5 &= -\frac{a_1}{2};\ \Gamma_{55}^6 = -\frac{a_2}{2};\ \Gamma_{55}^7 = -\frac{a_3}{2};\ \Gamma_{55}^8 = -\frac{a_4}{2};\\
\Gamma_{56}^1 &= \frac{a_6}{2};\ \Gamma_{56}^2 = \frac{a_5}{2};\ \Gamma_{57}^1 = \frac{a_7}{2};\ \Gamma_{57}^3 = \frac{b_5}{2};\ \Gamma_{58}^1 = \frac{a_8}{2};\ \Gamma_{58}^4 = \frac{b_5}{2};\\
\Gamma_{66}^1 &= \frac{aa_1}{2} - \frac{a_5}{2};\ \Gamma_{66}^2 = \frac{aa_2}{2} + \frac{a_6}{2};\ \Gamma_{66}^3 = \frac{ba_3}{2} - \frac{a_7}{2};\ \Gamma_{66}^4 = \frac{ba_4}{2} - \frac{a_8}{2};\\
\Gamma_{66}^5 &= -\frac{a_1}{2};\ \Gamma_{66}^6 = -\frac{a_2}{2};\ \Gamma_{66}^7 = -\frac{a_3}{2};\ \Gamma_{66}^8 = -\frac{a_4}{2};\\
\Gamma_{67}^2 &= \frac{a_7}{2};\ \Gamma_{67}^3 = \frac{b_6}{2};\ \Gamma_{68}^2 = \frac{a_8}{2};\ \Gamma_{68}^4 = \frac{b_6}{2}\\
\Gamma_{77}^1 &= \frac{ab_1}{2} - \frac{b_5}{2};\ \Gamma_{77}^2 = \frac{ab_2}{2} - \frac{b_6}{2};\ \Gamma_{77}^3 = \frac{bb_3}{2} + \frac{b_7}{2};\ \Gamma_{77}^4 = \frac{bb_4}{2} - \frac{b_8}{2};\\
\Gamma_{77}^5 &= -\frac{b_1}{2};\ \Gamma_{77}^6 = -\frac{b_2}{2};\ \Gamma_{77}^7 = -\frac{b_3}{2};\ \Gamma_{77}^8 = -\frac{b_4}{2};\ \Gamma_{78}^3 = \frac{b_8}{2};\ \Gamma_{78}^4 = \frac{b_7}{2};\\
\Gamma_{88}^1 &= \frac{ab_1}{2} - \frac{b_5}{2};\ \Gamma_{88}^2 = \frac{ab_2}{2} - \frac{b_6}{2};\ \Gamma_{88}^3 = \frac{bb_3}{2} - \frac{b_7}{2};\ \Gamma_{88}^4 = \frac{bb_4}{2} + \frac{b_8}{2};\\
\Gamma_{88}^5 &= -\frac{b_1}{2};\ \Gamma_{88}^6 = -\frac{b_2}{2};\ \Gamma_{88}^7 = -\frac{b_3}{2};\ \Gamma_{88}^8 = -\frac{b_4}{2}.
\end{aligned}$$

A straightforward calculation using the Lemma 3, we get the following:

The non zero components of the Levi-Civita connection of the Walker metric (3) and (4) is given by:

