Journal of the Nigerian Mathematical Society
Vol. 41, Issue 2, pp. 143-150, 2022

CONVERGENCE OF SOLUTIONS OF SOME SECOND-ORDER NONLINEAR DIFFERENTIAL EQUATIONS

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ABSTRACT. This paper is concerned with the convergence of solutions of the second-order differential equation

$$\ddot{x}+f(x)\dot{x}+g(x)=e(t,x,\dot{x}),$$

where $f(x),g(x)$ and $e(t,x,\dot{x})$ are continuous real-valued functions in their arguments. By using the direct method of Lyapunov and constructing a complete Lyapunov function, sufficient conditions which guarantee the convergence of solutions are obtained.

1. INTRODUCTION

Consider the second-order nonlinear ordinary differential equation

$$\ddot{x}+f(x)\dot{x}+g(x)=e(t,x,\dot{x}), \tag{1.1}$$

where $f(x),g(x)$ and $e(t,x,\dot{x})$ are continuous real-valued functions in their arguments, and dots denote differentiation with respect to t. Solutions of (1.1) exist for any pre-assigned initial data under the above conditions on f,g and e. Any two distinct solutions $x_1(t),x_2(t)$ are said to converge if

$$x_2(t)-x_1(t)\to 0,\ \dot{x}_2(t)-\dot{x}_1(t)\to 0 \tag{1.2}$$

as $t\to\infty$. The convergence property of solutions of nonlinear ordinary differential equations of order two and higher orders have been a subject of investigation in the literature, and the method of investigating this property is the direct method of Lyapunov, which also involves the construction of a quadratic function also known as the Lyapunov function ([1]-[5], [7]-[18]). However, the construction of this function remains a general problem. Perhaps, a reason behind few literature on the subject of convergence of solutions in ordinary differential equations. In particular, Ezeilo [7] and Loud [10] have investigated (1.1) in which $f(x)=c>0$ and $e(t,x,\dot{x})=e(t)$ for convergence of solutions. Loud [10] showed that if g' exists and satisfies

$$g'\geq b>0 \text{ for all } x$$

then all solutions $x(t)$ of

$$\ddot{x}+c\dot{x}+g(x)=e(t) \tag{1.3}$$

which ultimately lie in the range $|x|\leq A$ are convergent provided that

$$\max_{|x|\leq A} g'(x)<\frac{1}{2}c^2.$$

Ezeilo [7] in his own work showed that all solutions of (1.3) which satisfy $|x(t)|\leq A$ are convergent provided

$$H(A)<c^2,$$

where

$$H(A)=\max\frac{g(\xi_2)-g(\xi_1)}{\xi_2-\xi_1},\ \ \xi_2\neq\xi_1$$

Received by the editors May 04, 2022; Accepted: July 05, 2022
www.nigerianmathematicalsociety.org; Journal available online at https://ojs.ictp. it/jnms/
2000 *Mathematics Subject Classification.* 34D20.
Key words and phrases. Nonlinear differential equation; convergence of solution; second-order; direct method of Lyapunov.

for every pair of numbers ξ_2, ξ_1 $(\xi_2 \neq \xi_1)$ chosen from the closed interval $-A \leq x \leq A$. Ezeilo [7] further studied (1.3), and obtained sufficient conditions for convergence of solutions. On the other hand, Cartwright has investigated (1.1) for convergence of solutions. Cartwright [5] proved that if g'' exists and both f and g' are strictly positive for all x, then all ultimately bounded solutions of (1.1) converge provided $g(0) = 0$ and $|g''(x)|$ is sufficiently small.
Our present investigation is concerned with the convergence of solutions of (1.1). By using the direct method of Lyapunov, and constructing a complete Lyapunov function (See [6]) we establish sufficient conditions which guarantee the convergence of solutions of (1.1). Our results show that we do not require g' and g'' for the convergence of solutions of (1.1). To the best our knowledge, study on the convergence of solutions of (1.1) is scarce in literature.

