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APPROXIMATION OF FIXED POINTS OF A FINITE FAMILY OF MULTIVALUED LIPSCHITZ PSEUDO-CONTRACTIVE MAPPINGS IN BANACH SPACES

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ABSTRACT. In this work, approximation of fixed point of finite family of multi-valued Lipschitz pseudo-contractive mappings is studied in the setting of uniformly convex Banach spaces. Utilizing a result in [Ofoedu, E. U, Zegeye, H., *Iterative algorithm for multi-valued pseudo-contractive mappings in Banach spaces*, J. Math. Anal. Appl., **372** (2010), 68-76.], a sequence $\{y_n\}$ is generated by the implicit scheme: $y_1 \in \overline{D}$, $y_n = (1-t_n)y_{n-1} + t_n z_n$, $z_n \in T_n y_n$, $n = 2,3,4,\cdots, T_n = T_{n \bmod N}$, where $\{t_n\} \subseteq (0,1)$ such that $t_n \to 1$ as $n \to \infty$ and $T_i, i = 1,2,3,\cdots,N, N \in \mathbb{N}$, are multi-valued Lipschitz pseudo-contractive maps defined on $\overline{D}$ such that $T_i x$ is a proximinal subset of $\overline{D}$, D an open, convex and nonempty subset of a real uniformly convex Banach space. The strong convergence of $\{y_n\}$ to a common fixed point of T_i's, given existence of such common fixed point, is shown using the defining properties of uniformly convex spaces. Furthermore, using $\{y_n\}$ and certain characterization of pseudo-contractive maps, the explicit algorithm $x_1 \in \overline{D}$,

$$x_{n+1} = (1-\lambda_n)x_n + \lambda_n w_n - \lambda_n \theta_n (x_n - x_1),\ w_n \in Tx_n\ n = 1,2,3,\cdots,$$

where w_n is chosen appropriately and the parameters λ_n and θ_n satisfy certain conditions, is shown to converge strongly to a fixed point of a multi-valued Lipschitz pseudo-contractive map.

1. INTRODUCTION

The fixed point theory of multivalued mappings has attracted and continues to attract the attention of many researchers. Part of the reasons, perhaps, is the connection of the theory with many real-world applications. Apart from the theory of *differential inclusions*, used in studying differential equations with discontinuous right-hand sides, which was the initial mainstay of applications, other applications are found in *Game Theory*, *Optimization Theory,* etc. For instance, under some conditions, it has been shown that in noncooperative static games, the equilibrium points of such games coincide with fixed points of certain multivalued mappings (see, for example, [6] and [5] for more on these connections). Additionally, in solving the inclusion problem of the form $0 \in Au$, which has many applications in solving partial differential equations, a technique used is to convert the problem into a fixed point problem of an appropriate multivalued map.

Let E denote a normed linear space and E^* its dual (this will be assumed throughout the paper, unless otherwise stated). The *generalized duality mapping* $J_q : E \to 2^{E^*}$ is defined as

$$J_q(x) := \{f \in E^* : \langle x, f\rangle = \|x\|^q \ and\ \|f\| = \|x\|^{q-1}\}, q > 1,$$

where $\langle \cdot,\cdot\rangle$ denotes the duality pairing between elements of E and E^*. When $q = 2$, J_2 is called the *normalized duality mapping* and is denoted by J, that is,

$$J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^2 \ and\ \|f\| = \|x\|\}.$$

E is said to be uniformly convex if for any $\varepsilon \in (0,2]$, there exists $\delta_\varepsilon > 0$ such that if $x, y \in E$ with $\|x\| = \|y\| = 1$ and $\|x-y\| \geq \varepsilon$, then $\dfrac{\|x+y\|}{2} < 1-\delta_\varepsilon$.

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For a multi-valued mapping $T : \mathscr{D}(T) \subseteq E \to 2^E$, an element $x \in \mathscr{D}(T)$ is called a *fixed point of* T if $x \in T(x)$, where $\mathscr{D}(T) := \{x \in E : Tx \neq \emptyset\}$. A multi-valued mapping $T : \mathscr{D}(T) \subseteq E \to 2^E$ is called *pseudo-contractive* if the inequality

$$\|x-y\| \leq \|x-y+t((x-u)-(y-v))\|$$

holds for all $x, y \in \mathscr{D}(T),\ u \in Tx,\ v \in Ty$ and for all $t > 0$. Equivalently (following Kato [10]), a multivalued mapping T is pseudo-contractive if and only if for all $x, y \in \mathscr{D}(T)$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle u-v, j(x-y)\rangle \leq \|x-y\|^2 \ \forall\, u \in Tx,\ v \in Ty.$$

A multi-valued mapping $A : \mathscr{D}(A) \subseteq E \to 2^E$ is called *accretive* if

$$\|x-y\| \leq \|x-y+t(u-v)\| \ \forall x, y \in \mathscr{D}(A),\ u \in Ax,\ v \in Ay \text{ and for each } t > 0.$$

Following Kato [10], a mapping $A : \mathscr{D}(A) \subseteq E \to 2^E$ is *accretive* if and only if there exists $j(x-y) \in J(x-y)$ such that

$$\langle u-v, j(x-y)\rangle \geq 0$$

for each $x, y \in \mathscr{D}(A), u \in Ax, v \in Ay$. The *Hausdorff distance* h on 2^E is defined as

$$h(U,V) = \max\left\{\sup_{u\in U} \operatorname{dist}(u,V), \sup_{v\in V} \operatorname{dist}(v,U)\right\},$$

where $U, V \subseteq E$ and $\operatorname{dist}(x,K) := \inf\{d(x,u) : u \in K\}$ for $x \in E$, $K \subseteq E$. When restricted to $\mathscr{CB}(E)$, the family of nonempty closed and bounded subsets of E, h becomes a metric on $\mathscr{CB}(E)$. A multivalued mapping $T : \mathscr{D}(T) \subseteq E \to \mathscr{C}B(E)$ is called *Lipschitz* if there exists $L > 0$ such that

$$h(Tx, Ty) \leq L\|x-y\| \ \forall\, x, y \in \mathscr{D}(T).$$

The map T is called *nonexpansive* when $L = 1$. The pseudo-contractive mappings, introduced first by Browder [2], aside generalizing the nonexpansive mappings, are also closely related to the accretive mappings, in fact, T is pseudo-contractive if and only if $I - T$ is accretive, I being the identity map of E. Owing to the intimate connection between accretive mappings and the theory of ordinary and partial differential equations, the study of pseudo-contractive mappings has continued to attract a great deal of attention. For instance, in the evolution equation $\frac{du}{dt} + Au = 0, u(0) = u_0$ which describes certain phenomena, the operator A is accretive (see, e.g., [7], Ch. 8). Also, in the study of the more general evolution equation $\frac{du}{dt} + A(t)u(t) = f(t, u(t)), t \geq 0, u(0) = u_0$, a chief assumption on A is accretivity (see, e.g., Browder [1]). In fact, even for the initial-value problem for the 2nd-order *doubly nonlinear* equation: $\frac{d^2u}{dt^2} + A\left(\frac{du}{dt}\right) + Bu = f(t), u(0) = u_0,\ \frac{du}{dt}(0) = v_0$, accretivity of A, among others, is required to prove existence of a solution (see [16]).