$$
\begin{aligned}
\nabla_{\partial_5}\partial_5 &= \Big(\frac{aa_1}{2}+\frac{a_5}{2}\Big)\partial_1+\Big(\frac{aa_2}{2}-\frac{a_6}{2}\Big)\partial_2+\Big(\frac{ba_3}{2}-\frac{a_7}{2}\Big)\partial_3\\
&\quad+\Big(\frac{ba_4}{2}-\frac{a_8}{2}\Big)\partial_4-\frac{a_1}{2}\partial_5-\frac{a_2}{2}\partial_6-\frac{a_3}{2}\partial_7-\frac{a_4}{2}\partial_8;\\
\nabla_{\partial_6}\partial_6 &= \Big(\frac{aa_1}{2}-\frac{a_5}{2}\Big)\partial_1+\Big(\frac{aa_2}{2}+\frac{a_6}{2}\Big)\partial_2+\Big(\frac{ba_3}{2}-\frac{a_7}{2}\Big)\partial_3\\
&\quad+\Big(\frac{ba_4}{2}-\frac{a_8}{2}\Big)\partial_4-\frac{a_1}{2}\partial_5-\frac{a_2}{2}\partial_6-\frac{a_3}{2}\partial_7-\frac{a_4}{2}\partial_8;\\
\nabla_{\partial_7}\partial_7 &= \Big(\frac{ab_1}{2}-\frac{b_5}{2}\Big)\partial_1+(\frac{ab_2}{2}-\frac{b_6}{2}\Big)\partial_2+\Big(\frac{bb_3}{2}+\frac{b_7}{2}\Big)\partial_3\\
&\quad+\Big(\frac{bb_4}{2}-\frac{b_8}{2}\Big)\partial_4-\frac{b_1}{2}\partial_5-\frac{b_2}{2}\partial_6-\frac{b_3}{2}\partial_7-\frac{b_4}{2}\partial_8;\\
\nabla_{\partial_8}\partial_8 &= \Big(\frac{ab_1}{2}-\frac{b_5}{2}\Big)\partial_1+\Big(\frac{ab_2}{2}-\frac{b_6}{2}\Big)\partial_2+\Big(\frac{bb_3}{2}-\frac{b_7}{2}\Big)\partial_3\\
&\quad+\Big(\frac{bb_4}{2}+\frac{b_8}{2}\Big)\partial_4-\frac{b_1}{2}\partial_5-\frac{b_2}{2}\partial_6-\frac{b_3}{2}\partial_7-\frac{b_4}{2}\partial_8;\\
\nabla_{\partial_5}\partial_6 &= \frac{a_6}{2}\partial_1+\frac{a_5}{2}\partial_2;\ \nabla_{\partial_5}\partial_7=\frac{a_7}{2}\partial_1+\frac{b_5}{2}\partial_3;\\
\nabla_{\partial_5}\partial_8 &= \frac{a_8}{2}\partial_1+\frac{b_5}{2}\partial_4;\ \nabla_{\partial_6}\partial_7=\frac{a_7}{2}\partial_2+\frac{b_6}{2}\partial_3;\\
\nabla_{\partial_6}\partial_8 &= \frac{a_8}{2}\partial_2+\frac{b_6}{2}\partial_4;\ \nabla_{\partial_7}\partial_8=\frac{b_8}{2}\partial_3+\frac{b_7}{2}\partial_4.
\end{aligned}
$$

From now, we assume the a and b are functions that depend on u_1, u_2, u_3 and u_4, then by the Lemma 3, we have the following:

$$
\begin{aligned}
\nabla_{\partial_5}\partial_5 &= \frac{aa_1}{2}\partial_1+\frac{aa_2}{2}\partial_2+\frac{ba_3}{2}\partial_3+\frac{ba_4}{2}\partial_4-\frac{a_1}{2}\partial_5-\frac{a_2}{2}\partial_6-\frac{a_3}{2}\partial_7-\frac{a_4}{2}\partial_8;\\
\nabla_{\partial_6}\partial_6 &= \frac{aa_1}{2}\partial_1+\frac{aa_2}{2}\partial_2+\frac{ba_3}{2}\partial_3+\frac{ba_4}{2}\partial_4-\frac{a_1}{2}\partial_5-\frac{a_2}{2}\partial_6-\frac{a_3}{2}\partial_7-\frac{a_4}{2}\partial_8;\\
\nabla_{\partial_7}\partial_7 &= \frac{ab_1}{2}\partial_1+\frac{ab_2}{2}\partial_2+\frac{bb_3}{2}\partial_3+\frac{bb_4}{2}\partial_4-\frac{b_1}{2}\partial_5-\frac{b_2}{2}\partial_6-\frac{b_3}{2}\partial_7-\frac{b_4}{2}\partial_8;\\
\nabla_{\partial_8}\partial_8 &= \frac{ab_1}{2}\partial_1+\frac{ab_2}{2}\partial_2+\frac{bb_3}{2}\partial_3+\frac{bb_4}{2}\partial_4-\frac{b_1}{2}\partial_5-\frac{b_2}{2}\partial_6-\frac{b_3}{2}\partial_7-\frac{b_4}{2}\partial_8.
\end{aligned}
$$

