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FURTHER STABILITY CRITERIA FOR CERTAIN SECOND-ORDER DELAY DIFFERENTIAL EQUATIONS WITH MIXED COEFFICIENTS

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ABSTRACT. This work investigates the asymptotic stability of the trivial solution of the second-order linear delay differential equation

$$\ddot{x}(t) = a_1\dot{x}(t) + a_2\dot{x}(t-\tau) + b_1x(t) + b_2x(t-\tau),$$

where $\tau > 0, a_1, a_2, b_1, b_2$ are real numbers. By reducing the equation to a linear second-order ordinary differential equation with constant coefficients, sufficient conditions which guarantee the asymptotic stability of the trivial solution are obtained in a very simple form.

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1. Introduction

Consider the differential equation

$$\ddot{x}(t) = a_1\dot{x}(t) + a_2\dot{x}(t-\tau) + b_1x(t) + b_2x(t-\tau), \tag{1.1}$$

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where $\tau > 0, a_i, b_i$ $(i = 1, 2)$ are constants, which has applications in science and technology ([1] - [14]). Equation 1.1 has been a subject of interest in literature recently. Cahlon and Schmidt [3],[4], and Yeniceriolu [14] have investigated 1.1 for asymptotic stability of solutions. Cahlon and Schmidt [3] have studied 1.1 for stability of solution using the Pontryagin's theory of quasi-polynomials, which is rather long and complicated. In [3], the authors showed that solutions are not stable if $a_1a_2 \geq 0$ and $b_1 > 0, b_2 > 0$, while in [4], they investigated the asymptotic stability of the trivial solution of 1.1 whenever $a_1a_2 \geq 0$ and $b_1b_2 < 0$, and presented some examples to demonstrate their results. Yeniceriolu [14], also studied equation 1.1 together with a given initial value for the stability, asymptotic stability and instability of solutions. In the present paper, we study the stability of solutions of 1.1 by reducing it to an ordinary differential equation with constant coefficients, and then deduce stability criteria according to the zeros of the characteristic polynomial arising from the differential equation. These criteria are easily determined and are more general than those obtained in [3],[4] and [14], and we do not necessarily require any algorithm test [3],[4] nor initial data [9] to obtain them.

To the best of our ability, this approach is scarce in literature to investigate the stability of solutions of 1.1, and does not require any knowledge of solution.

This paper is outlined thus: §2 consists of some preliminary stability results, §3 contains the main result of the paper, and examples are presented in §4.

2. PRELIMINARY RESULTS

Consider the linear second-order ordinary differential equation

$$\ddot{y}(t) + p_1\dot{y}(t) + p_2y(t) = 0, \tag{2.1}$$

where p_j $(j = 1, 2)$ are real arbitrary constants. The characteristics polynomial associated with 2.1 is

$$\lambda^2 + p_1\lambda + p_2 = 0, \tag{2.2}$$

obtained by assuming that $y = e^{\lambda t}$ is a solution of 2.1. The eigenvalues λ_j $(j = 1, 2)$ (which are solutions) of 2.2 determine the stability properties of the solutions of 2.1.

Lemma 2.1. [2]: *Suppose the eigenvalues λ_j $(j = 1, 2)$ of 2.2 satisfy one of the following:*

(i) $\lambda_1 < \lambda_2 = 0$;
(ii) $\lambda_{1,2} = \pm i\mu,\ \mu \in \mathbb{R}$,

then the trivial solution $y = 0$ *of 2.1 is stable.*

Lemma 2.2. [2]: *If the eigenvalues* λ_j $(j = 1, 2)$ *of 2.2 satisfy one of the following:*

(i) $\lambda_1 = \lambda_2 < 0$;
(ii) $\lambda_1 < \lambda_2 < 0$;
(iii) $\lambda_{1,2} = \gamma \pm i\mu,\ \gamma < 0, \mu \in \mathbb{R}$,

then the trivial solution $y = 0$ *of 2.1 is asymptotically stable.*

Lemma 2.3. [2]: *The trivial solution* $y = 0$ *of 2.1 is not stable or unstable if the eigenvalues* λ_j $(j = 1, 2)$ *of 2.2 satisfy one of the following:*

(i) $\lambda_1 > \lambda_2 > 0$;
(ii) $\lambda_1 > \lambda_2 = 0$;
(iii) $\lambda_1 < 0 < \lambda_2$;
(iv) $\lambda_1 = \lambda_2 > 0$;
(v) $\lambda_{1,2} = \gamma \pm i\mu,\ \gamma > 0, \mu \in \mathbb{R}$.