A lot of work can be found in the literature concerning approximation, especially for singlevalued mappings. With regard to approximation of fixed points of multivalued mappings, the question of whether the result of Browder [3] for (single-valued) nonexpansive mappings in Hilbert spaces can be obtained for multivalued nonexpansive mappings was answered in the negative (see Pietramala [14]). In [13], Ofoedu and Zegeye were able to extend the work of Morales and Jung [12] (we recall that the work of Morales and Jung [12] was an extension of the result of Reich [15] from the setting of uniformly smooth Banach spaces and for nonexpansive mappings to that of reflexive Banach spaces having uniformly Gâteuax differentiable norms and for continuous pseudo-contractive mappings) to multi-valued settings. Ofoedu and Zegeye, precisely, proved the following theorem, which will be used in the sequel.

Theorem 1.1 ([13])**.** Let D be a nonempty open convex subset of a real Banach space X and $T : \overline{D} \to CB(X)$ be a continuous (relative to Hausdorff metric) pseudo-contractive mapping satisfying weakly inward condition and let $u \in \overline{D}$ be fixed. Then for each $t \in (0,1)$ there exists $y_t \in \overline{D}$ satisfying $y_t \in tTy_t + (1-t)u$. If, in addition, X is reflexive and has a uniformly *Gâteaux* differentiable norm such that every closed convex bounded subset of $\overline{D}$ has fixed point property for nonexpansive self-mappings, then

T has a fixed point if and only if $\{y_t\}$ remains bounded as $t \to 1^-$. In this case, $\{y_t\}$ converges strongly to a fixed point of T.

In the same paper [13], Ofoedu and Zegeye obtained the following result.

Theorem 1.2 ([13])**.** Let X be a reflexive real Banach space having a uniformly Gâteaux differentiable norm, D be a nonempty, open and convex subset of X, such that every closed, convex, bounded and nonempty subset of $\overline{D}$ has the fixed point property for nonexpansive self-mappings. Let $T : \overline{D} \to K(\overline{D})$ be a pseudo-contractive Lipschitz mapping with Lipschitz constant $L > 0$, where $K(\overline{D})$ denotes the family of all nonempty compact subsets of $\overline{D}$, and let $u \in \overline{D}$ be fixed. Let $\{x_n\}$ be a sequence generated iteratively from arbitrary $x_0 \in \overline{D},\ w_0 \in Tx_0$ by

$$x_{n+1} := (1-\lambda_n)x_n + \lambda_n w_n - \lambda_n \theta_n (x_n - x_0),\ w_n \in Tx_n. \tag{1.1}$$

Suppose that $\|w_n - w_{n-1}\| = \text{dist}(w_{n-1}, Tx_n),\ n \geq 1$. If $F(T) \neq \emptyset$, then $\{\text{x}_n\}$ converges strongly to a fixed point of T.

Chidume *et al.* [5] improved the result of Ofoedu and Zegeye [13] above in the class of q-uniformly smooth Banach spaces, by weakening the Lipschitz assumption to continuity and boundedness on the mapping and making the choice of w_n easier. One notes that the choice of w_n in the scheme of Ofoedu and Zeyege [13] is not arbitrary. For the sigle-valued case, the perturbation of Mann algorithm given by the explicit algorithm (1.1), which we use in Theorem 3.6 above, has the advantage of giving strong convergence to fixed point for the class of Lipschitz pseudo-contractive mappings without having to require λ_n and θ_n to be *acceptably paired* (see, e.g., [7] Chapter 11). Chidume *et al. [5]* proved the following theorem.

Theorem 1.3 ([5])**.** Let X be a q-uniformly smooth real Banach space and D be a nonempty, open and convex subset of X. Assume that $T : \overline{D} \to CB(\overline{D})$ is a multi-valued continuous (with respect to the Hausdorff metric), bounded and pseudo-contractive mapping with $F(T) \neq \emptyset$. Let $\{\text{x}_n\}$ be a sequence generated iteratively from arbitrary $x_1 \in \overline{D}$ by

$$x_{n+1} := (1-\lambda_n)x_n + \lambda_n u_n - \lambda_n \theta_n (x_n - x_1),\ u_n \in Tx_n,\ n \geq 1.$$

Then, there exists a real constant $\gamma_0 > 0$ such that if $\lambda_n^{q-1} < \gamma_0 \theta_n,\ \forall n \geq 1$, the sequence $\{\text{x}_n\}$ converges strongly to a fixed point of T, where the sequences $\{\lambda_n\}$, $\{\theta_n\}$ satisfy certain conditions.

Following the work of Chidume and Shahzad [9] and that of Zhou and Chang [19], Song [17] constructed the following implicit scheme in the setting of uniformly convex Banach spaces.

Theorem 1.4 (see [17])**.** Suppose K is a nonempty closed convex subset of a uniformly convex Banach space X and $T_i : K \to K,\ i = 1,2,3,...,N$, are Lipschitz pseudo-contractive mappings with Lipschitz constant $L \geq 0$ such that $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. Let $\{\alpha_n\}$ be a real sequence satisfying the conditions $\alpha_n \in (0,b] \subseteq (0,1)$ for some $b \in (0,1)$. Let $\{x_n\}$ be defined by

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n) T_n x_n,\ n \geq 1,\ T_n = T_{n \bmod N}.$$

Then, $\lim_{n\to\infty} \|x_n - T_i x_n\| = 0,\ \forall\ i \in\ I = \{1,2,3,...,N\}$. In addition, the sequence $\{x_n\}$ converges strongly to a common fixed point of the mappings $\{T_i\}$ if and only if $\{x_n\}$ has a strongly convergent subsequence.

It is our purpose in this paper to obtain a multivalued version of Theorem 1.4 and use it to prove the convergence of an explicit scheme to a fixed point of a multivalued Lipschitz pseudo-contracive mapping in uniformly convex Banach spaces.

2. Preliminaries

A subset A of E is called proximinal if for each $x \in E$ there exists $u \in A$ such that

$$||x-u|| = \text{dist}(x, A).$$

Examples of proximinal sets are nonempty, closed and convex subsets of a Hilbert space. We denote by $\mathscr{P}(A)$ the family of all proximinal and bounded subsets of A.

Lemma 2.1 ([13])**.** Let D be a nonempty open convex subset of a real Banach space E and $T : \overline{D} \to \mathscr{CB}(E)$ be a continuous (relative to the Hausdorff metric) pseudo-contractive mapping satisfying weakly inward condition and let $u \in \overline{D}$ be fixed. Then for each $t \in (0,1)$ there exists $y_t \in \overline{D}$ satisfying $y_t \in tTy_t + (1-t)u$.

Lemma 2.2. (See, e.g., [12]) Let J be the normalized duality mapping on E. Then, for any $x, y \in E$, the following inequality holds:

$$||x+y||^2 \leq ||x||^2 + 2\langle y, j(x+y)\rangle, \ \forall \ j(x+y) \in \ J(x+y).$$

We shall also need the following lemma.