From the above relations, after a long but straightforward calculation, the nonzero components of the $(1,3)$-curvature operator of any Walker metric (3) and (4) whith $a = a(u_1,\ldots,u_4)$ and $b = b(u_1,\ldots,u_4)$ are determined by

$$
\begin{aligned}
R(\partial_5,\partial_6)\partial_5 &= \frac{aa_1a_2}{4}\partial_1+\frac{aa_2^2}{4}\partial_2+\frac{ba_2a_3}{4}\partial_3+\frac{ba_2a_4}{4}\partial_4\\
&\quad -\frac{a_1a_2}{4}\partial_5-\frac{a_2^2}{4}\partial_6-\frac{a_2a_3}{4}\partial_7-\frac{a_2a_4}{4}\partial_8;\\
R(\partial_5,\partial_6)\partial_6 &= -\frac{aa_1^2}{4}\partial_1-\frac{aa_1a_2}{4}\partial_2-\frac{ba_1a_3}{4}\partial_3-\frac{ba_1a_4}{4}\partial_4\\
&\quad +\frac{a_1^2}{4}\partial_5+\frac{a_1a_2}{4}\partial_6+\frac{a_1a_3}{4}\partial_7+\frac{a_1a_4}{4}\partial_8;\\
R(\partial_5,\partial_7)\partial_5 &= \frac{aa_3b_1}{4}\partial_1+\frac{aa_3b_2}{4}\partial_2+\frac{ba_3b_3}{4}\partial_3+\frac{ba_3b_4}{4}\partial_4\\
&\quad -\frac{a_3b_1}{4}\partial_5-\frac{a_3b_2}{4}\partial_6-\frac{a_3b_3}{4}\partial_7-\frac{a_3b_4}{4}\partial_8;\\
R(\partial_5,\partial_7)\partial_7 &= -\frac{aa_1b_1}{4}\partial_1-\frac{aa_2b_1}{4}\partial_2-\frac{bb_1a_3}{4}\partial_3-\frac{bb_1a_4}{4}\partial_4\\
&\quad +\frac{b_1a_1}{4}\partial_5+\frac{a_2b_1}{4}\partial_6+\frac{a_3b_1}{4}\partial_7+\frac{a_4b_1}{4}\partial_8;\\
R(\partial_5,\partial_8)\partial_5 &= \frac{ab_1a_4}{4}\partial_1+\frac{aa_4b_2}{4}\partial_2+\frac{bb_3a_4}{4}\partial_3+\frac{bb_4a_4}{4}\partial_4\\
&\quad -\frac{b_1a_4}{4}\partial_5-\frac{b_2a_4}{4}\partial_6-\frac{b_3a_4}{4}\partial_7-\frac{b_4a_4}{4}\partial_8;\\
R(\partial_5,\partial_8)\partial_8 &= -\frac{ab_1a_1}{4}\partial_1-\frac{ab_1a_2}{4}\partial_2-\frac{bb_1a_3}{4}\partial_3-\frac{bb_1a_4}{4}\partial_4\\
&\quad +\frac{a_1b_1}{4}\partial_5+\frac{a_2b_1}{4}\partial_6+\frac{a_3b_1}{4}\partial_7+\frac{a_4b_1}{2}\partial_8;\\
R(\partial_6,\partial_7)\partial_6 &= \frac{aa_3b_1}{4}\partial_1+\frac{aa_3b_2}{4}\partial_2+\frac{ba_3b_3}{4}\partial_3+\frac{ba_3b_4}{4}\partial_4\\
&\quad -\frac{b_1a_3}{4}\partial_5-\frac{b_2a_3}{4}\partial_6-\frac{a_3b_3}{4}\partial_7-\frac{b_4a_3}{4}\partial_8;\\
R(\partial_6,\partial_7)\partial_7 &= -\frac{ab_2a_1}{4}\partial_1-\frac{ab_2a_2}{4}\partial_2-\frac{bb_2a_3}{2}\partial_3-\frac{bb_2a_4}{4}\partial_4\\
&\quad +\frac{a_1b_2}{4}\partial_5+\frac{a_2b_2}{4}\partial_6+\frac{a_3b_2}{4}\partial_7+\frac{a_4b_2}{4}\partial_8;\\
R(\partial_6,\partial_8)\partial_6 &= \frac{aa_4b_1}{4}\partial_1+\frac{aa_4b_2}{4}\partial_2+\frac{ba_4b_3}{4}\partial_3+\frac{ba_4b_4}{4}\partial_4\\
&\quad -\frac{b_1a_4}{4}\partial_5-\frac{b_2a_4}{4}\partial_6-\frac{b_3a_4}{4}\partial_7-\frac{b_4a_4}{4}\partial_8;\\
R(\partial_6,\partial_8)\partial_8 &= -\frac{aa_1b_2}{4}\partial_1-\frac{aa_2b_2}{4}\partial_2-\frac{bb_2a_3}{4}\partial_3-\frac{bb_2a_4}{4}\partial_4\\
&\quad +\frac{a_1b_2}{4}\partial_5+\frac{a_2b_2}{4}\partial_6+\frac{b_2a_3}{4}\partial_7+\frac{b_2a_4}{2}\partial_8;
\end{aligned}
$$

