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ON THE LATTICE POINTS OF DILATIONS OF THE STANDARD 2-SIMPLEX AND THE GRASSMANNIAN Gr(2,n)

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ABSTRACT. The connection between the combinatorics of the lattice points of the dilation $r\Delta_2$ of the standard 2-simplex Δ_2 and the cohomology ring of the Grassmmannian $\mathrm{Gr}(2, r+2)$ is explored. Specifically, two important refinements of the Ehrhart polynomial $L_{\Delta_2}(r)$ are realized from this connection. One of the refinements interprets the Poincaré polynomial $\mathrm{P}(\mathrm{Gr}(2, r+2), z)$ as the number of lattice points on each of the slicing lines of $r\Delta_2$ with respect to a fixed weight.

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1. Introduction

Grassmannians are among the well studied algebraic varieties due to their rich algebraic, combinatorial and geometric structures. Their cohomology theory, though classical has become one of the central themes in modern algebraic combinatorics. It is well known that the ring of the cohomology of the Grassmannian $Gr(d, n)$ of d-subspaces in an n-dimensional complex vector space is generated by Schubert cycles σ_λ indexed by the fitted partitions λ, that is, every partition λ has at most d parts each of which cannot exceed $n-d$. The multiplication in this ring is induced by the homomorphism from the ring of symmetric functions to the cohomology ring of Grassmannian which sends Schur function s_λ to σ_λ [3, 7,8,11,13]. On the other hand, the standard d-simplex Δ_d is the convex hull of the set $\{\underline{0}, e_1, \ldots, e_d\}$ where $e_i's$, $1 \leq i \leq d$ are the

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standard vectors in ^{d} and $\underline{0}$ is the origin. That is,

$$\Delta_d := \mathrm{conv}(0, e_1, \ldots, e_d) = \{\mathbf{x} \in^d : \mathbf{x} \cdot e_i \geq 0, \ \ \sum_{i=1}^{d} \mathbf{x} \cdot e_i \leq 1\} \tag{1}$$

and the dilation $r\Delta_d$, is given by

$$r\Delta_d = \{\mathbf{x} \in^d : \mathbf{x} \cdot e_i \geq 0, \ \ \sum_{i=1}^{d} \mathbf{x} \cdot e_i \leq r, \quad r \in\} \tag{2}$$

Lattice points are the points whose coordinates are integers. Counting the lattice points on $r\Delta_d$ is equivalent to asking for the number of integer solutions for the inequality

$$\sum_{i=1}^{d} \mathbf{x} \cdot e_i \leq r \tag{3}$$

The number of lattice points on any given lattice polytope is well known. This is central theme of Ehrhart polynomials, [2, 4, 11, 12, 17]. In fact the number of the lattice points on $r\Delta_d$ is given by

$$|r\Delta_d \cap^d_{\geq 0}| = \binom{r+d}{d} \tag{4}$$

and its generating function by

$$\mathrm{P}(r\Delta_d, z) = \sum_{r=0}^{\infty} A_r z^r = \frac{1}{(1-z)^{d+1}}, \text{ where } \ A_r = \binom{r+d}{d} \tag{5}$$

The main focus of this short paper is the exploration of some remarkable connections between the lattice points of the dilations $r\Delta_2$ of the standard 2-simplex and the Grassmannians $\mathrm{Gr}(2, 2+r)$. It is well known that the multiplicative generators of the cohomology of the Grassmannian $\mathrm{Gr}(d, d+r)$ are given by the special Schubert cycles σ_λ, see [3]. These cycles are indexed by one-row partitions $\lambda = (k), 1 \leq k \leq r$ and they constitute the total Chern class of the quotient bundle $\mathcal{Q}$, that is,

$$c(\mathcal{Q}) = 1 + \sigma_{7pt(1)} + \sigma_{7pt(2)} + \cdots + \sigma_{7pt(2)\cdots 7pt(1)_{1\times r}}$$

We study the monomials identified with the semi standard tableaux of these one-diagrams for $d = 2$ and realize a natural graded polynomial $T_r(t)$ which is indeed a refinement of the Ehrhart polynomial for the dilation $r\Delta_2$ using the weight $(1, 1)$.