Lemma 2.3. ([11]) Let $\{\alpha_n\}$, $\{\lambda_n\}$ and $\{\gamma_n\}$ be sequences of nonnegative numbers satisfying the conditions: $\lim_{n\to\infty} \alpha_n = 0$, $\lim_{n\to\infty} \frac{\gamma_n}{\alpha_n} = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Suppose the recursive inequality

$$\lambda_{n+1}^2 \leq \lambda_n^2 - \alpha_n \psi(\lambda_{n+1}) + \gamma_n, \ n = 1,2,3,...$$

is satisfied, where $\psi : [0,\infty) \to [0,\infty)$ is a strictly increasing function such that it is positive on $(0,\infty)$ and $\psi(0) = 0$. Then $\lambda_n \to 0$ as $n \to \infty$.

Lemma 2.4. See, e.g., [18] Let $\{a_n\}$ be a sequence of real numbers satisfying the condition that for some positive integer N, $a_{kN+j} \to p_j$ as $k \to \infty$, $j = 0,1,2,\cdots,N-1$. Then $\{a_n\}$ converges if and only if $a_{n+1} - a_n \to 0$ as $n \to \infty$.

3. Main Results

We first use Lemma 2.1 (due to Ofoedu and Zegeye) to obtain the following lemma which is a multi-valued analogue of Lemma 2.1(i) of [4] which will be used in the main results.

Lemma 3.1. Let E be a Banach space and $D \subseteq E$ be nonempty open and convex. Let $T_i : \overline{D} \to \mathscr{CB}(\overline{D})$, $i = 1,2,3,...,N$, be continuous pseudo-contractive mappings such that $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. Let $\{t_n\}_{n=1}^{\infty} \subseteq (0,1)$ and $\{y_n\}_{n=1}^{\infty}$ be a sequence defined by

$$\begin{cases} y_0 \in \overline{D}, \\ y_n = (1-t_n)y_{n-1} + t_n z_n, \ z_n \in \ T_n y_n, \ n \geq 1, \ T_n = T_{n \ mod \ N}. \end{cases}$$

Then, $\lim_{n\to\infty} \|y_n - p\|$ exists for each $p \in F$.

Proof. Using Lemma 2.1 above, given y_{n-1}, taking $t = t_n$, $u = y_{n-1}$, and $T = T_n$, we obtain $y_n = (1-t_n)y_{n-1} + t_n z_n$ for some $z_n \in T_n y_n$. Let $p \in F$. Then, using the definition of y_n above, Lemma 2.1 and definition of the duality mapping, we obtain the following inequality

$$\begin{aligned} \|y_n - p\|^2 &= \langle t_n z_n + (1-t_n)y_{n-1} - p, j(y_n - p)\rangle \\ &= (1-t_n)\langle y_{n-1} - p, j(y_n - p)\rangle + t_n\langle z_n - p, j(y_n - p)\rangle \\ &\leq (1-t_n)\|y_{n-1} - p\|\|y_n - p\| + t_n\|y_n - p\|^2. \end{aligned}$$

It follows that, for all $p \in F$,

$$\|y_n - p\| \leq \|y_{n-1} - p\| \ \forall n \geq 1.$$

Thus,

$$\lim_{n\to\infty} \|y_n - p\| \text{ exists for each } p \in F.$$

□

Theorem 3.2. Let E be a uniformly convex Banach space and $D \subseteq E$ be nonempty, open and convex. Let $T_i : \overline{D} \to \mathscr{P}(\overline{D})$, $i = 1,2,3,...,N$, be multi-valued Lipschitz pseudo-contractive mappings such that $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. Let $\{y_n\}$ be a sequence defined by

$$\begin{cases} y_1 \in \overline{D} \\ y_n = (1-t_n)y_{n-1} + t_n z_n, \ z_n \in T_n y_n, \ n \geq \ 2, \ T_n = T_{n \ mod \ N}, \end{cases} \tag{3.1}$$

where $\{t_n\}_{n=1}^{\infty} \subseteq (0,1)$ such that $t_n \to 1$ as $n \to \infty$. Suppose that $T_i(p) = \{p\}, \ \forall \ p \in \ F$. Then, $\lim_{n\to\infty} \text{dist}(y_n, T_l y_n) = 0, \forall \ l \ \in \ \{1,2,3,...,N\}$.

Proof. Let $p \in F$. By Lemma 3.1, $\lim_{n\to\infty} \|y_n - p\|$ exists. This implies that $\{y_n\}$ is bounded and since $t_n \to 1$, it follows that $\{z_n\}$ is bounded. Also for $y \in T_l y_n$,

$$\begin{aligned}\|y_n - y\| &\le \|y_n - p\| + \|p - y\| \\ &= \|y_n - p\| + \operatorname{dist}(y, T_l p) \\ &\le \|y_n - p\| + \max_{w \in T_l y_n} \operatorname{dist}(w, T_l p) \\ &\le \|y_n - p\| + h(T_l y_n, T_l p) \\ &\le (1+L)\|y_n - p\|.\end{aligned}$$

Thus, $\inf_{y\in T_l y_n} \|y_n - y\| \le (1+L)\|y_n - p\|$ for all $l \in \{1,2,3,\dots,N\}$, where $L := \max\{L_i : i = 1,2,3,\dots,N\}$ and L_i is a Lipschitz constant of $T_i,\ i = 1,2,3,\dots,N$. Hence, for all $l \in \{1,2,3,\dots,N\}$,

$$\operatorname{dist}(y_n, T_l y_n) \le (1+L)\|y_n - p\| \ \forall\, n \in \mathbb{N}.$$

Therefore, if $\|y_n - p\| \to 0$ as $n \to \infty$, the conclusion holds. We now assume that $\|y_n - p\| \nrightarrow 0$ as $n \to \infty$. Then, $\|y_n - p\| \to \sigma > 0$ as $n \to \infty$. We first show that $\lim_{n\to\infty} \|y_n - y_{n-1}\| = 0$. For contradiction, suppose $\|y_n - y_{n-1}\| \nrightarrow 0$ as $n \to \infty$. Then, for any $n_0 \in \mathbb{N}$ fixed, we have

$$\left\| \frac{y_n - y_{n-1}}{\|y_{n_0} - p\|} \right\| \nrightarrow 0 \text{ as } n \to \infty.$$

Hence, there exist $\varepsilon_0 \in (0,2]$, a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that

$$\left\| \frac{y_{n_k} - y_{n_k-1}}{\|y_{n_0} - p\|} \right\| = \left\| \frac{(y_{n_k} - p)}{\|y_{n_0} - p\|} - \frac{(y_{n_k-1} - p)}{\|y_{n_0} - p\|} \right\| \ge \varepsilon_0 \text{ for all } k.$$

By uniform convexity of E, there exists $\delta_{\varepsilon_0} > 0$ such that

$$\|(y_{n_k} - p) + (y_{n_k-1} - p)\| \le 2(1 - \delta_{\varepsilon_0})\|y_{n_0} - p\| \ \forall k \text{ large enough.} \tag{3.2}$$

We note that as $\{\|y_n - p\|\}$ is monotone decreasing, $\|y_{n_k} - p\| \le \|y_{n_0} - p\|$ for all k large enough. It then follows that $\left\| \frac{y_{n_k} - p}{\|y_{n_0} - p\|} \right\| \le 1$ for all k large enough. Now, using inequality (3.2) we have

$$\begin{aligned}\langle (y_{n_k} - p) + (y_{n_k-1} - p), j(y_{n_k} - p)\rangle &\le \|(y_{n_k} - p) + (y_{n_k-1} - p)\|\|y_{n_k} - p\| \\ &\le 2(1 - \delta_{\varepsilon_0})\|y_{n_0} - p\|\|y_{n_k} - p\|\end{aligned}$$

which implies that

$$\langle y_{n_k-1} - p, j(y_{n_k} - p)\rangle \le 2(1 - \delta_{\varepsilon_0})\|y_{n_0} - p\|\|y_{n_k} - p\| - \|y_{n_k} - p\|^2.$$