2. STATEMENT OF MAIN RESULT

It will be assumed throughout what follows that the functions $f(x), g(x)$ and $e(t,x,\dot{x})$ are continuous in their respective arguments. Our first result is as follows:

Theorem 2.1. *Suppose that* $g(0) = 0$ *and that*
(i) there are finite constants $\varepsilon > 0, a_0 > 0, b_0 > 0, a > 0, b > 0$ *such that*

$$\varepsilon < a \leq f(x_2 - x_1) \leq a_0 \tag{2.1}$$

$$b \leq \frac{g(x_2) - g(x_1)}{x_2 - x_1} \leq b_0,\ x_2 \neq x_1; \tag{2.2}$$

(ii) there is a finite constant $\Delta > 0$ *satisfying*

$$|e(t,x_2,\dot{x}_2) - e(t,x_1,\dot{x}_1)| \leq \Delta\{(x_2 - x_1)^2 + (\dot{x}_2 - \dot{x}_1)^2\}^{\frac{1}{2}}. \tag{2.3}$$

In addition, let α_i, β_i $(i = 1,2,3)$ *be positive constants such that* $\sum_{i=1}^{3} \alpha_i < 1,\ \sum_{i=1}^{3} \beta_i < 1$, *and set*

$$A = \varepsilon\alpha_1\beta_1 a[b - \frac{1}{4}\varepsilon(a_0 - \varepsilon)], \tag{2.4}$$

$$B = \min\left\{\varepsilon\alpha_2\beta_2 a[b - \frac{1}{4}\varepsilon(a_0 - \varepsilon)], \frac{4}{\varepsilon}\alpha_3\beta_3 a[b - \frac{1}{4}\varepsilon(a_0 - \varepsilon)]\right\}. \tag{2.5}$$

Then, there is a finite constant $\nu > 0$ *such that if the constant* Δ *in* (2.3) *satisfies* $\Delta < \nu$, *then any two distinct solutions* $x_1 = x_1(t), x_2 = x_2(t)$ *of* (1.1) *for which*

$$[g(x_2 - x_1) - \{g(x_2) - g(x_1)\}]^2 \leq A(x_2 - x_1)^2, \tag{2.6}$$

$$[f(x_2 - x_1)(\dot{x}_2 - \dot{x}_1) - \{f(x_2)\dot{x}_2 - f(x_1)\dot{x}_1\}]^2 \leq B\left\{(x_2 - x_1)^2 + (\dot{x}_2 - \dot{x}_1)^2\right\} \tag{2.7}$$

for all $t \geq t_0$ $(0 < t_0 < \infty)$ *necessarily converge.*

We shall also show that the theorem itself still holds with (2.3) replaced by the more weaker condition

$$|e(t,x_2,\dot{x}_2) - e(t,x_1,\dot{x}_1)| \leq \psi(t)\{(x_2 - x_1)^2 + (\dot{x}_2 - \dot{x}_1)^2\}^{\frac{1}{2}}, \tag{2.8}$$

where for some k such that $0 \leq k \leq 2, \psi \geq 0$ is such that

$$\int_{-\infty}^{\infty} \psi^k(t)dt < \infty. \tag{2.9}$$

Sufficient conditions which guarantee the convergence of all solutions of (1.1) were also obtained.
However, it will be convenient to deal first with Theorem 2.1 in its present form, and then, later (see §5), to indicate what modifications are necessary to convert our methods to the case in which $e(t,x,\dot{x})$ satisfies (2.8).

3. PRELIMINARY

Consider

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -f(x)y - g(x) + e(t,x,y), \end{aligned} \tag{3.1}$$

the equivalent system form of (1.1) obtained by setting $\dot{x} = y$. Let $(x_1(t), y_1(t))$, $(x_2(t), y_2(t))$ be two distinct solutions of system (3.1) such that

$$[g(x_2 - x_1) - \{g(x_2) - g(x_1)\}]^2 \leq A(x_2 - x_1)^2 \tag{3.2}$$

$$[f(x_2 - x_1)(y_2 - y_1) - \{f(x_2)y_2 - f(x_1)y_1\}]^2 \leq B\left\{(x_2 - x_1)^2 + (y_2 - y_1)^2\right\} \tag{3.3}$$

for all $t \geq t_0$ $(0 < t_0 < \infty)$, where A and B are given by (2.4) and (2.5) respectively. Then, in view of (1.2), it is enough, in order to prove Theorem 2.1, to show that