Theorem 1.1.

(i.) The size $L^2(r)$ of the 2-filling set $\mathcal{C}^2_{2,r}$ is $\binom{r+2}{2}$ and the sequence $(L^2(r))_{r=0}^{\infty}$ of cardinalities as r grows is recorded by the generating function

$$P(\mathcal{C}^2_{(2,r)}, z) = \frac{1}{(1-z)^3}$$

(ii.) More is true, there is a graded counting polynomial of the semi standard tableaux in $\mathcal{C}^2_{2,r}$ given by

$$T^2_r(t) = \sum_{k=0}^{r} (k+1)t^k$$

that is, a k-box row diagram gives $(k+1)$ semi standard Young tableaux. This has a generating function

$$G(t,z) = \frac{z}{(1-z)(1-tz)^2}.$$

Theorem 1.2.
There is a bijection between the set $\mathcal{C}^2_{2,r}$ and the set $r\Delta_2 \cap^2_{\geq 0}$ of the lattice points of the dilation $r\Delta_2$. Furthermore, the triangular polynomial $T_r(t)$ is precisely the slicing of the dilation $r\Delta_2$ with lines $y+x=c,\ 0 \leq c \leq r$.

It turns out that the exponent vectors of the monomials encode some information about the indexing partitions of the Schubert varieties of Grassmannian $\mathrm{Gr}(2, 2+r)$ and Grassmannian permutations. This ultimately expresses the Poincaré polynomial of the Grassmannian $\mathrm{Gr}(2, 2+r)$ as another refinement of the Ehrhart polynomial of the dilation $r\Delta_2$ using the weight $(1,2)$. The refinement is simply the slicing of $r\Delta_2$ by lines $x+2y=m,\ 0 \leq m \leq 2r$. This generalizes to higher dimensions using hyperplanes [1]

Theorem 1.3. Every β-partition λ^* identified with each of the monomials $\mathbf{t}^{\mathbf{a}} \in W^r_2$ fits into the $2 \times r$ rectangle $\square_{2\times r}$.

Corollary 1.4. The set of β-partitions λ^* identified with monomials in W^r_2 index the Schubert varieties in the Grassmannian $\mathrm{Gr}(2, 2+r)$.

Theorem 1.5. Let λ^* be the β-partition identified with the monomial $\mathbf{t}^{\mathbf{a}} \in W^r_2$ then the length $\ell(w(\lambda^*))$ of the Grassmannian permutation $w(\lambda^*)$ is the weight $w_{\mathbf{a}}$ of the exponent vector $\mathbf{a} \in r\Delta_2 \cap^2_{\geq 0}$.

Theorem 1.6. Let $P^h_{r\Delta_2}(z)$ be the weighted polynomial of the lattice points of the dilation $r\Delta_2$. Then the Poincaré polynomial $P(Gr(2, 2+$

$r), t)$ of the Grassmannian $Gr(2, 2+r)$ coincides with the weighted polynomial $P^h_{r\Delta_2}(z)$.

In Section 2, we review some cohomology of Grassmannian as it affects our discussion and describe the first refinement by constructing the 2-filling set $\mathcal{C}^2_{2,r}$ and give some of its properties. The set consists of all semi standard tableaux identifies with one-row Young diagrams. The size of this set is given in terms of a polynomial. A generating function for these polynomial is constructed as r grows. The bijection between the 2-filling set $\mathcal{C}^2_{2,r}$ and lattice points $r\Delta_2 \cap^2_{\geq 0}$ is established. In Section 3 we identify the monomials associated with semi standard Young tableaux of $\mathcal{C}^2_{2,r}$ as Grassmannians and construct two important partitions from each of their exponent vectors. This leads to our second refinement of the Ehrhart polynomial $L_{\Delta_2}(r)$ of the standard 2-simplex Δ_2. This is precisely the Poincaré polynomial of the Grassmannian $\mathrm{Gr}(2, 2+r)$.