Using the last inequality and pseudo-contractiveness of the mappings we get

$$\begin{aligned}\|y_{n_k} - p\|^2 &= \langle y_{n_k} - p, j(y_{n_k} - p)\rangle \\ &= \langle (1 - t_{n_k})y_{n_k-1} + t_{n_k} z_{n_k} - p, j(y_{n_k} - p)\rangle \\ &= (1 - t_{n_k})\langle y_{n_k-1} - p, j(y_{n_k} - p)\rangle + t_{n_k}\langle z_{n_k} - p, j(y_{n_k} - p)\rangle \\ &\le 2(1 - t_{n_k})(1 - \delta_{\varepsilon_0})\|y_{n_0} - p\|\|y_{n_k} - p\| - (1 - t_{n_k})\|y_{n_k} - p\|^2 \\ &\quad + t_{n_k}\|y_{n_k} - p\|^2\end{aligned}$$

from which we obtain the following inequality

$$\begin{aligned}2(1 - t_{n_k})\|y_{n_k} - p\|^2 &\le 2(1 - t_{n_k})(1 - \delta_{\varepsilon_0})\|y_{n_0} - p\|\|y_{n_k} - p\| - t_{n_k}\|y_{n_k} - p\|^2 \\ &\le 2(1 - t_{n_k})(1 - \delta_{\varepsilon_0})\|y_{n_0} - p\|\|y_{n_k} - p\|.\end{aligned}$$

Therefore, $\|y_{n_k} - p\| \le (1 - \delta_{\varepsilon_0})\|y_{n_0} - p\|$ for k large enough. Taking limit as $k \to \infty$, we get $\sigma \le (1 - \delta_{\varepsilon_0})\|y_{n_0} - p\|$. Since n_0 was arbitrary, we have

$$\sigma \le (1 - \delta_{\varepsilon_0}) \inf_{n\in\mathbb{N}} \|y_n - p\| = (1 - \delta_{\varepsilon_0}) \lim_{n\to\infty} \|y_n - p\| = (1 - \delta_{\varepsilon_0})\sigma < \sigma,$$

a contradiction. Therefore, $\lim_{n\to\infty}\|y_n-y_{n-1}\|=0$. Hence,

$$\begin{aligned}\|y_{n+i}-y_n\| &\leq \|y_{n+i}-y_{n+i-1}+y_{n+i-1}-+\ldots-y_{n+1}+y_{n+1}-y_n\|\\ &\leq \|y_{n+i}-y_{n+i-1}\|+\|y_{n+i-1}-y_{n+i-2}\|+\ldots+\|y_{n+1}-y_n\|\to 0\end{aligned}$$

as $n\to\infty$ for all $i\in\{1,2,3,...,N\}$. Thus,

$$\lim_{n\to\infty}\|y_{n+i}-y_n\|=0,\ \forall\ i\in\{1,2,3,...,N\}.$$

Using the recursive formula (3.1), $\lim_{n\to\infty}\|y_n-y_{n-1}\|=0$ and the fact that $t_n\to 1$, we obtain

$$\lim_{n\to\infty}\|y_n-z_n\|=0.$$

Therefore, for $z_n^i\in T_{n+i}y_n$ such that $\mathrm{dist}(z_{n+i},T_{n+i}y_n)=\|z_n^i-z_{n+i}\|$ (z_n^i exists since $T_jx\in\mathscr{P}(\overline{D})$ for all j), we obtain

$$\begin{aligned}\|y_n-z_n^i\| &\leq \|y_n-y_{n+i}\|+\|y_{n+i}-z_{n+i}\|+\|z_{n+i}-z_n^i\|\\ &=\|y_n-y_{n+i}\|+\|y_{n+i}-z_{n+i}\|+\mathrm{dist}(z_{n+i},T_{n+i}y_n)\\ &\leq \|y_n-y_{n+i}\|+\|y_{n+i}-z_{n+i}\|+\sup_{v\in T_{n+i}y_{n+i}}\mathrm{dist}(v,T_{n+i}y_n)\\ &\leq \|y_n-y_{n+i}\|+\|y_{n+i}-z_{n+i}\|+h(T_{n+i}y_n,T_{n+i}y_{n+i})\\ &\leq (1+L)\|y_n-y_{n+i}\|+\|y_{n+i}-z_{n+i}\|\to 0 \text{ as } n\to\infty.\end{aligned}$$

Thus,

$$\lim_{n\to\infty}\|y_n-z_n^i\|=0,\ \forall\ i\in\{1,2,3,...,N\}. \tag{3.3}$$

For any $l\in\{1,2,3,...,N\}$ fixed, setting $n_k=kN, k=1,2,3,\cdots$, it follows that

$$z_{n_k}^l\in T_{n_k+l}y_{n_k}=T_{kN+l}y_{n_k}=T_ly_{n_k}.$$

Using (3.3),

$$\mathrm{dist}(y_{n_k},T_ly_{n_k})\to 0 \text{ as } k\to\infty\ \forall\ l\in\{1,2,3,...,N\}.$$

For $j\in\mathbb{N}\cup\{0\}$ fixed, let $u\in T_ly_{n_k+j}$ such that $\|u-z_{n_k}^l\|=\mathrm{dist}(z_{n_k}^l,T_ly_{n_k+j})$. Then,

$$\begin{aligned}\Big|\|y_{n_k+j}-u\|-\|y_{n_k}-z_{n_k}^l\|\Big| &\leq \|y_{n_k+j}-y_{n_k}\|+\|u-z_{n_k}^l\|\\ &=\|y_{n_k+j}-y_{n_k}\|+\mathrm{dist}(z_{n_k}^l,T_ly_{n_k+j})\\ &\leq \|y_{n_k+j}-y_{n_k}\|+h(T_ly_{n_k},T_ly_{n_k+j})\\ &\leq (1+L)\|y_{n_k+j}-y_{n_k}\|\to 0 \text{ as } k\to\infty.\end{aligned}$$

Therefore,

$$\lim_{k\to\infty}\left(\|y_{n_k+j}-u\|-\|y_{n_k}-z_{n_k}^l\|\right)=0.$$

Since $\lim_{k\to\infty}\|y_{n_k}-z_{n_k}^l\|=0$, we must have $\lim_{k\to\infty}\|y_{n_k+j}-u\|=0$. Thus,

$$\lim_{k\to\infty}\mathrm{dist}(y_{n_k+j},T_ly_{n_k+j})=0,\ j\in\mathbb{N}\cup\{0\}. \tag{3.4}$$

We note that for any $w\in T_ly_{n+1}, v\in T_ly_n$,

$$\|y_{n+1}-w\|\leq\|y_{n+1}-y_n\|+\|y_n-v\|+\|v-w\|$$

so that taking infimum over v we obtain

$$\|y_{n+1}-w\|\leq\|y_{n+1}-y_n\|+\mathrm{dist}(y_n,T_ly_n)+\mathrm{dist}(w,T_ly_n)$$

from which we conclude that

$$\|y_{n+1}-w\|\leq\|y_{n+1}-y_n\|+\mathrm{dist}(y_n,T_ly_n)+h(T_ly_{n+1},T_ly_n).$$

