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ON THE STABILITY AND BOUNDEDNESS OF SOLUTIONS OF CERTAIN KIND OF SECOND ORDER DELAY DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper, we study the second order non-autonomous nonlinear delay differential equation

$$x'' + b(t)g(x, x') + c(t)h(x(t-r))m(x') = p(t, x, x')$$

for asymptotic stability of solutions when $p(t, x, x') = 0$ and the boundedness of solutions when $p(t, x, x') \neq 0$. By using a suitable Lyapunov-Krasovskiĭ functional with sufficient conditions we investigate the stability of solutions while the mean value theorem and the integral method with sufficient conditions are hereby employed to establish the boundedness of solutions result. This work improved on some earlier results in the literature.

Keywords and phrases: Stability; Lyapunov-Krasovskiĭ functional; boundedness; mean value theorem; integral method; nonlinear delay differential equations of second order
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1. INTRODUCTION

Consider the second order non-autonomous nonlinear delay differential equations of the form

$$x'' + b(t)g(x, x') + c(t)h(x(t-r))m(x') = p(t, x, x'), \tag{1}$$

where $b, c \in C(I, \mathbb{R}^+)$, $g \in C(\mathbb{R}^2, \mathbb{R}), h \in C(\mathbb{R}, \mathbb{R}), m \in C(\mathbb{R}, \mathbb{R}^+)$, $p \in C(I \times \mathbb{R}^2, \mathbb{R})$ are real valued functions which depend on the arguments displayed explicitly and r is a positive constant. It would be assumed that solutions of the class of delay differential equations

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being considered exist, b, c, g, h, m, and p are continuous in their respective arguments (see Rao [15]).

It is interesting to note that stability and boundedness of solutions are fundamental to the theory and applications of differential equations. While the study of qualitative theory of solutions to nonlinear differential equations with or without delay have attracted the attention of many researchers and have achieved many results in this area (see [2], [11], [13], [19]). In these references, the authors made use of Lyapunov's direct method to obtain the conditions which establish the stability and boundedness of solutions. In particular for delay differential equations, Zhang [20] considered the retarded Liènard equation

$$x'' + f(x)x' + g(x(t-h)) = 0, \tag{2}$$

in which h is a nonnegative constant and g, f are continuous with $f(x) > 0$ for all $x \in \mathbb{R}$. The author obtained conditions for the boundedness and global asymptotic stability results. In [21], Zhang examined equation (2) and gave results on the uniform boundedness, uniform ultimate boundedness and oscillation of solutions. Peng [14] studied the second-order nonlinear system with delay:

$$x''(t) + f(x(t), x'(t)) + g(x(t), x'(t))\psi(x(t-\tau)) = p(t),$$

where f,g,p are continuous functions, ψ is a differentiable function, τ is a positive constant and gave four theorems on the stability of zero solution, the boundedness of the solutions, the existence of the periodic solutions, the existence and uniqueness of the stationary oscillation. Besides, in [17], Tunç established some results for the stability and the boundedness of solutions of non-autonomous differential equations of second order with a variable deviating argument of the form:

$$x''(t) + f(t, x(t), x'(t))x'(t) + b(t)g(x(t-\tau(t))) = q(t),$$

where $\tau(t)$ is variable deviating argument; f, b, g and q are continuous functions in their arguments on $\mathbb{R}^+ \times \mathbb{R}^2$, $\mathbb{R}$, $\mathbb{R}$ and $\mathbb{R}^+$ respectively, where $\mathbb{R}^+ = [0, \infty)$.

Also, Ogundare *et al.* [13] considered the second order nonlinear differential equations of the form:

$$x'' + a(t)f(x, x') + g(x(t-\tau)) = p(t, x, x'),$$

where a, f, g and p are continuous functions that depend (at most) only on the arguments displayed explicitly and $\tau \in [0, h]$ $(\tau > 0)$. The authors obtained results for the global asymptotic stability, boundedness and ultimate boundedness of the solutions respectively.