In the next section, we state and prove a result which ensures that the trivial solution of 2.1 is asymptotically stable.

3. MAIN RESULT

The main result of this paper is the following theorem.

Theorem 3.1. : *Let* $\tau > 0, a_i, b_i$ $(i = 1, 2)$ *be arbitrary real constants satisfying:*

(i) $\dfrac{b_1 + b_2}{1 + \tau a_2} < 0$,
(ii) $\dfrac{a_1 + a_2 - \tau b_2}{1 + \tau a_2} < 0$, *where* $1 + \tau a_2 \neq 0$.

Then, the trivial solution of equation (1.1) is asymptotically stable.

Proof. Let us rewrite1.1 as

$$\ddot{x}(t) = (a_1 + a_2)\dot{x}(t) + (b_1 + b_2)x(t) - a_2 \int_{t-\tau}^{t} \ddot{x}(s)ds - b_2 \int_{t-\tau}^{t} \dot{x}(s)ds,$$

where

$$x(t - \tau) = x(t) - \int_{t-\tau}^{t} \dot{x}(s)ds \text{ and } \dot{x}(t - \tau) = \dot{x}(t) - \int_{t-\tau}^{t} \ddot{x}(s)ds$$

have been substituted accordingly in equation 1.1. By some rearrangements, we obtain

$$\int_{t-\tau}^{t} [a_2\ddot{x}(s) + b_2\dot{x}(s)]\, ds = (a_1 + a_2)\dot{x}(t) + (b_1 + b_2)x(t) - \ddot{x}(t).$$

Simplifying further, we obtain

$$\int_{t-\tau}^{t} [a_2\ddot{x}(s) + b_2\dot{x}(s)]\, ds = \int_{t-\tau}^{t} \frac{1}{\tau}\left[(a_1 + a_2)\dot{x}(t) + (b_1 + b_2)x(t) - \ddot{x}(t)\right] ds.$$

Thus,

$$(1 + \tau a_2)\ddot{x}(t) - (a_1 + a_2 - \tau b_2)\dot{x}(t) - (b_1 + b_2)x(t) = 0.$$

Dividing the last equation by $1 + \tau a_2 \neq 0$, we obtain

$$\ddot{x}(t) - \left(\frac{a_1 + a_2 - \tau b_2}{1 + \tau a_2}\right)\dot{x}(t) - \left(\frac{b_1 + b_2}{1 + \tau a_2}\right)x(t) = 0, \qquad (3.1)$$

where $1 + \tau a_2 \neq 0$.

This implies that the problem of investigating the stability properties of the equation 1.1 is equivalent to investigating the stability properties of equation 3.1.

Suppose $x(t) = e^{\lambda t}$ for $t \in \mathbb{R}$, is a solution of 3.1, we see that λ is a root (zero) of the characteristic polynomial

$$\lambda^2 - \left(\frac{a_1 + a_2 - \tau b_2}{1 + \tau a_2}\right)\lambda - \left(\frac{b_1 + b_2}{1 + \tau a_2}\right) = 0,\ 1 + \tau a_2 \neq 0. \qquad (3.2)$$

The zeros of 3.2 are given by

$$\lambda_{1,2} = \frac{1}{2}\left\{\left(\frac{a_1 + a_2 - \tau b_2}{1 + \tau a_2}\right) \pm \sqrt{\left(\frac{a_1 + a_2 - \tau b_2}{1 + \tau a_2}\right)^2 + 4\left(\frac{b_1 + b_2}{1 + \tau a_2}\right)}\right\},$$

where $1 + \tau a_2 \neq 0$.

Obviously, $\lambda_{1,2}$ possess negative real part if conditions (i) and (ii) of Theorem 3.1 are satisfied. Hence the trivial solution of 1.1 (or equivalently 3.1) is asymptotically stable. □

REMARKS

(1) If condition (ii) of Theorem3.1 is replaced with $\frac{a_1 + a_2 - \tau b_2}{1 + \tau a_2} = 0$, then $\lambda_{1,2}$ have no real roots. Then, following Lemma2.1 the trivial solution of 1.1 (or equivalently 3.1) is stable.