Taking infimum over w gives

$$\mathrm{dist}(y_{n+1},T_ly_{n+1})-\mathrm{dist}(y_n,T_ly_n)\leq(1+L)\|y_{n+1}-y_n\|.$$

Similar arguments yield

$$\text{dist}(y_n, T_l y_n) - \text{dist}(y_{n+1}, T_l y_{n+1}) \leq (1+L)\|y_{n+1} - y_n\|.$$

Therefore,

$$|\text{dist}(y_{n+1}, T_l y_{n+1}) - \text{dist}(y_n, T_l y_n)| \leq (1+L)\|y_{n+1} - y_n\| \to 0 \text{ as } n \to \infty \tag{3.5}$$

By virtue of (3.4) and (3.5), Lemma 2.4 gives $\lim_{n\to\infty} \text{dist}(y_n, T_l y_n) = 0,\ \forall\, l \in \{1,2,3,...,N\}$. □

A mapping $T : D \to \mathscr{CB}(D)$ is called *hemicompact* if for any sequence $\{x_n\}$ in D such that $d(x_n, Tx_n) \to 0$ as $n \to \infty$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to p \in D$.

Remark 3.3. If D is compact, every multi-valued mapping $T : D \to \mathscr{CB}(D)$ is hemicompact.

Corollary 3.4. Let E, D, T_i, $\{y_n\}$ and $\{t_n\}$ be as in Theorem 3.2. If T_{i_0} is hemicompact for some $i_0 \in \{1,2,3,...,N\}$, then $\{y_n\}$ converges strongly to a common fixed point of the mappings $\{T_i\}_{i=1}^N$.

Proof. From Theorem 3.2, $dist(y_n, T_{i_0} y_n) \to 0$ as $n \to \infty$. Hemicompactness of T_{i_0} guarantees existence of a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $y_{n_k} \to q \in \overline{D}$ as $k \to \infty$. Continuity of T_i, $i = 1,2,3,...,N$ (with respect to h) implies that $T_i y_{n_k} \to T_i q$ as $k \to \infty\ \forall i \in \{1,2,3,...,N\}$. Now,

$$\text{dist}(q, T_i q) \leq \|q - y_{n_k}\| + \text{dist}(y_{n_k}, T_i y_{n_k}) + h(T_i y_{n_k}, T_i q) \to 0 \text{ as } k \to \infty\ \forall\, i.$$

Consequently, $\text{dist}(q, T_i q) = 0$ for each $i \in \{1,2,3,...,N\}$ and this gives $q \in T_i q$ (proximinal sets are closed) for each $i \in \{1,2,3,...,N\}$. Thus, $q \in F$. By Lemma 3.1, $\lim_{n\to\infty} \|y_n - q\|$ exists. Since $\lim_{k\to\infty} \|y_{n_k} - q\| = 0$, it follows that $\lim_{n\to\infty} \|y_n - q\| = 0$. □

Corollary 3.5. Let E, D, T_i, $\{y_n\}$ and $\{t_n\}$ be as in Theorem 3.2. If D is relatively compact, then $\{y_n\}$ converges strongly to a common fixed point of the mappings $\{T_i\}_{i=1}^N$.

Proof. Since $\overline{D}$ is compact, each T_i is hemicompact and Corollary 3.4 applies. □

3.1. **Explicit Algorithm.** In this section, we shall use the implicit scheme (3.1) of Theorem 3.2 to prove strong convergence theorem for an explicit scheme for a multivalued Lipschitz pseudo-contractive mapping. Our theorem below, Theorem 3.6, gives an extension of the main result (Theorem 3.1) of [8] for a singlevalued map, due to Chidume and Zegeye, from the setting of real Banach spaces having uniformly Gâteaux differentiable norms to that of uniformly convex real Banach spaces and for a multivalued map.

Theorem 3.6. Let D be a nonempty, open and convex subset of a uniformly convex Banach space E and $T : \overline{D} \to \mathscr{P}(\overline{D})$ be a multi-valued Lipschitz pseudo-contractive mapping with Lipschitz constant $L \geq 0$ such that $F = F(T) \neq \emptyset$ and $T(p) = \{p\}\ \forall p \in F$. Let $\{\lambda_n\}$ and $\{\theta_n\}$ be real sequences in $(0,1]$ such that
(i) $\lim_{n\to\infty} \theta_n = 0$, (ii) $\lambda_n(1+\theta_n) \leq 1$, $\sum \lambda_n \theta_n = \infty$, $\lim\limits_{n\to\infty} \dfrac{\lambda_n}{\theta_n} = 0$ and
$(iii) \lim\limits_{n\to\infty} \dfrac{(\frac{\theta_{n-1}}{\theta_n} - 1)}{\lambda_n \theta_n} = 0$. Let $x^* \in F$ and $\{x_n\}$ be a sequence generated from arbitrary $x_1 \in \overline{D}$ by

$$x_{n+1} = (1-\lambda_n)x_n + \lambda_n w_n - \lambda_n \theta_n (x_n - x_1),\ w_n \in Tx_n\ \forall\, n \geq 1, \tag{3.6}$$

where $\|w_n - w_{n-1}\| = \text{dist}(w_{n-1}, Tx_n), n \geq 2$.Then, $\lim_{n\to\infty} \text{dist}(x_n, Tx_n) = 0$.

Proof. Since $\frac{\lambda_n}{\theta_n} \to 0$ as $n \to \infty$, we can find an $N_0 \in \mathbb{N}$ such that $\lambda_n \leq d\theta_n\ \forall n \geq N_0$, where $d := \frac{1}{2(2+L)(\frac{5}{2}+L)}$. Let $x^* \in F(T)$ and let $r > 0$ be sufficiently large such that $x_{N_0} \in B_r(x^*)$ and $x_1 \in B_{\frac{r}{2}}(x^*)$.

Step 1. Here, we prove that the sequence $\{x_n\}$ is bounded. To show this, it suffices to show that $x_n \in B := \overline{B_r}(x^*)$, $\forall n \geq N_0$. We now proceed by induction. The choice of r guarantees that $x_{N_0} \in B := \overline{B_r}(x^*)$.