However, in the paper by Athanassov [3], a generalized Lienard differential equations of the form

$$x'' + b(t)f(x, x') + c(t)g(x)h(x') = p(t, x, x'), \tag{3}$$

where b, c, f, g, h, and p are real valued functions which depend at most on the arguments displayed explicitly was considered. The Gronwall's inequality and two forms of second mean value theorem for integrals were used to established the boundedness of all solutions and their first derivatives.

The motivation for this work comes from the paper by Athanassov [3] in which we extend equation (3) to a system of delay differential equation and investigate the stability of all solutions for the case in which $p(t, x, x') = 0$ and the boundedness of all solutions for the case $p(t, x, x') \neq 0$. We shall employ a suitable Lyapunov Krasovskiǐ functional with sufficient conditions to establish the stability of all solutions while the mean value theorem and the two forms of second mean value theorem for integrals with sufficient conditions will be used to investigate boundedness of solutions respectively.

2. PRELIMINARIES AND RESULT ON STABILITY OF SOLUTIONS

Here, we give some basic information for the general non-autonomous differential system with delay (see Burton [4], Burton and Markay [5], Tunç [16], see also Kolmanovskii and Myšhkis [8], Kolmanovskii and Nosov [9], Krasovskiǐ [10] and Yoshizawa [18]).

Consider the general non-autonomous differential system with delay:

$$x'(t) = f(t, x_t),\ x_t(s) = x(t+s),\ -r \leq s \leq 0,\ t \geq 0 \quad ' = d/dt \tag{4}$$

where $f : [0, \infty) \times C_H \to \mathbb{R}^n$ is continuous and takes bounded sets into bounded sets and $f(t, 0) = 0$. Here, $(C, \|\cdot\|)$ is the Banach space of continuous functions $\phi : [-r, 0] \to \mathbb{R}^n$ with supremum norm, r is a non-negative constant, C_H is the open H-ball in $C_H := \{\phi \in (C[-r, 0], \mathbb{R}^n) : \|\phi\| < H\}$.

Standard existence (see Burton [4]) shows that if $\phi \in C_H$ and $t \geq 0$, then there is at least one continuous solution $x(t, t_0, \phi)$ on $[t_0, t_0+\alpha)$ satisfying (4) for $t > t_0$, $x_t(t_0, \phi) = \phi$ and α some positive constant; if there is a closed subset $B \subset C_H$ such that the solution remains in B, then $\alpha = \infty$. In addition, $\|\cdot\|$ denotes the norm in $\mathbb{R}^n$ with $\|x\| = \max\limits_{1 \leq i \leq n} |x_i|$.

Definition 2.1[4]. A continuous function $W : [0, \infty) \to [0, \infty)$ with $W(0) = 0$, $W(s) > 0$ if $s > 0$, and W strictly increasing is a wedge. Let wedges be denoted by W or W_i, where i is an integer.

Definition 2.2[4]. Let D be an open set in $\mathbb{R}^n$ with $0 \in D$. A function $V : [0, \infty) \times D \to [0, \infty)$ is called positive definite if $V(t, 0) = 0$ and if there is a wedge W_1 with $V(t, x) \geq W_1(|x|)$, and is called decrescent if there is a wegde W_2 with $V(t, x) \leq W_2(|x|)$.

Definition 2.3[4]. Let $f(t, 0) = 0$. The zero solution of equation (4) is:

(i): stable if for each $\varepsilon > 0$ and $t_1 \geq t_0$ there exists $\delta > 0$ such that $[\phi \in C(t_1),\ \|\phi\| < \delta,\ t \geq t_1]$ implies that $|x(t, t_1, \phi)| < \varepsilon$.

(ii): asymptotically stable if it is stable and if for each $t_1 \geq t_0$ there is an $\eta > 0$ such that $[\phi \in C(t_1), \|\phi\| < \eta]$ implies that $x(t, t_0, \phi) \to 0$ as $t \to \infty$.