(2) If condition (i) of Theorem3.1 is replaced with $\frac{b_1 + b_2}{1 + \tau a_2} = 0$, then the zeros of 3.2 are given by $\lambda_1 < 0$ and $\lambda_2 = 0$.

Then, following Lemma2.1 the trivial solution of 1.1 (or equivalently 3.1) is stable.

(3) If $a_1a_2 \geq 0$ and $b_1 > 0, b_2 > 0$, then either $a_1 \leq 0, a_2 \leq 0, b_1 > 0, b_2 > 0$ or $a_1 \geq 0, a_2 \geq 0, b_1 > 0, b_2 > 0$. In either case conditions (i) and (ii) of Theorem3.1 are not satisfied. Hence the trivial solution of 1.1 (or equivalently 3.1) is not stable. This corroborates results obtained by Cahlon and Schmidt in [3].

(4) Obviously, conditions (i) and (ii) of Theorem3.1 are easy to apply to examples presented in [4] and [14]. We do not need any algorithm test nor initial data to obtain our results.

4. EXAMPLES

Here, we present the following examples which are also found in [4] and [14].

Example 4.1. Consider

$$\ddot{x}(t) = 0.3\dot{x}(t) + 0.6\dot{x}(t-1) - 11x(t) + x(t-1), \tag{4.1}$$

where we deduce that

$$a_1 = 0.3, a_2 = 0.6, b_1 = -11, b_2 = 1, \tau = 1, a_1a_2 = 0.18 > 0, b_1b_2 = -11 < 0,$$

$$\frac{b_1 + b_2}{1 + \tau a_2} = -6.25 < 0$$

and $\dfrac{a_1 + a_2 - \tau b_2}{1 + \tau a_2} = -0.0625 < 0$. Following equation 3.1, we obtain

$$\ddot{x}(t) + 0.0625\dot{x}(t) + 6.25x(t) = 0,$$

with solution

$$x(t) = e^{-0.03125t}\left(C_1 \cos 2.4998046799t + C_2 \sin 2.4998046799t\right),$$

C_1, C_2 are real constants. Hence the trivial solution of 4.1 is asymptotically stable since all hypotheses of Theorem3.1 are satisfied. See Figure 1.

Example 4.2. Consider

$$\ddot{x}(t) = 0.6\dot{x}(t) + 0.3\dot{x}(t-1) - 2x(t) + x(t-1), \tag{4.2}$$

where we deduce that

$$a_1 = 0.6, a_2 = 0.3, b_1 = -2, b_2 = 1, \tau = 1, a_1a_2 = 0.18 > 0, b_1b_2 = -2 < 0,$$

$$\frac{b_1 + b_2}{1 + \tau a_2} = -0.769 < 0$$

and $\dfrac{a_1 + a_2 - \tau b_2}{1 + \tau a_2} = -0.0769 < 0$. Following equation 3.1, we obtain

$$\ddot{x}(t) + 0.0769\dot{x}(t) + 0.769x(t) = 0,$$

with solution

$$x(t) = e^{-0.03846153385t} \left(C_1 \cos 0.87621488t + C_2 \sin 0.87621488t \right),$$

C_1, C_2 are real constants. All the conditions of Theorem 3.1 are satisfied, hence the trivial solution of 4.2 is asymptotically stable. See Figure 2.

Example 4.3. Consider

$$\ddot{x}(t) = 2\dot{x}(t-1) - 3x(t) + x(t-1), \tag{4.3}$$

where we deduce that

$$a_1 = 0, a_2 = 2, b_1 = -3, b_2 = 1, \tau = 1, a_1 a_2 = 0, b_1 b_2 = -3 < 0,$$

$$\frac{b_1 + b_2}{1 + \tau a_2} = -0.667 < 0,$$

and $\dfrac{a_1 + a_2 - \tau b_2}{1 + \tau a_2} = 0.3333 > 0$. Following equation 3.1, we get

$$\ddot{x}(t) + 0.3333\dot{x}(t) + 0.667x(t) = 0,$$

with solution

$$x(t) = e^{0.1666667t} \left(C_1 \cos 0.799305253t + C_2 \sin 0.799305253t \right),$$

C_1, C_2 are real constants. Clearly, condition (ii) of Theorem 3.1 is not satisfied. Therefore, the trivial solution of 4.3 is not stable. See Figure 3.