$$\begin{aligned} R(\partial_7,\partial_8)\partial_7 &= \frac{ab_1b_4}{4}\partial_1+\frac{ab_2b_4}{4}\partial_2+\frac{bb_3b_4}{4}\partial_3+\frac{bb_4^2}{4}\partial_4\\ &\quad -\frac{b_1b_4}{4}\partial_5-\frac{b_2b_4}{4}\partial_6-\frac{b_3b_4}{4}\partial_7-\frac{b_4^2}{4}\partial_8;\\ R(\partial_7,\partial_8)\partial_8 &= -\frac{ab_1b_3}{4}\partial_1-\frac{ab_2b_3}{4}\partial_2-\frac{bb_3^2}{4}\partial_3-\frac{bb_3b_4}{4}\partial_4\\ &\quad +\frac{b_1b_3}{4}\partial_5+\frac{b_2b_3}{4}\partial_6+\frac{b_3^2}{4}\partial_7+\frac{b_3b_4}{4}\partial_8. \end{aligned}$$

From the above relations, after a long but straightforward calculation, the nonzero components of the $(0,4)$-curvature tensor of any Walker metric (3) and (4) whith $a=a(u_1,\dots,u_4)$ and $b=b(u_1,\dots,u_4)$ are determined by

$$\begin{aligned} R_{1556} &= R_{5662}=\frac{a_1a_2}{4};\ R_{5772}=R_{5882}=\frac{a_2b_1}{4};\ R_{6771}=R_{6881}=\frac{a_1b_2}{4};\\ R_{1557} &= R_{5773}=R_{5883}=R_{1667}=\frac{a_3b_1}{4};\ R_{7881}=\frac{b_1b_3}{4};;\ R_{5663}=\frac{a_1a_3}{4},\\ R_{1558} &= R_{5774}=R_{5884}=R_{1668}=\frac{a_4a_1}{4};\ R_{1778}=\frac{b_1b_4}{4};\ R_{5664}=\frac{a_1a_4}{4},\\ R_{2557} &= R_{6773}=R_{6883}=R_{2667}=\frac{a_3b_2}{4};\ R_{7882}=\frac{b_2b_3}{4};\ R_{3556}=\frac{a_2a_3}{4},\\ R_{2558} &= R_{6774}=R_{6884}=R_{2668}=\frac{a_4b_2}{4};\ R_{2778}=\frac{b_2b_4}{4};\ R_{4556}=\frac{a_2a_4}{4},\\ R_{3558} &= R_{3668}=\frac{a_4b_3}{4};\ R_{3778}=R_{7884}=\frac{b_3b_4}{4};\ R_{4557}=R_{4667}=\frac{a_3b_4}{4},\\ R_{1665} &= \frac{a_1^2}{4};\ R_{1775}=R_{1885}=\frac{a_1b_1}{4};\ R_{2556}=\frac{a_2^2}{4};\ R_{2776}=R_{2886}=\frac{a_2b_2}{4},\\ R_{3557} &= R_{3667}=\frac{a_3b_3}{4};\ R_{3887}=\frac{b_3^2}{4};\ R_{4558}=R_{4668}=\frac{a_4b_4}{4};\ R_{4778}=\frac{b_4^2}{4}, \end{aligned}$$

Next, let $\rho(X,Y)=\operatorname{trace}\{Z\mapsto R(X,Z)Y\}$ be the Ricci tensor. Then, we have

$$\begin{aligned} \rho_{55} &= \frac{a_2^2}{2}+\frac{a_3b_3}{2}+\frac{a_4b_4}{2};\quad \rho_{66}=\frac{a_1^2}{2}+\frac{a_3b_3}{2}+\frac{a_4b_4}{2}\\ \rho_{77} &= \frac{a_1b_1}{2}+\frac{a_2b_2}{2}+\frac{b_4^2}{2};\quad \rho_{88}=\frac{a_1b_1}{2}+\frac{a_2b_2}{2}+\frac{b_3^2}{2};\\ \rho_{56} &= -\frac{a_1a_2}{2},\quad \rho_{57}=-\frac{a_3b_1}{2},\quad \rho_{58}=-\frac{a_4b_1}{2}\\ \rho_{67} &= -\frac{a_3b_2}{2},\quad \rho_{68}=-\frac{a_4b_2}{2},\quad \rho_{78}=-\frac{b_3b_4}{2}. \end{aligned} \tag{5}$$

From (5) the scalar curvature $\mathrm{Sc}=\sum_1^8 g^{ij}\rho_{ij}$ of the Walker metric given by (3) and (4) is zero.