Definition 2.4[4]. Let $V(t, \phi) = V$ be a continuous functional defined for $t \geq 0$, $\theta \in C_H$. The derivative of V along solutions of (4) will be denoted by $\dot{V}$ and is defined by the following relation:

$$\dot{V} = \limsup_{h \to 0} \frac{V(t + h, x_{t+h}(t + \phi)) - V(t, x_t(t_0, \phi))}{h},$$

where $x(t_0, \phi)$ is the solution of (4) with $x_{t_0}(t_0, \phi) = \phi$.

The main result for the stability of solutions of the non-autonomous nonlinear second order delay differential equations is hereby given. First, we state the equivalent system for equation (1) with $p(t, x, x') = 0$ as follows: Let

$$\begin{aligned} x' &= y, \\ y' &= -b(t)g(x(t), y(t)) - c(t)h(x(t))m(y(t)) \\ &\quad + c(t)m(y(t)) \int_{t-r}^{t} h'(x(s))y(s)ds. \end{aligned} \tag{5}$$

Theorem 2.5. Suppose that $b(t)$ and $c(t)$ are continuously differentiable on $[0, \infty)$ and the following conditions are satisfied

(c_1) $1 \leq c(t) \leq b(t) \leq \Phi$, $c'(t) \leq 0$, $\Phi > 0$, $t \in [0, \infty)$;

(c_2) $h(0) = 0$, $\dfrac{h(x)}{x} \geq \delta_0 > 0$ $(x \neq 0)$, and $h'(x) \leq c$ for all x;

(c_3) $\dfrac{g(x,y)}{y} \geq \eta > 0,\ (y \neq 0)$, for all $x, y \in \mathbb{R}$;

(c_4) $\dfrac{1}{m(y)} \geq \xi$ $(m(y) \neq 0)$ for all $y \in \mathbb{R}$;

(c_5) $\int_0^\infty |c'(t)|dt < \infty,\ c'(t) \to 0$ as $t \to \infty$.

Then the solution $x = 0$ of (5) is asymptotically stable provided that

$$0 < r < \frac{\xi\eta}{\Phi c}.$$

To prove the **Theorem 2.5**, we define the Lyapunov functional $V \equiv V(t, x_t, y_t)$ as:

$$V = c(t)\int_0^x h(\tau)d\tau + \int_0^y \tau/m(\tau)d\tau + \lambda\int_{-r}^0\int_{t+s}^t y^2(\tau)d\tau ds, \quad (6)$$

where λ is a positive constant to be determined later. We now show that equation (6) is positive definite, we have

$$V = c(t)\int_0^x \frac{h(\tau)}{\tau}\tau d\tau + \int_0^y \frac{1}{m(\tau)}\tau d\tau + \lambda\int_{-r}^0\int_{t+s}^t y^2(\tau)d\tau ds.$$

Applying the assumptions (c_2) to (c_4), we obtain

$$V \geq c(t)\delta_0\int_0^x \tau d\tau + \xi\int_0^y \tau d\tau + \lambda\int_{-r}^0\int_{t+s}^t y^2(\tau)d\tau ds.$$

Then

$$V \geq \delta_0\frac{x^2}{2} + \xi\frac{y^2}{2} + \lambda\int_{-r}^0\int_{t+s}^t y^2(\tau)d\tau ds,$$

which shows that the functional is positive definite.