We now assume that $x_n \in B$ for some $n \geq N_0$. Suppose, for the sake of contradiction, that $x_{n+1} \notin B$. Using the recursive formula (3.6) and Lemma 2.2 we obtain

$$\begin{aligned} ||x_{n+1}-x^*||^2 &= ||(x_n-x^*)-\lambda_n((x_n-w_n)+\theta_n(x_n-x_1))||^2 \\ &\leq ||x_n-x^*||^2-2\lambda_n\langle(x_n-w_n)+\theta_n(x_n-x_1),j(x_{n+1}-x^*)\rangle \\ &= ||x_n-x^*||^2-2\lambda_n\theta_n||x_{n+1}-x^*||^2 \\ &\quad +2\lambda_n\langle\theta_n(x_{n+1}-x_n)-(x_n-w_n)+\theta_n(x_1-x^*) \\ &\quad +(x_{n+1}-w_{n+1})-(x_{n+1}-w_{n+1}),j(x_{n+1}-x^*)\rangle. \end{aligned} \tag{3.7}$$

Since T is pseudo-contractive, we have $\langle x_{n+1}-w_{n+1}, j(x_{n+1}-x^*)\rangle \geq 0 \ \forall\ n \geq 1$. Thus, from (3.7), we get the following estimates

$$\begin{aligned} ||x_{n+1}-x^*||^2 \leq\ & ||x_n-x^*||^2-2\lambda_n\theta_n||x_{n+1}-x^*||^2 \\ & +2\lambda_n\langle\theta_n(x_{n+1}-x_n)+\theta_n(x_1-x^*)+(x_{n+1}-x_n) \\ & +(w_n-w_{n+1}),j(x_{n+1}-x^*)\rangle \end{aligned} \tag{3.8}$$

$$\begin{aligned} \leq\ & ||x_n-x^*||^2-2\lambda_n\theta_n||x_{n+1}-x^*||^2+2\lambda_n\,[2||x_{n+1}-x_n|| \\ & +\mathrm{dist}(w_n,Tx_{n+1})+\theta_n||x_1-x^*||]\,||x_{n+1}-x^*|| \\ \leq\ & ||x_n-x^*||^2-2\lambda_n\theta_n||x_{n+1}-x^*||^2+2\lambda_n\,[2||x_{n+1}-x_n|| \\ & +h(Tx_n,Tx_{n+1})+\theta_n||x_1-x^*||]\,||x_{n+1}-x^*|| \\ \leq\ & ||x_n-x^*||^2-2\lambda_n\theta_n||x_{n+1}-x^*||^2+2\lambda_n\,[\theta_n||x_1-x^*|| \\ & +(2+L)||(x_{n+1}-x_n||\,]\,||x_{n+1}-x^*|| \\ \leq\ & ||x_n-x^*||^2-2\lambda_n\theta_n||x_{n+1}-x^*||^2+2\lambda_n\,[\theta_n||x_1-x^*|| \\ & +(2+L)\lambda_n||(x_n-x^*+x^*-w_n)+\theta_n(x_n-x^*+x^*-x_1)||] \\ & \times||x_{n+1}-x^*|| \\ \leq\ & ||x_n-x^*||^2-2\lambda_n\theta_n||x_{n+1}-x^*||^2+2\lambda_n\,[\theta_n||x_1-x^*|| \\ & +(2+L)\lambda_n((2+L)||(x_n-x^*||+\theta_n||x_1-x^*||)\,]||x_{n+1}-x^*|| \\ \leq\ & ||x_n-x^*||^2-2\lambda_n\theta_n||x_{n+1}-x^*||^2 \\ & +2\lambda_n\left[\lambda_n(2+L)\left(\frac{5}{2}+L\right)r+\frac{\theta_n}{2}r\right]||x_{n+1}-x^*|| \end{aligned} \tag{3.9}$$

since $x_n \in B$, $x_1 \in B_{\frac{r}{2}}(x^*)$ and $\theta_n \leq 1$. We now have from (3.9) that

$$\begin{aligned} 2\lambda_n\theta_n||x_{n+1}-x^*||^2 &\leq ||x_n-x^*||^2-||x_{n+1}-x^*||^2 \\ &\quad +2\lambda_n\left[\lambda_n(2+L)\left(\frac{5}{2}+L\right)r+\frac{\theta_n}{2}r\right]||x_{n+1}-x^*||. \end{aligned}$$

Thus, since $||x_{n+1}-x^*|| > r \geq ||x_n-x^*||$, we get

$$\theta_n||x_{n+1}-x^*|| \leq \left[\lambda_n(2+L)\left(\frac{5}{2}+L\right)r+\frac{\theta_n}{2}r\right].$$

So, it follows from the last inequality that

$$\begin{aligned} ||x_{n+1}-x^*|| &\leq \left[\frac{\lambda_n}{\theta_n}(2+L)\left(\frac{5}{2}+L\right)r+\frac{r}{2}\right] \\ &\leq \left[\frac{1}{2(2+L)(\frac{5}{2}+L)}\left(2+L\right)\left(\frac{5}{2}+L\right)r+\frac{r}{2}\right]=r \end{aligned}$$

since

$$\frac{\lambda_n}{\theta_n} \leq \frac{1}{2(2+L)(\frac{5}{2}+L)},\ \forall\ n \geq N_0.$$

This is a contradiction since $x_{n+1} \notin B$. Thus, $x_n \in B$ for all positive integers $n \geq N_0$. The sequence $\{x_n\}$ is therefore bounded.

Step 2. We prove that $||x_n - y_n|| \to 0$ as $n \to \infty$, where $\{y_n\}$ is the sequence obtained from Lemma 3.1 with $t_n = 1 - \theta_n$ and $y_0 = x_1$. Using the recursive formula (3.6) and Lemma 2.2, we have

$$\begin{aligned}
||x_{n+1} - y_n||^2 &= ||(x_n - y_n) - \lambda_n((x_n - w_n) + \theta_n(x_n - x_1))||^2 \\
&\leq ||x_n - y_n||^2 - 2\lambda_n\langle (x_n - w_n) + \theta_n(x_n - x_1), j(x_{n+1} - y_n)\rangle \\
&= ||x_n - y_n||^2 - 2\lambda_n\theta_n\langle (x_{n+1} - y_n), j(x_{n+1} - y_n)\rangle \\
&\quad +2\lambda_n\langle \theta_n(x_{n+1} - y_n) - (x_n - w_n) - \theta_n(x_n - x_1), j(x_{n+1} - y_n)\rangle \\
&= ||x_n - y_n||^2 - 2\lambda_n\theta_n||x_{n+1} - y_n||^2 + 2\lambda_n\langle \theta_n(x_{n+1} - x_n) \\
&\quad +[\theta_n(x_1 - y_n) - (y_n - z_n)] - [(x_{n+1} - w_{n+1}) - (y_n - z_n)] \\
&\quad +[(x_{n+1} - w_{n+1}) - (x_n - w_n)], j(x_{n+1} - y_n)\rangle,\ w_{n+1} \in Tx_{n+1}. \qquad (3.10)
\end{aligned}$$