Example 4.4. Consider

$$\ddot{x}(t) = \dot{x}(t-1) - 3x(t) + x(t-1), \tag{4.4}$$

where we deduce that

$$a_1 = 0, a_2 = 1, b_1 = -3, b_2 = 1, \tau = 1, a_1 a_2 = 0, b_1 b_2 = -3 < 0,$$

$\frac{b_1 + b_2}{1 + \tau a_2} = -1 < 0$ and $\dfrac{a_1 + a_2 - \tau b_2}{1 + \tau a_2} = 0$. Following equation 3.1, we obtain

$$\ddot{x}(t) + x(t) = 0,$$

with solution

$$x(t) = C_1 \cos t + C_2 \sin t,$$

C_1, C_2 are real constants. The trivial solution of 4.4 is stable. See Figure 4.

Example 4.5. Consider

$$\ddot{x}(t) = -4\dot{x}(t) + \frac{1}{e}\dot{x}(t-\frac{1}{2}) - 3x(t) + \frac{1}{e}x(t-\frac{1}{2}), \qquad (4.5)$$

where we deduce that

$a_1 = -4,\ a_2 = \frac{1}{e},\ b_1 = -3,\ b_2 = \frac{1}{e}, \tau = \frac{1}{2}, a_1a_2 = -\frac{4}{e} < 0, b_1b_2 = -\frac{3}{e} < 0,$

$\frac{b_1+b_2}{1+\tau a_2} = -2.3786 < 0$ and and $\dfrac{a_1 + a_2 - \tau b_2}{1 + \tau a_2} = -3.2232 < 0.$ Following equation 3.1, we obtain

$$\ddot{x}(t) + 3.2232\dot{x}(t) + 2.3786x(t) = 0,$$

with solution

$$x(t) = C_1e^{-2.8304628117t} + C_2e^{-0.3927251709t},$$

C_1, C_2 are real constants. Obviously, the trivial solution of 4.5 is asymptotically stable.See Figure 5.

Example 4.6. Consider

$$\ddot{x}(t) = -\frac{e}{2}\dot{x}(t) - \frac{1}{2}\dot{x}(t-1) + x(t) - x(t-1), \qquad (4.6)$$

where we deduce that

$a_1 = -\frac{e}{2},\ a_2 = -\frac{1}{2},\ b_1 = 1,\ b_2 = -1, \tau = 1, a_1a_2 = \frac{e}{4} > 0, b_1b_2 = -1 < 0, \dfrac{b_1 + b_2}{1 + \tau a_2} = 0$

and $\dfrac{a_1 + a_2 - \tau b_2}{1 + \tau a_2} = 1 - e < 0$. Following equation 3.1, we obtain

$$\ddot{x}(t) + (1 - e)\dot{x}(t) = 0,$$

with solution

$$x(t) = C_1 + C_2e^{-1.7182881825t},$$

C_1, C_2 are real constants. The trivial solution of 4.6 is stable.See Figure 6.

Example 4.7. Consider

$$\ddot{x}(t) = 3\dot{x}(t) - \dot{x}(t-\frac{\pi}{2}) - 2x(t) + x(t-\frac{\pi}{2}), \qquad (4.7)$$

where we deduce that

$na_1 = 3,\ a_2 = -1,\ b_1 = -2,\ b_2 = 1, \tau = \frac{\pi}{2}, a_1a_2 = -3 < 0, b_1b_2 = -2 < 0,$

$\frac{b_1+b_2}{1+\tau a_2} = 0.9339 > 0$ and $\dfrac{a_1 + a_2 - \tau b_2}{1 + \tau a_2} = -0.7519 < 0$. Following equation 3.1, we obtain

$$\ddot{x}(t) + 0.7519\dot{x}(t) - 09339x(t) = 0,$$

with solution

$$x(t) = C_1 e^t + C_2 e^{-1.751938394t},$$

C_1, C_2 are real constants. The trivial solution of 4.7 is not stable. See Figure 7.

Example 4.8. Consider

$$\ddot{x}(t) + a\dot{x}(t) + bx(t - \tau) = 0, \tag{4.8}$$

where a, b, τ are positive real numbers. Following §3, equation 4.8 reduces to

$$\ddot{x} + (a - \tau b)\dot{x} + bx = 0,$$

whose trivial solution (equivalently 4.8) is asymptotically stable provided $a - \tau b > 0$. This corroborates results obtained in ([1],Example 4.2.7.,page 253).