Differentiating V along the equation (5), we have:

$$\begin{aligned} V' &= c'(t)\int_0^x h(\tau)d\tau + c(t)h(x)y + y/m(y)\cdot y' + \lambda r y^2 - \lambda\int_{t-r}^t y^2(\tau)d\tau \\ &= c'(t)\int_0^x h(\tau)d\tau + c(t)h(x)y - b(t)\frac{g(x,y)}{m(y)}y - c(t)h(x)y \\ &\quad + c(t)y\int_{t-r}^t h'(x(\tau))y(\tau)d\tau + \lambda r y^2 - \lambda\int_{t-r}^t y^2(\tau)d\tau. \end{aligned}$$

In the light of the inequality $2|uv| \leq u^2 + v^2$, we have

$$c(t)y\int_{t-r}^t h'(x(\tau))y(\tau)d\tau \leq \frac{\Phi c}{2}ry^2 + \frac{\Phi c}{2}\int_{t-r}^t y^2(\tau)d\tau.$$

So,

$$
\begin{aligned}
V' &\leq c'(t)\int_0^x h'(\tau)d\tau - b(t)\frac{g(x,y)}{m(y)}y + [\frac{\Phi c}{2}ry^2 + \frac{\Phi c}{2}\int_{t-r}^{t} y^2(\tau)d\tau] \\
&\quad +\lambda r y^2 - \lambda\int_{t-r}^{t} y^2(\tau)d\tau \\
&\leq c'(t)\int_0^x h(\tau)d\tau - \frac{1}{m(y)}\eta y^2 + \frac{1}{2}(\Phi c + 2\lambda)ry^2 + \{\frac{1}{2}\Phi c - \lambda\}\int_{t-r}^{t} y^2(\tau)d\tau.
\end{aligned}
$$

If we take $\lambda = \frac{1}{2}\Phi c > 0$, and by (c_4), we have

$$V' \leq c'(t)\int_0^x h(\tau)d\tau - (\xi\eta - \Phi c r)y^2.$$

If we choose

$$0 < r < \frac{\xi\eta}{\Phi c}$$

and there exist a $\delta > 0$ such that

$$V' \leq c'(t)\int_0^x h(\tau)d\tau - \delta y^2.$$

Thus, the solutions of system (5) is asymptotically stable. Hence, Theorem 2.5 is satisfied.

Remark 2.6. From equation (1), if $h(x(t-r)) = h(x)$, where $t > r,\ r > 0$ and $p(t,x,x') \neq 0$, this become similar to the equation (3) considered by Athanasov [3].

3. FURTHER PRELIMINARIES AND RESULT ON THE BOUNDEDNESS OF SOLUTIONS

Firstly, we denote $\mathbb{R}$ the real line, by $\mathbb{R}^+$ and I the intervals $(0,\infty)$ and $[0,\infty)$ respectively. The $|.|$ is an absolute value, $C(A,\mathbb{R})$ denote the set of $\mathbb{R}$-valued continuous function defined on the set A. While $L_1(A)$ denotes the set of Lebesque integrable functions on A.

We state the following lemmas and theorem. The following are two forms of second mean value theorem for integrals. For example, one can refer to Hildebrandt [7] and Athanassov [3].

Lemma 3.1. If $u \in L_1[\alpha,\beta]$ and v is a positive, bounded and nonincreasing function $[\alpha,\beta]$, then there is a number $\delta \in [\alpha,\beta]$ such that

$$\int_\alpha^\beta u(s)v(s)ds = v(\alpha+0)\int_\alpha^\delta u(s)ds.$$

Lemma 3.2. If $u \in L_1[\alpha, \beta]$ and v is a positive, bounded and nondecreasing function $[\alpha, \beta]$, then there is a number $\delta \in [\alpha, \beta]$ such that

$$\int_{\alpha}^{\beta} u(s)v(s)ds = v(\beta - 0) \int_{\delta}^{\beta} u(s)ds.$$

Theorem 3.3. (Mean Value Theorem [6]). Let $[\alpha, \beta]$ be a closed, bounded interval, i.e., $-\infty < \alpha < \beta < \infty$. Let $f : [\alpha, \beta] \to \mathbb{R}$ be continuous and let f' exist on (α, β). Then there exists a number $\theta \in (\alpha, \beta)$ such that

$$f(\beta) - f(\alpha) = f'(\theta)(\beta - \alpha)$$

(see also Afuwape [1] and Meng [12]).