$$x_2(t) - x_1(t) \to 0, \; y_2(t) - y_1(t) \to 0 \tag{3.4}$$

as $t \to \infty$. Our proof of this will rest mainly on the properties of the function $V = V(x_2 - x_1, y_2 - y_1)$ defined thus:

$$2V = [\varepsilon(x_2 - x_1) + (y_2 - y_1)]^2 + (y_2 - y_1)^2 + 4\int_0^{x_2 - x_1} g(s)ds, \tag{3.5}$$

where ε $(0 < \varepsilon < 1)$ is a constant.
Clearly, subject to hypothesis (i) of Theorem 2.1, the function V satisfies

$$D_1\left\{(x_2 - x_1)^2 + (y_2 - y_1)^2\right\} \leq V \leq D_2\left\{(x_2 - x_1)^2 + (y_2 - y_1)^2\right\}, \tag{3.6}$$

where $D_1 = \min\{b, \frac{1}{2}\} > 0$ and $D_2 = \max\{b_0 + \varepsilon, \frac{1}{2}\varepsilon + 1\} > 0$ are finite constants.
Next, we want to show that for any two distinct solutions $(x_1, y_1), (x_2, y_2)$ of (3.1) such that the inequalities (3.2) and (3.3) (equivalently (2.6), (2.7)) hold, the function $V(t) \equiv V(x_2(t) - x_1(t), y_2(t) - y_1(t))$ satisfies

$$V(t) \to 0 \text{ as } t \to \infty. \tag{3.7}$$

This consequently implies (3.4) since $V(t)$ is positive definite. To arrive at (3.7), we first prove the following:

Lemma 3.1. *Let the conditions of Theorem 2.1 hold and let* $(x_1(t), y_1(t))$, $(x_2(t), y_2(t))$ *be any two distinct solutions of* (3.1) *such that*

$$[g(x_2 - x_1) - \{g(x_2) - g(x_1)\}]^2 \leq A(x_2 - x_1)^2 \tag{3.8}$$

$$[f(x_2 - x_1)(y_2 - y_1) - \{f(x_2)y_2 - f(x_1)y_1\}]^2 \leq B\left\{(x_2 - x_1)^2 + (y_2 - y_1)^2\right\} \tag{3.9}$$

for all $t \geq t_0$ $(0 < t_0 < \infty)$, *where A and B are given by* (2.4) *and* (2.5) *respectively. Then there exists constants* $\nu > 0, D_3 > 0$ *such that if the constant* Δ *in* (2.3) *satisfies* $\Delta < \nu$, *then the function* $V(t) \equiv V(x_2(t) - x_1(t), y_2(t) - y_1(t))$ *satisfies*

$$\dot{V}(t) + D_3 V(t) \leq 0 \tag{3.10}$$

for all $t \geq t_0$.

Proof. The proof of Lemma 3.1 rests on the function $V(t) \equiv V(x_2(t) - x_1(t), y_2(t) - y_1(t))$ (in (3.5)) defined for any two distinct solutions $(x_1(t), y_1(t))$, $(x_2(t), y_2(t))$ of (3.1). On differentiating V with respect to t and using the system

$$\begin{aligned} \dot{x}_2 - \dot{x}_1 &= y_2 - y_1 \\ \dot{y}_2 - \dot{y}_1 &= -\{f(x_2)y_2 - f(x_1)y_1\} - \{g(x_2) - g(x_1)\} + \theta, \end{aligned} \tag{3.11}$$

where $\theta = e(t,x_2,y_2) - e(t,x_1,y_1)$, we have, after further simplification that

$$\begin{aligned} \dot{V}(t) &= -\varepsilon\{g(x_2)-g(x_1)\}(x_2-x_1) - f(x_2-x_1)(y_2-y_1)^2 - U \\ &\quad +\varepsilon(x_2-x_1)G(x_2,y_2,x_1,y_1) + 2(y_2-y_1)G(x_2,y_2,x_1,y_1) \\ &\quad +2(x_2-x_1)H(x_2,x_1) + \{\varepsilon(x_2-x_1)+2(y_2-y_1)\}\theta, \end{aligned} \tag{3.12}$$