Recall that a Walker metric is said to be Einstein Walker metric if its Ricci tensor is a scalar multiple of the metric at each point. We have the following result.

A Walker metric given by (3) and (4) is not Einstein if the function a and b depends only on $(u_1, \dots, u_4)$.

Proof. The Einstein equations defined by $G_{ij} = \rho_{ij} - \frac{Sc}{8} g_{ij}$ for the Walker metric given by (3) and (4) whith $a = a(u_1, \dots, u_4)$ and $b = b(u_1, \dots, u_4)$ are as follows:

$$\begin{aligned} G_{56} &= -\frac{a_1 a_2}{2} = 0, \quad G_{57} = -\frac{a_3 b_1}{2} = 0, \quad G_{58} = -\frac{a_4 b_1}{2} = 0 \\ G_{67} &= -\frac{a_3 b_2}{2} = 0, \quad G_{68} = -\frac{a_4 b_2}{2} = 0, \quad G_{78} = -\frac{b_3 b_4}{2}, \end{aligned}$$

and

$$\begin{aligned} G_{55} &= \frac{1}{2}(a_2^2 + a_3 b_3 + a_4 b_4) = 0, \quad G_{66} = \frac{1}{2}(a_1^2 + a_3 b_3 + a_4 b_4) = 0 \\ G_{77} &= \frac{1}{2}(a_1 b_1 + a_2 b_2 + b_4^2) = 0, \quad G_{88} = \frac{1}{2}(a_1 b_1 + a_2 b_2 + b_3^2) = 0. \end{aligned}$$

This complete the proof. □

Let W denote the Weyl conformal curvature tensor given by

$$\begin{aligned} W(X,Y,Z,T) : &= R(X,Y,Z,T) \\ &+ \frac{Sc}{(n-1)(n-2)} \Big\{ g(Y,Z)g(X,T) - g(X,Z)g(Y,T) \Big\} \\ &+ \frac{1}{n-2} \Big\{ \rho(Y,Z)g(X,T) - \rho(X,Z)g(Y,T) \\ &- \rho(Y,T)g(X,Z) + \rho(X,T)g(Y,Z) \Big\}. \end{aligned}$$

A pseudo-Riemannian manifold is locally conformally flat if and only if its Weyl tensor vanishes. The nonzero components of Weyl conformal tensor of a Walker metric defined by (3) and (4) with $a = a(u_1, \dots, u_4)$