We now investigate the boundedness of solutions for the non-autonomous nonlinear delay differential equation (1).

Theorem 3.4. We hereby consider the following basic conditions for the boundedness of solutions result, we have
(c_1) $g(x, x')x' > 0$ for all $(x, x') \in \mathbb{R}^2, x' \neq 0$;
(c_2) $H(x) \to \infty$ as $|x| \to \infty$, where $H(x) = \int_0^x h(\tau)d\tau \geq 0$;
(c_3) $M(x') \to \infty$ as $|x'| \to \infty$, where $M(x') = \int_0^{x'} \frac{\tau}{m(\tau)}d\tau$;
(c_4) $h(x(t - r)) = h(x(t)) - rh'(x(t))x'(t)$, $t > r$, $r > 0$ and $x \in \mathbb{R}$;

(c_5) there exist the function $(x'(t))^2 \in L^2(I)$ and positive constants β_2 and T such that $0 < |h'(x(t))|(x'(t))^2 \leq \beta_2, \ |h'(x(t))| \leq T$.

(c_6) There is a nonnegative function $e(t) \in L^1(I)$ such that
$|p(t, x, y)y| \leq e(t)m(x')$ for all $(t, x, x') \in I \times \mathbb{R}^2$;

Then any solution $x(t)$ of (7) is bounded. If $c(t)$ is non-decreasing on I and is bounded above on I, then $x'(t)$ is also bounded.

Proof. Now, with the concept of mean value theorem we apply (c_4) where appropriate, such that equation (1) in which $p(t, x, x') \neq 0$ is equivalent to the following: let $x' = y$, then

$$y' + b(t)g(x, y) + c(t)m(y)[h(x(t)) - rh'(x(t))y(t)] = p(t, x, y). \quad (7)$$

Multiplying (7) by $y(t)/[c(t)m(y(t))]$, and integrating both sides from 0 to t, we obtain

$$\begin{aligned}\int_0^t y(\tau)y'(\tau)/[c(\tau)m(y(\tau))]d\tau &+ \int_0^t b(\tau)g(x(\tau),y(\tau))y(\tau)/[c(\tau)m(y(\tau))]d\tau \\ &+ \int_0^t h(x(\tau))y(\tau)d\tau - r\int_0^t h'(x(\tau))(y(\tau))^2 d\tau \\ &= \int_0^t p(\tau,x(\tau),y(\tau))y(\tau)/[c(\tau)m(y(\tau))]d\tau.\end{aligned}$$

Since the integral in the second term on the left is nonnegative and by (c_1), we have as follows

$$\begin{aligned}&\int_0^t y(\tau)y'(\tau)/[c(\tau)m(y(\tau))]d\tau + \int_0^t h(x(\tau))y(\tau)d\tau \\ &\leq \int_0^t |p(\tau,x(\tau),y(\tau))y(\tau)|/[c(\tau)m(y(\tau))]d\tau + r\int_0^t |h'(x(\tau))|(y(\tau))^2 d\tau.\end{aligned}$$

By Lemma 3.1, it follows that there is $\delta \in [0,t]$ such that

$$\begin{aligned}&\frac{1}{c(0)}\int_0^\delta y(\tau)/m(y(\tau))\frac{dy(\tau)}{d\tau}d\tau + \int_0^t h(x(\tau))\frac{dx(\tau)}{d\tau}d\tau \\ &\leq \frac{1}{c(0)}\int_0^\delta |p(\tau,x(\tau),y(\tau))y(\tau)|/m(y(\tau))d\tau + r\int_0^t |h'(x(\tau))|(y(\tau))^2 d\tau.\end{aligned}$$

Applying (c_2), (c_3), (c_5) and (c_6) then

$$\frac{1}{c(0)}[M(y(\delta)) - M(y(0))] + H(x(t)) - H(x(0)) \leq \frac{1}{c(0)}\int_0^\infty e(\tau)d\tau + r\beta_2.$$

Now by (c_3) and the assumption on $c(t)$ such that $c(t)$ is non-decreasing on I lead to the estimate

$$\begin{aligned}H(x(t)) &\leq H(x(t)) + \frac{1}{c(0)}M(y(\delta)) \\ &\leq H(x(0)) + r\beta_2 + \frac{1}{c(0)}[M(y(0)) + \int_0^\infty e(\tau)d\tau].\end{aligned}$$

The right side of the last inequality is a constant independent of t, say K, therefore (c_2) implies that $x(t)$ is bounded on I.