where

$$U = \{f(x_2-x_1)-\varepsilon\}(y_2-y_1)^2 + \varepsilon\{f(x_2-x_1)-\varepsilon\}(x_2-x_1)(y_2-y_1),$$

$$H(x_2,x_1) = g(x_2-x_1) - \{g(x_2)-g(x_1)\},$$

$$G(x_2,y_2,x_1,y_1) = f(x_2-x_1)(y_2-y_1) - \{f(x_2)y_2 - f(x_1)y_1\},$$

$$\theta = e(t,x_2,y_2) - e(t,x_1,y_1). \tag{3.13}$$

Obviously,

$$\varepsilon\{g(x_2)-g(x_1)\}(x_2-x_1) \geq \varepsilon b(x_2-x_1)^2$$

by (2.2) and,

$$f(x_2-x_1)(y_2-y_1)^2 \geq a(y_2-y_1)^2$$

by (2.1). We also have that

$$U = \{f(x_2-x_1)-\varepsilon\}\left\{(y_2-y_1)^2 + \varepsilon(x_2-x_1)(y_2-y_1)\right\}.$$

By completing the square for the quadratic on the right hand side and using (2.1), we obtain

$$U \geq -\frac{\varepsilon^2}{4}(a_0-\varepsilon)(x_2-x_1)^2 \;\text{ if }\; y_2 \neq y_1.$$

This estimate is always true since $U = 0$ when $y_2 = y_1$.
Substituting various estimates in (3.12) yields

$$\begin{aligned} \dot{V}(t) &\leq -D_4(x_2-x_1)^2 - D_5(y_2-y_1)^2 + 2(y_2-y_1)H(x_2,x_1) \\ &\quad +\{\varepsilon(x_2-x_1)+2(y_2-y_1)\}G(x_2,y_2,x_1,y_1) \\ &\quad +D_6\left\{(x_2-x_1)^2+(y_2-y_1)^2\right\}^{\frac{1}{2}}|\theta|, \end{aligned} \tag{3.14}$$

where $D_4 = \varepsilon\left[b - \frac{1}{4}\varepsilon(a_0-\varepsilon)\right] > 0, D_5 = a > 0, D_6 = D_7^{\frac{1}{2}} > 0, D_7 = \max\{2\varepsilon+\varepsilon^2, 2\varepsilon+4\} > 0$.
Now, let us choose some real constants $\alpha_i > 0, \beta_i > 0$ $(i = 1,2,3)$ such that $\sum_{i=1}^{3}\alpha_i < 1,\ \sum_{i=1}^{3}\beta_i < 1$. The inequality (3.14) is then rewritten

$$\begin{aligned} \dot{V}(t) &\leq -\left(1-\sum_{i=1}^{3}\alpha_i\right)D_4(x_2-x_1)^2 - \left(1-\sum_{i=1}^{3}\beta_i\right)D_5(y_2-y_1)^2 \\ &\quad -\sum_{i=1}^{3}U_i + D_6\left\{(x_2-x_1)^2+(y_2-y_1)^2\right\}^{\frac{1}{2}}|\theta|, \end{aligned} \tag{3.15}$$

where

$$\begin{aligned} U_1 &= \alpha_1 D_4(x_2-x_1)^2 + \beta_1 D_5(y_2-y_1)^2 - 2(y_2-y_1)H(x_2,x_1), \\ U_2 &= \alpha_2 D_4(x_2-x_1)^2 + \beta_2 D_5(y_2-y_1)^2 - 2(y_2-y_1)G(x_2,y_2,x_1,y_1), \\ U_3 &= \beta_3 D_5(y_2-y_1)^2 + \alpha_3 D_4(x_2-x_1)^2 - \varepsilon(x_2-x_1)G(x_2,y_2,x_1,y_1). \end{aligned}$$

Clearly,

$$\begin{aligned} U_1 &= \alpha_1 D_4 (x_2 - x_1)^2 + \beta_1 D_5 \left\{ (y_2 - y_1) - \frac{1}{\beta_1 D_5} H(x_2, x_1) \right\}^2 \\ &\quad - \tfrac{1}{\beta_1 D_5} H^2(x_2, x_1) \\ &\geq \alpha_1 D_4 (x_2 - x_1)^2 - \frac{1}{\beta_1 D_5} H^2(x_2, x_1). \end{aligned}$$