From the pseudo-contractiveness of T, we have for some $j(x_{n+1} - y_n) \in J(x_{n+1} - y_n)$ that, $\langle (x_{n+1} - w_{n+1}) - (y_n - z_n), j(x_{n+1} - y_n)\rangle \geq 0\ \forall n \geq 1$.
Also, using the definition of y_n we obtain

$$\theta_n(x_1 - y_n) - (y_n - z_n) = 0,\ z_n \in Ty_n.$$

Therefore, from (3.10), we get

$$\begin{aligned}
||x_{n+1} - y_n||^2 &\leq ||x_n - y_n||^2 - 2\lambda_n\theta_n||x_{n+1} - y_n||^2 + 2\lambda_n\langle \theta_n(x_{n+1} - x_n) \\
&\quad +[(x_{n+1} - w_{n+1}) - (x_n - w_n)], j(x_{n+1} - y_n)\rangle \\
&\leq ||x_n - y_n||^2 - 2\lambda_n\theta_n||x_{n+1} - y_n||^2 + 2\lambda_n \mathrm{dist}(w_n, Tx_{n+1}) \\
&\quad \times ||x_{n+1} - y_n|| + 4\lambda_n^2||x_n - w_n + \theta_n(x_n - x_1)||||x_{n+1} - y_n|| \\
&\leq ||x_n - y_n||^2 - 2\lambda_n\theta_n||x_{n+1} - y_n||^2 + 2\lambda_n h(Tx_n, Tx_{n+1}) \\
&\quad \times ||x_{n+1} - y_n|| + 4\lambda_n^2||x_n - w_n + \theta_n(x_n - x_1)||||x_{n+1} - y_n|| \\
&\leq ||x_n - y_n||^2 - 2\lambda_n\theta_n||x_{n+1} - y_n||^2 \\
&\quad +2\lambda_n(2+L)||x_{n+1} - x_n||||x_{n+1} - y_n|| \\
&\leq ||x_n - y_n||^2 - 2\lambda_n\theta_n||x_{n+1} - y_n||^2 \\
&\quad +2\lambda_n^2(2+L)||x_n - w_n + \theta_n(x_n - x_1)||||x_{n+1} - y_n||. \qquad (3.11)
\end{aligned}$$

Since x_n and $\{y_n\}$ are bounded, a real number $M > 0$ exists such that from (3.11),

$$||x_{n+1} - y_n||^2 \leq ||x_n - y_n||^2 - 2\lambda_n\theta_n||x_{n+1} - y_n||^2 + 2\lambda_n^2(2+L)M. \qquad (3.12)$$

Using pseudo-contractiveness of T again, we have

$$\begin{aligned}
||y_{n-1} - y_n|| &\leq ||y_{n-1} - y_n + \frac{1}{\theta_n}((y_{n-1} - z_{n-1}) - (y_n - z_n))|| \\
&= ||y_{n-1} - y_n + \left(\frac{\theta_{n-1}}{\theta_n}x_1 - \frac{\theta_{n-1}}{\theta_n}y_{n-1}\right) - (x_1 - y_n)|| \\
&\leq \left(\frac{\theta_{n-1}}{\theta_n} - 1\right)(||x_1|| + ||y_{n-1}||) \\
&\leq \left(\frac{\theta_{n-1}}{\theta_n} - 1\right)M_1,\ \text{for some } M_1 > 0. \qquad (3.13)
\end{aligned}$$

From (3.12) and (3.13), we get

$$\begin{aligned}
||x_{n+1} - y_n||^2 &\leq ||x_n - y_{n-1}||^2 - 2\lambda_n\theta_n||x_{n+1} - y_n||^2 + M_1\left(\frac{\theta_{n-1}}{\theta_n} - 1\right) \\
&\quad +2\lambda_n^2(2+L)M.
\end{aligned}$$

The conditions on $\{\lambda_n\}$ and $\{\theta_n\}$, the foregoing inequality and Lemma 2.3 imply $x_{n+1} - y_n \to 0$ as $n \to \infty$. Thus, from the Lipschitz continuity of T and the fact that $\lambda_n \to 0$, it follows that $x_n - y_n \to 0$ as $n \to \infty$.

Step 3. We prove that $\text{dist}(x_n, Tx_n) \to 0$ as $n \to \infty$. Since $\text{dist}(y_n, Ty_n) \to 0$ as $n \to \infty$ from Theorem 3.2, we have

$$\begin{aligned}\text{dist}(x_n, Tx_n) &\leq ||x_n - y_n|| + \text{dist}(y_n, Ty_n) + h(Ty_n, Tx_n) \\ &\leq ||x_n - y_n|| + \text{dist}(y_n, Ty_n) + L||x_n - y_n|| \\ &= (1+L)||x_n - y_n|| + \text{dist}(y_n, Ty_n) \to 0 \text{ as } n \to \infty.\end{aligned}$$

□

Remark 3.7. Examples of λ_n and θ_n satisfying the conditions (*i*), (*ii*) and (*iii*) of Theorem 3.6 are: $\lambda_n := \frac{1}{(n+1)^a}, \theta_n := \frac{1}{(n+1)^b}$, with $0 < b < a$ and $a + b < 1$, see, for example, [8].

Corollary 3.8. Let D, E, T, $\{x_n\}$, $\{\lambda_n\}$ and $\{\theta_n\}$ be as in Theorem 3.6. Suppose T is hemicompact, then $\{x_n\}$ converges strongly to a fixed point of T.

Proof. From Theorem 3.6, we get $\text{dist}(x_n, Tx_n) \to 0$ as $n \to \infty$. Since T is hemicompact, there exists a subsequence $\{x_{n_k}\}$ of the sequence $\{x_n\}$ such that $x_{n_k} \to x^* \in \overline{D}$ *as* $k \to \infty$. Now, $\text{dist}(x^*, Tx^*) \leq ||x^* - x_{n_k}|| + \text{dist}(x_{n_k}, Tx_{n_k}) + h(Tx_{n_k}, Tx^*) \to 0$ as $k \to \infty$. This implies that $\text{dist}(x^*, Tx^*) = 0$ and this gives $x^* \in \overline{Tx^*} = Tx^*$ which in turn implies that $x^* \in F$. From **Step 2** of the proof of Theorem 3.6, $y_{n_k} \to x^*$ as $k \to \infty$. Since $\{\|y_n - x^*\|\}$ has a limit by Lemma 3.1, it follows that $y_n \to x^*$ as $n \to \infty$. Consequently,

$$||x_n - x^*|| \leq ||x_n - y_n|| + ||y_n - x^*|| \to 0 \text{ as } n \to \infty$$

which shows that $\{x_n\}$ converges strongly to x^*. □

Corollary 3.9. Let D, E, T, $\{x_n\}$, $\{\lambda_n\}$ and $\{\theta_n\}$ be as in Theorem 3.6. If, in addition, D is relatively compact, then $\{x_n\}$ converges strongly to a fixed point of T.

Proof. Compactness of $\overline{D}$ makes T hemicompact and Corollary 3.8 applies. □

A mapping $T : D \to \mathscr{CB}(D)$ is said to satisfy *condition (I)* if there exists a strictly increasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(x) > 0$ for all $x \in (0,\infty)$ such that $d(x, T(x)) \geq f(d(x, F(T))\ \forall x \in \mathscr{D}(T)$.

Corollary 3.10. Let D, E, T, $\{x_n\}$, $\{\lambda_n\}$ and $\{\theta_n\}$ be as in Theorem 3.6, with $N = 1$ and $T_1 = T$. If T satisfies condition (I), then $\{x_n\}$ converges strongly to a fixed point of T.