We now suppose that $c(t) \leq c_0$ on I. Multiply on both sides of (7) by $y(t)/m(y(t))$ and integrating from 0 to t, we obtain

$$\begin{aligned}&\int_0^t y(\tau)y'(\tau)/m(y(\tau))d\tau + \int_0^t b(\tau)g(x(\tau),y(\tau))y(\tau)/m(y(\tau))d\tau \\ &+ \int_0^t c(\tau)h(x(\tau))y(\tau)d\tau \leq \int_0^t |p(\tau,x(\tau),y(\tau))y(\tau)|/m(x'(\tau))d\tau \\ &+ r\int_0^t c(\tau)|h'(x(\tau))|(y(\tau))^2 d\tau.\end{aligned}$$

The integral in the second term on the left is nonnegative and by Lemma 3.2 there exists $\delta \in [0,t]$ such that

$$\int_0^t y(\tau)/m(y(\tau))\frac{dy(\tau)}{d\tau}d\tau + c(t)\int_\delta^t h(x(\tau))\frac{dx(\tau)}{d\tau}d\tau$$
$$\leq \int_0^t |p(\tau,x(\tau),y(\tau))y(\tau)|/m(y(\tau))d\tau + rc(t)\int_\delta^t |h'(x(\tau))|(y(\tau))^2 d\tau.$$

Applying (c_2) - (c_5), then

$$M(y(t)) - M(y(0)) + c(t)[H(x(t)) - H(x(\delta))] \leq \int_0^\infty e(\tau)d\tau + r\beta_2.$$

Since $c(t)H(x(t))$ is nonnegative on I, so we have

$$\begin{aligned} M(y(t)) &\leq M(y(t) + c(t)H(x(t)) \\ &\leq M(y(0)) + c(t)H(x(\delta)) + \int_0^\infty e(\tau)d\tau + r\beta_2 \\ &\leq M(y(0)) + c_0K + r\beta_2 + \int_0^\infty e(\tau)d\tau = L, \end{aligned}$$

a constant independent of t. Hence, (c_3) implies that $y(t)$ is bounded on I.

Remark 3.5. From equation (1), if $h(x(t-r)) = h(x)$, where $t > r$, $r > 0$, this reduces to the one considered by Athanasov [3]. Interestingly, Theorem 3.4 shows a significant improvement to that of Theorem 1 considered by Athanassov [3].

4. CONCLUSION

This work investigated the asymptotic stability of solutions for the equation (1) for which $p(t,x,x') = 0$ by using a suitable Lyapunov-Krasovskiĭ functional. In addition, we employed the concept of mean value theorem (specifically, the chain rule) and the two forms of mean value theorem for integrals in which the case $p(t,x,x') \neq 0$ to establish our result on the boundedness of solutions.

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REFERENCES

(1) Afuwape, A. U. *Ultimate Boundedness Results for a Certain System of Third Order Nonlinear Differential Equations*, Journal of Mathematical Analysis and Applications **97** 140-150, 1983. https://doi.org/10.1016/0022-247X(83)90243-3

(2) Afuwape, A. U. and Omeike, M. O. *On the stability and boundedness of solutions of a kind of third order delay differential equations.* Applied Mathematics and Computation **200** 444 - 451, 2008. https://doi.org/10.1016/j.amc.2007.11.037, www.elsevier.com/locate/amc

(3) Athanassov, Z. S. *Boundedness criteria for solutions of certain second order non-linear differential equations.* Journal of Mthematical Analysis and Applications **123** 461-479, 1987. https://doi.org/10.1016/0022-247X(87)90324-6

(4) Burton, T. A. *Stability and periodic solutions of ordinary and functional differential equations.* Mathematics in Science and Engineering **178**, Academic Press, Inc., Orlando, FL, 1985.