and $b = b(u_1, \ldots, u_4)$ are given by

$$\begin{aligned}
W_{1556} &= \frac{a_1a_2}{3}, \quad W_{1557} = \frac{a_3b_1}{3}, \quad W_{1558} = \frac{a_4b_1}{3},\\
W_{1665} &= \frac{1}{12}(4a_1^2 + a_3b_3 + a_4b_4), \quad W_{1667} = \frac{a_3b_1}{4}, \quad W_{1668} = \frac{a_4b_1}{4},\\
W_{1775} &= \frac{1}{12}(4a_1b_1 + a_2b_2 + b_4^2), \quad W_{1776} = \frac{a_1b_2}{4}, \quad W_{1778} = \frac{b_1b_4}{4},\\
W_{1885} &= \frac{1}{12}(4a_1b_1 + a_2b_2 + b_3^2), \quad W_{1886} = \frac{a_1b_2}{4}, \quad W_{1887} = \frac{b_1b_3}{4},\\
W_{2556} &= \frac{1}{12}(4a_2^2 + a_3b_3 + a_4b_4), \quad W_{2557} = \frac{a_3b_2}{4}, \quad W_{2558} = \frac{a_4b_2}{4},\\
W_{2665} &= \frac{a_1a_2}{3}, \quad W_{2667} = \frac{a_3b_2}{3}, \quad W_{2668} = \frac{a_4b_2}{3},\\
W_{2775} &= \frac{a_2b_1}{4}, \quad W_{2776} = \frac{1}{12}(a_1b_1 + 4a_2b_2 + b_4^2), \quad W_{2778} = \frac{b_2b_4}{4},\\
W_{2885} &= \frac{a_2b_1}{4}, \quad W_{2886} = \frac{1}{12}(a_1b_1 + 4a_2b_2 + b_3^2), \ quad W_{2887} = \frac{b_2b_3}{4},\\
W_{3556} &= \frac{a_2a_3}{4}, \quad W_{3557} = \frac{1}{12}(a_2^2 + 4a_3b_3 + a_4b_4), \quad W_{3558} = \frac{a_4b_3}{4},\\
W_{3665} &= \frac{a_1a_3}{4}, \quad W_{3667} = \frac{1}{12}(a_1^2 + 4a_3b_3 + a_4b_4), \quad W_{3668} = \frac{a_4b_3}{4},\\
W_{3775} &= \frac{a_3b_1}{3}, \quad W_{3776} = \frac{a_3b_2}{3}, \quad W_{3778} = \frac{b_3b_4}{3},\\
W_{3885} &= \frac{a_3b_1}{4}, \quad W_{3886} = \frac{a_3b_2}{4}, \quad W_{3887} = \frac{1}{12}(a_1b_1 + a_2b_2 + 4b_3^2),\\
W_{4556} &= \frac{a_2a_4}{4}, \quad W_{4557} = \frac{a_3b_4}{4}, \quad W_{4558} = \frac{1}{12}(a_2^2 + a_3b_3 + 4a_4b_4),\\
W_{4665} &= \frac{a_1a_4}{4}, \quad W_{4667} = \frac{a_3b_4}{4}, \quad W_{4668} = \frac{1}{12}(a_1^2 + a_3b_3 + a_4b_4),\\
W_{4775} &= \frac{a_4b_1}{4}, \quad W_{4776} = \frac{a_4b_2}{4}, \quad W_{4778} = \frac{1}{12}(a_1b_1 + a_2b_2 + 4b_4^2),\\
W_{4885} &= \frac{a_4b_1}{3}, \quad W_{4886} = \frac{a_4b_2}{3}, \quad W_{4887} = \frac{b_3b_4}{3}. \qquad (6)
\end{aligned}$$

Now it is possible to obtain the form of a locally conformal flat Walker metric defined by (3) and (4) with $a = a(u_1, \ldots, u_4)$ and $b = b(u_1, \ldots, u_4)$. A Walker metric given by (3) and (4) with $a = a(u_1, \ldots, u_4)$ and $b = b(u_1, \ldots, u_4)$ is locally conformal flat if the functions a and b are constants.

Proof. From (6) after a straightforward calculation. □

In [14], Pişcoran and Mishra investigate the S-curvature, the Landsberg curvature, mean Landsberg curvature, Cartan torsion and mean Cartan torsion for recently introduced (α, β)-metric $F = \beta + \frac{a\alpha^2+\beta^2}{\alpha}$, where $\alpha = \sqrt{a_{ij}y^i y^j}$ is a Riemannian metric ; β is a 1-form and $a \in (0, 1]$ is a real positive scalar. They find the necessary and sufficient condition under which this class of Finsler metrics is Riemannian or locally Minkowskian. In [15], the authors investigate a new (α, β)-metric $F = \beta + \frac{a\alpha^2+\beta^2}{\alpha}$, where $\alpha = \sqrt{a_{ij}y^i y^j}$ is a Riemannian metric ; $\beta = b_i y^i$ is a 1-form and $a \in (1/4, +\infty)$ is a real scalar. They investigate the variational problem in Lagrange spaces endowed with this type of metrics. Also, they study the dually local flatness for this type of metric and they proof that this kind of metric can be reduced to a locally Minkowskian metric. In [16], Pişcoran and Mishra investigate the relationship between the geodesic coefficients of the new (α, β)-metric $F = \beta + \frac{a\alpha^2+\beta^2}{\alpha}$ and the corresponding geodesic coefficients of the metric α.

4. Conclusion

Various geometric quantities are computed explicitly in terms of metrics coefficients, including the Christoffel symbols, curvature operator, Ricci curvature and Weyl tensor. Using these formulas, we have obtained a class of eight-dimensional Walker metrics which are Einstein and locally conformally flat..

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