Hence $U_1 \geq 0$ provided that

$$H^2(x_2, x_1) \leq \alpha_1 \beta_1 D_4 D_5 (x_2 - x_1)^2.$$

Similarly,

$$\begin{aligned} U_2 &= \alpha_2 D_4 (x_2 - x_1)^2 + \beta_2 D_5 \left\{ (y_2 - y_1) - \frac{1}{\beta_2 D_5} G(x_2, y_2, x_1, y_1) \right\}^2 \\ &\quad - \frac{1}{\beta_2 D_5} G^2(x_2, y_2, x_1, y_1) \\ &\geq \alpha_2 D_4 (x_2 - x_1)^2 - \frac{1}{\beta_2 D_5} G^2(x_2, y_2, x_1, y_1). \end{aligned}$$

Thus, $U_2 \geq 0$ provided

$$G^2(x_2, y_2, x_1, y_1) \leq \alpha_2 \beta_2 D_4 D_5 (x_2 - x_1)^2.$$

In the same vein,

$$\begin{aligned} U_3 &= \beta_3 D_5 (y_2 - y_1)^2 + \alpha_3 D_4 \left\{ (x_2 - x_1) - \frac{\varepsilon}{2\alpha_3 D_4} G(x_2, y_2, x_1, y_1) \right\}^2 \\ &\quad - \frac{\varepsilon^2}{\alpha_3 D_4} G^2(x_2, y_2, x_1, y_1) \\ &\geq \beta_3 D_5 (y_2 - y_1)^2 - \frac{\varepsilon^2}{\alpha_3 D_4} G^2(x_2, y_2, x_1, y_1). \end{aligned}$$

Therefore, $U_3 \geq 0$ provided that

$$G^2(x_2, y_2, x_1, y_1) \leq \frac{4}{\varepsilon^2} \alpha_3 \beta_3 D_4 D_5 (y_2 - y_1)^2.$$

On combining these estimates with (3.15), we have, in view of (3.13) and (2.3)

$$\dot{V}(t) \leq -(D_8 - D_6 \Delta) \left\{ (x_2 - x_1)^2 + (y_2 - y_1)^2 \right\}$$

provided that

$$\begin{aligned} H^2(x_2, x_1) &\leq D_9 (x_2 - x_1)^2, \\ G^2(x_2, y_2, x_1, y_1) &\leq D_{10} \left\{ (x_2 - x_1)^2 + (y_2 - y_1)^2 \right\}, \end{aligned} \tag{3.16}$$

where $D_8 = \min\left\{ \left(1 - \sum_{1=1}^{3} \alpha_i\right) D_4, \left(1 - \sum_{1=1}^{3} \beta_i\right) D_5 \right\} > 0$, $D_9 = \alpha_1 \beta_1 D_4 D_5 > 0$, $D_{10} = \min\left\{ \alpha_2 \beta_2 D_4 D_5, \frac{4}{\varepsilon^2} \alpha_3 \beta_3 D_4 D_5 \right\} > 0$, and choose

$$\nu = \frac{D_8}{D_6}. \tag{3.17}$$

Then, provided $\Delta < \nu$, and if (3.16) holds, there exists a constant $D_{11} > 0$ such that

$$\dot{V}(t) \leq -D_{11} \left\{ (x_2 - x_1)^2 + (y_2 - y_1)^2 \right\}$$

which by (3.6), implies that

$$\dot{V}(t) \leq -D_{12}V(t)$$

for some constant $D_{12} > 0$. □

4. COMPLETION OF THE PROOF OF THEOREM 2.1

Let $\nu > 0$ and let t_0 $(0 < t_0 < \infty)$ be fixed as in Lemma 3.1. From Lemma 3.1, for any two distinct solutions $(x_1(t), y_1(t)), (x_2(t), y_2(t))$ of (3.1) (satisfying (3.11)) for which (3.8) and (3.9) hold, the function $V(t) \equiv V(x_2(t) - x_1(t), y_2(t) - y_1(t))$ satisfies

$$\dot{V}(t) + D_{12}V(t) \leq 0$$

for all $t \geq t_0$ provided $\Delta < \nu$. Integrating this inequality between t_0 and t, yields

$$V(t) \leq V(t_0)e^{-D_{12}(t-t_0)},\ t \geq t_0,$$

which implies

$$V(t) \to 0 \text{ as } t \to \infty$$

since $V(t_0)$ is bounded and the function V is positive definite. In view of the preceding remarks in §3, this proves (3.4) and thus the theorem is proved with ν given by (3.17).