(5) Burton, T. A. and Markay, G. *Asymptotic stability for functional differential equations.* Acta Math. Hung. **65** (3) 243-251, 1994. https://doi.org/10.1007/BF01875152

(6) Driver, R. D. *Ordinary and Delay Differential Equations.* Applied Mathematical Sciences; volume 20. Springer-Verlag, New York Inc., 1977. https://doi.org/10.1007/978-1-4684-9467-9

(7) Hildebrandt, T. H. *Introduction to the theory of integration.* Academic Press, New York, 1963.

(8) Kolmanovskii, V. and Myshkis, A. *Introduction to the Theory and Applications of Functional Differential Equations.* Klumer Academic, Dordrecht, 1999.

(9) Kolmanovskii, V. B. and Nosov, V. R. *Stability of functional-differential equations.* Mathematics in Science and Engineering **180**, 1986. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], London. https://doi.org/10.1016/S0076-5392(08)62050-0

(10) Krasovskiĭ, N. N. *Stability of Motion. Applications of Lyapunov's Second Method to Differential Systems and Equations with Delay.* Stanford University Press, Stanford, California, 1963.

(11) Lalli, B. S. *On the boundedness of solutions of certain second order differential equations.* J. Math. Anal. Appl. **25** 182-188, 1989. https://doi.org/10.1016/0022-247X(69)90221-2

(12) Meng, F. W. *Ultimate Boundedness Results for a Certain System of Third Order Nonlinear Differential Equations.* Journal of Mathematical Analysis and Applications **177** 496-509, 1993. https://doi.org/10.1006/jmaa.1993.1273

(13) Ogundare, B. S., Ademola, A. T., Ogundiran, M. O. and Adesina, O. A. *On the qualitative behaviour of solutions to certain second order nonlinear differential equation with delay.* Annali dell'Università di Ferrara **63** 333-351, 2017. https://doi.org/10.1007/s11565-016-0262-y

(14) Peng, Q. *Qualitative analysis for a class of second-order nonlinear system with delay.* Applied Mathematics and Mechanics. Published by Shanghai University, Shangai, China. English Edition **22** (7) 842-845, 2001. https://doi.org/10.1023/A:1016373806172

(15) Rao, M. R. M. *Ordinary Differential Equations.* Affiliated East-West Private Limited London, 1980.

(16) Tunç, C. *On the qualitative behaviours of solutions to a kind of nonlinear third order differential equations with retarded argument.* Italian Journal of Pure and Applied Mathematics (28) 273-284, 2011.

(17) Tunç, C. *A note on the stability and boundedness of non-autonomous differential equations of second order with a variable deviating arguments.* Afrika Matematika **25** 417-425, 2014. https://doi.org/10.1007/s13370-012-0126-2

(18) Yoshizawa, T. *Stability theory by Lyapunov's second method.* The Mathematical Society of Japan, Tokyo, 1966.
(19) Zarghamee, M. S. and Mehri, B. *A note on boundedness of solutions of certain second order differential equations.* J. Math. Anal. Appl. **31** 504-508, 1970. https://doi.org/10.1016/0022-247X(70)90003-X
(20) Zhang, B. *On the retarded Liénard equation.* Proceedings of the American Mathematical Society **115** (3) 779-785, 1992. Published by American Mathematical Society. https://doi.org/10.2307/2159227
(21) Zhang, B. *Necessary and Sufficient Conditions for the Boundedness and Oscillation in the Retarded Liénard Equation.* Journal of Mathematical Analysis and Applications **200** 453-473, 1996. https://doi.org/10.1006/jmaa.1996.0216

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