Example 4.9. Consider

$$\dot{x}(t) = a_0 x(t) + a_1 x(t - \tau), \tag{4.9}$$

where $\tau > 0, a_0, a_1$ are real constants. Following §3, equation 4.9 reduces to

$$\dot{x}(t) - \left(\frac{a_0 + a_1}{1 + a_1\tau}\right) x(t) = 0,$$

with solution

$$x(t) = e^{\left(\frac{a_0 + a_1}{1 + a_1\tau}\right)t}, \ 1 + a_1\tau \neq 0. \tag{4.10}$$

The trivial solution of equation 4.9 is asymptotically stable provided $a_0 + a_1 < 0$ and $1 + a_1\tau > 0$. This corroborates results obtained in ([5],Example 2.4.3.,page 58). We can also deduce from 4.10 that solutions of 4.9 are asymptotically stable provided $a_0 + a_1 > 0$ and $1 + a_1\tau < 0$.

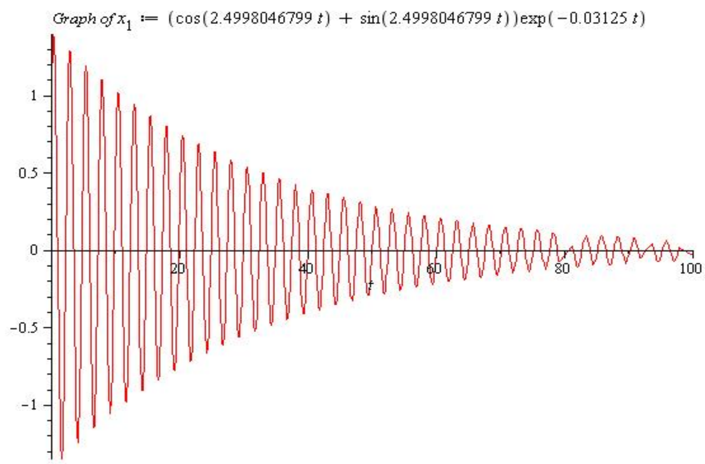


FIGURE 1

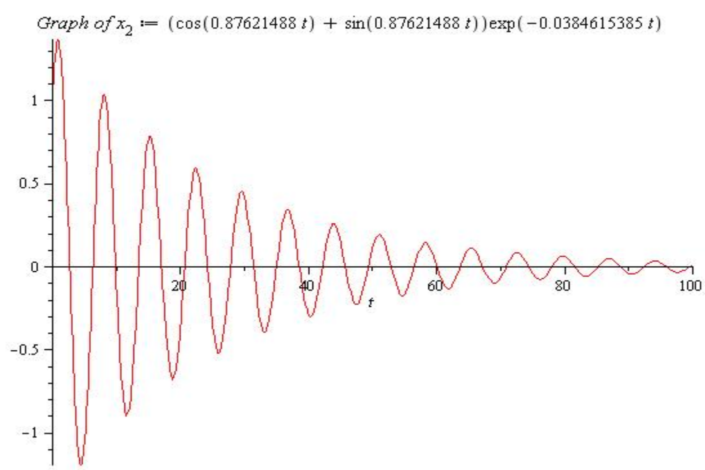


FIGURE 2

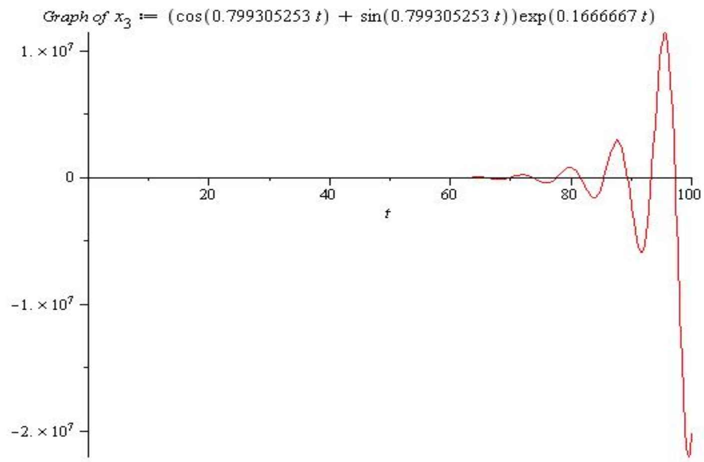


FIGURE 3

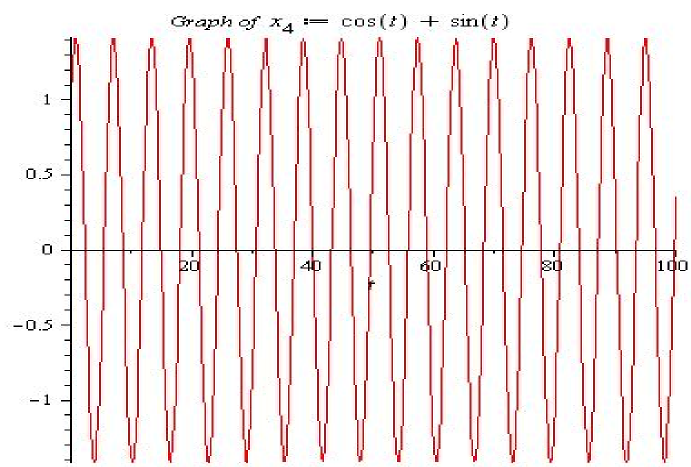


FIGURE 4

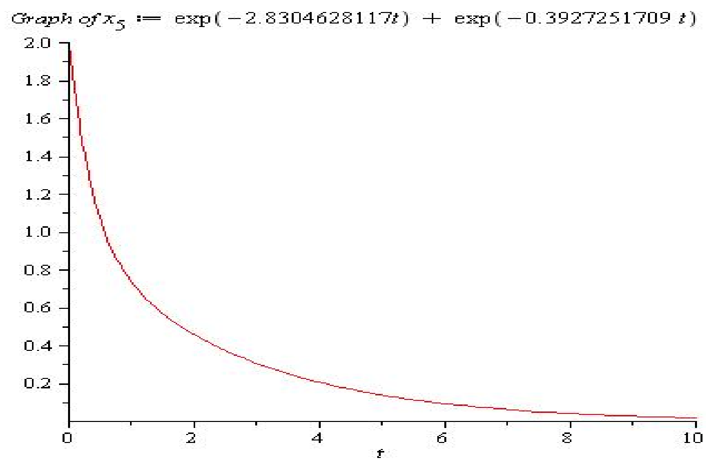
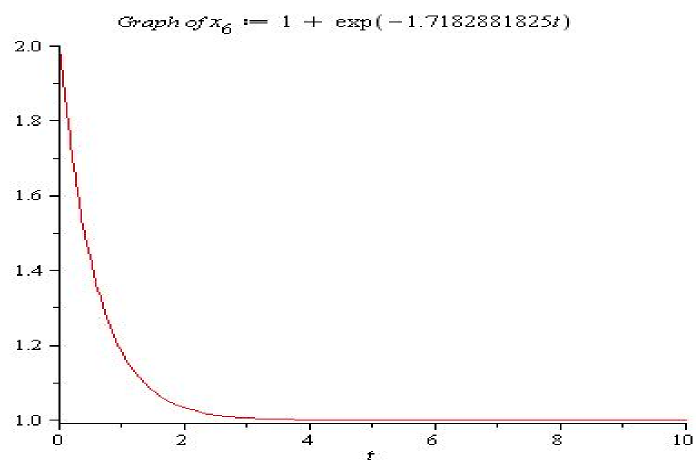


FIGURE 5



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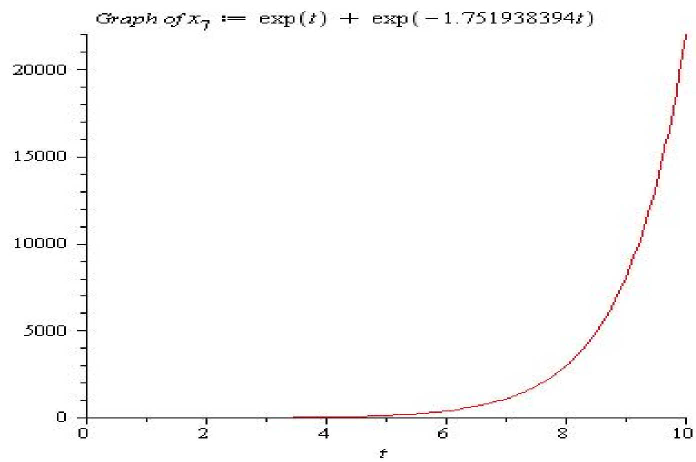

FIGURE 7

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