5. CASE FOR WHICH $e(t,x,y)$ SATISFIES (2.8)

Let us now turn to the case in §1 in which $e(t,x,y)$ satisfies (2.8). The proof of Theorem 2.1 in this case follows the lines indicated in §3-§4 except for some minor modifications which we now outline. Assume that (3.2), (3.3) hold. Then by (3.5), (3.6), (3.12), (3.13) and (3.15), $\dot{V}(t)$ satisfies

$$\dot{V}(t) \leq -2D_{13}V(t) + D_{14}\{V(t)\}^{\frac{1}{2}}|\Phi|, \tag{5.1}$$

where

$$\Phi = e(t,x_2,y_2) - e(t,x_1,y_1).$$

Let k_2 be any constant in the range $1 \leq k_2 \leq 2$. Then, by proceeding as in [14,§5], using (2.8), it can be shown that

$$-D_{13}V(t) + D_{14}\{V(t)\}^{\frac{1}{2}}|\Phi| \leq D_{15}\psi^{k_2}(t)V(t)$$

for some D_{15}. On combining this with (5.1) yields

$$\dot{V}(t) + D_{13}V(t) \leq D_{15}\psi^{k_2}(t)V(t) \tag{5.2}$$

for all $t \geq t_0$. A straightforward integration of (5.2) between t_0 and t now yields

$$V(t)\exp\left[D_{13}t - D_{15}\int_0^t \psi^{k_2}(\tau)d\tau\right] \leq V(t_0)\exp\left[D_{13}t - D_{15}\int_0^{t_0} \psi^{k_2}(\tau)d\tau\right],$$

$t \geq t_0$, which, in view of (2.9), implies that

$$V(t) \leq D_{16}V(t_0)e^{-D_{17}(t-t_0)},\ t \geq t_0$$

for some constant $D_{17}, 0 < D_{17} < \infty$. The proof of the theorem may now be completed by proceeding as in §4.

6. ULTIMATE BOUNDEDNESS OF SOLUTION

Having established the convergence result, Theorem 2.1, we can now state and prove a result on the ultimate boundedness of solution of (1.1).

Theorem 6.1. *Suppose that* $g(0) = 0$ *and that*
(i) there are positive constants a_0, b_0, a, b *such that*

$$\varepsilon < a \le f(x) \le a_0,$$

$$b \le \frac{g(x)}{x} \le b_0,\ x \neq 0;$$

(ii) $e(t,x,y)$ *satisfies*

$$|e(t,x,y)| \le A_0 + A_1(x^2+y^2)^{\frac{1}{2}}$$

for all t,x,y*, where* $A_0 \ge 0, A_1 \ge 0$ *are finite constants. Then, there exists a finite constant* $\mu > 0$ *such that if* $A_1 < \mu$*, every solution* $x(t)$ *of* (1.1) *ultimately satisfies*

$$x^2 + y^2 \le D,$$

where $D > 0$ *is a finite constant whose magnitude depends only on constants* a_0, b_0, a, b, A_0, A_1*, as well as on the functions* f *and* g*.*

Proof. Obviously, if we choose $x_1 = y_1 = 0$ then conditions (i), (ii) of Theorem 2.1 reduce to (i), (ii) of Theorem 6.1 with $A_0 = 0$. Thus, by Theorem 6.1 subject to the conditions of Theorem 2.1, every solution $x(t)$ of (3.1) ultimately satisfies

$$|x(t)| \le D_{18},\ |y(t)| \le D_{18},$$

where $D_{18} = D_{18}(a_0, b_0, a, b, \Delta) > 0$ is a finite constant. □

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