Proof. Let $\{y_n\}$ be the sequence obtained from Lemma 3.1 with $t_n := 1 - \theta_n$ and $y_0 = x_1$. From Theorem 3.6, we have $\lim_{n\to\infty} \text{dist}(y_n, Ty_n) = 0$. Since T satisfies condition (I), we have $\lim_{n\to\infty} f(\text{dist}(y_n, F)) = 0$. Hence, there exists a sequence $\{p_k\} \subseteq F$ and a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that

$$\|y_{n_k} - p_k\| < \frac{1}{2^k}\ \forall\ k.$$

Since the sequence $\{\|y_n - p\|\}$ is monotone non-increasing, we have

$$\|y_{n_{k+1}} - p_k\| \leq \|y_{n_k} - p_k\| < \frac{1}{2^k}\ \forall\ k.$$

Thus,

$$\begin{aligned}\|p_{k+1} - p_k\| &\leq \|p_{k+1} - y_{n_{k+1}}\| + \|y_{n_{k+1}} - p_k\| \\ &< \frac{1}{2^{k+1}} + \frac{1}{2^k} \\ &< \frac{1}{2^{k-1}}.\end{aligned} \tag{3.14}$$

This shows that $\{p_k\}$ is a Cauchy sequence in $\overline{D}$. Hence, $p_k \to q \in \overline{D}$. From the fact that $p_k \to q$ as $n \to \infty$ and T si Lipschitz, we get

$$\text{dist}(q, Tq) \leq \|q - p_k\| + \text{dist}(p_k, Tp_k) + h(T_i p_k, Tq) \to 0 \text{ as } k \to \infty. \tag{3.15}$$

This implies that $\text{dist}(q, Tq) = 0$. It follows that $q \in F$. Now,

$$\begin{aligned} \|y_{n_k} - q\| &\leq \|y_{n_k} - p_k\| + \|p_k - q\| \\ &< \frac{1}{2^k} + \|p_k - q\| \to 0 \text{ as } k \to \infty. \end{aligned}$$

This shows that $y_{n_k} \to q$ as $k \to \infty$. From Lemma 3.1, $\lim_{n\to\infty} \|y_n - q\|$ exists. Thus, $\lim_{k\to\infty} \|y_{n_k} - q\| = \lim_{n\to\infty} \|y_n - q\| = 0$ and this gives $y_n \to q$ as $n \to \infty$. The proof is complete by applying **Step 2** of the proof of Theorem 3.6. □

We give an example below where our algorithm is used to approximate a solution of a constrained minimization problem:

$$\begin{cases} \min f(x) \\ x \in K \subseteq H, \end{cases} \tag{3.16}$$

where $f : H \to \mathbb{R} \cup \{\infty\}$ is a convex function, H a real Hilbert space and K a convex nonempty subset of H.

Example. Let $f : H \to \mathbb{R}$ be a convex and Fréchet differentiable function with $\nabla f : H \to H$ Lipschitz. Suppose K is compact, convex and nonempty. Then starting from any $x_1 \in K$, the sequence

$$x_{n+1} = x_n - \lambda_n \nabla f(x_n) - \lambda_n \theta_n (x_n - x_1), n = 1, 2, 3, \cdots, \tag{3.17}$$

converges strongly to a solution x^* of (3.16) given that x^* is in the interior of K.

Indeed, we note that convexity of f makes ∇f accretive. Therefore the map T defined $T = I - \nabla f$ is pseudo-contractive. Since ∇f is Lipschitz, T is also Lipschitz. Moreover, since K is closed and bounded and f is continuous, then problem (3.16) has a solution x^* such that $\nabla f(x^*) = 0$. Therefore $Tx^* = x^*$, i.e., T has a fixed point. Compactness of K makes T hemicompact. Thus, by Corollary 3.8, the sequence $\{x_n\}$ defined by (3.6) converges strongly to a fixed point x^* of T which gives $\nabla f(x^*) = 0$ making x^* a minimizer of f on K. Now the sequence $\{x_n\}$ given by (3.6) with $T = I - \nabla f$ reduces to (3.17).

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REFERENCES

[1] F.E. Browder, *Nonlinear operators and nonlinear equations of evolution in Banach spaces*, Proc. of Symposia in pure Math. Vol. XVIII, part 2, 1976.

[2] F.E. Browder, *Nonlinear mappings of nonexpansive and accretive type in Banach spaces*, Bull. Amer. Math. Soc. **73** (1967), 875 - 882.

[3] F.E. Browder, *Convergence of approximants to fixed points of nonexpansive nonlinear mappings in Banach spaces*, Arch. Ration. Mech. Anal, **24** (1967), 82 - 90.

[4] R. Chen, Y. Song and H. Zhou, *Convergence theorems for implicit iteration process for a finite family of continuous pseudo-contractive mappings*, J. Math. Anal. Appl., **314** (2006), 701 - 709.

[5] C.E. Chidume, C.O. Chidume, N. Djitte and M.S. Minjibir, *Iterative algorithm for fixed points of multi-valued pseudo-contractive mappings in Banach spaces*, J. Nonlinear Convex Anal. **15** (2014), 241 - 255.

[6] C. E. Chidume, C. O. Chidume, N. Djitte and M. S. Minjibir, *Convergence theorems for fixed points of multi-valued strictly pseudo-contractive mappings in Hilbert spaces*, Abst. Appl. Anal. Volume 2013, Article ID 629468, doi: 10.1155/2013/629468.

[7] C. Chidume, *Geometric Properties of Banach spaces and Nonlinear Iterations*, Springer Verlag Series: Lecture Notes in Mathematics, Vol.1965 (2009), ISBN 978-1-84882-189-7.

[8] C. E. Chidume and H. Zegeye, *Approximate fixed point sequences and convergence theorems for Lipschitz pseudo-contractive maps*, Proc. Amer. Math. Soc. **132** (2003), no. 3, 831 - 840.

[9] C.E. Chidume and N. Shahzad, *Strong convergence of an implicit iteration process for a finite family of nonexpansive mappings*, Nonlinear Anal., **62** (2005), 1149 - 1156.

[10] T. Kato, *Nonlinear semi groups and evolution equations*, J. Math. Soc. Japan **19** (1967), 508 - 520.

[11] C. Moore and B. V. C. Nnoli, *Iterative solutions of nonlinear equations involving set-valued uniformly accretive operators*, Comput. Math. Appl. **42** (2001), 131 - 140.

[12] C.H. Morales and J.S. Jung, *Convergence of paths for pseudo-contractive mappings in Banach spaces*, Proc. Amer. Math. Soc. **128** (2000), 3411 - 3419.

[13] E. U. Ofoedu and H. Zegeye, *Iterative algorithm for multi-valued pseudo-contractive mappings in Banach spaces*, J. Math. Anal. Appl., **372** (2010), 68 - 76.

[14] P. Pietramala, *Convergence of approximating fixed points sets for multi-valued nonexpansive mappings*, Comment. Math. Univ. Carolin. , **32** (1991), 697 - 701.

[15] S. Reich, *Strong convergence theorems for resolvents of accretive operators in banach spaces*, J. Math. Anal. Appl., **75** (1980), 287 - 292.

[16] T. Roubíček, *Evolution governed by accretive mappings.* In: Nonlinear Partial Differential Equations with Applications. International Series of Numerical Mathematics, vol 153 (2013). Birkhäuser, Basel.

[17] Y.S. Song, *An iterative process for a finite family of pseudo-contractive mappings*, Acta Math. Sin, **25** (2009), 293 - 298.

[18] H.-K. Xu and R.G. Ori, *An implicit iteration process for nonexpansive mapings*, Numer. Funct. Anal. and Optimz., **22**(5&6) (2001), 767 - 773.

[19] Y. Zhou and S.S. Chang, *Convergence of implicit iteration process for a finite family of asymptotically nonexpansive mappings in Banach spaces*, Numer. Funct. Anal. Optim., **23** (2002), 911 - 921.

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