2. The Cohomology of the Grassmannian $\mathrm{G}r(2,n)$

A flag $F_\bullet$ is a nested sequence of vector subspaces

$$F_\bullet := (\{0\} \subset F_1 \subset F_2 \subset \cdots \subset F_m)$$

of ^{n}. The flag $F_\bullet$ is said to be complete if the chain is maximal, that is, $\mathrm{dim} F_i = i$ and $m = n$. The set of complete flags in ^{n} is called the full flag variety $\mathcal{F}\ell_n()$ and its dimension is $\frac{n(n-1)}{2}$. The Grassmannian $Gr(2, n)$ is the special case of the flag variety being the set of all 2-dimensional subspaces in ^{n} and its dimension is $2(n-2)$. There is a forgetful map

$$\pi : \mathcal{F}\ell_n() \longrightarrow Gr(2, n) \tag{6}$$

from the full flag variety $\mathcal{F}\ell_n()$ to the Grassmannian $Gr(2, n)$ with $\pi^{-1}(X_\lambda(F_\bullet)) = X_{w(\lambda)}(F_\bullet)$, where $X_\lambda(F_\bullet)$ is a Schubert variety in the Grassmannian $Gr(2, n)$ defined as the closure of a certain Schubert cell. The partition λ is called fitted in the sense that it has at most length 2 and each part cannot exceed $n-2$. The permutation $w(\lambda)$ identified with the partition λ is given by

$$w_i = i + \lambda_{3-i},\ 1 \leq i \leq 2 \text{ and } w_j < w_{j+1},\ 3 \leq j \leq n. \tag{7}$$

The projection π induces a monomorphism π^* at the level of cohomology.

$$\pi^* : H^*(Gr(2, n),) \longrightarrow H^*(\mathcal{F}\ell_n(),) \tag{8}$$

which takes cycle σ_λ to the cycle $\sigma_{w(\lambda)}$. The cohomology ring of the Grassmannian $Gr(2, n)$ is generated by the Schubert cycles σ_λ. These are Poincaré dual of the fundamental classes in the homology of Schubert varieties. The $Gr(2, n)$ admits many important vector bundles. There is a universal short exact sequence: $0 \longrightarrow \mathcal{S} \longrightarrow^n \times \mathrm{Gr}(2, n) \longrightarrow \mathcal{Q} \longrightarrow 0$ of bundles on $Gr(2, n)$ which makes it easy to compute the Chern class $c(\mathcal{Q})$ of the quotient bundle $\mathcal{Q}$ on the Grassmannian $Gr(2, n)$. Recall that $\mathcal{Q}$ is a globally generated vector bundle of rank $n-2$ and all its global sections are from the trivial bundle $^n \times \mathrm{Gr}(2,n)$. The total Chern class is the sum over all the one-row partitions inside the rectangle $\square_{2\times n-2}$. That is,

$$c(\mathcal{Q}) = 1 + \sigma_{7pt(1)} + \sigma_{7pt(2)} + \cdots + \sigma_{7pt(2)\cdots 7pt(1)_{1\times r}} \tag{9}$$

It turns out that the set of all one-row Young diagrams indexing the multiplicative generators of the cohomology of the Grassmannian $Gr(2,n)$ are deeply connected with the lattice points of $r\Delta_2$. This becomes evident in what follows. Let $\mathcal{C}_{(2,r)}$ denote the set of all one row Young diagrams whose number of boxes cannot exceed r with the empty set $\emptyset$. That is,

$$\mathcal{C}_{(2,r)} = \{\square_{1\times d} :\ 1 \leq d \leq r\} \cup \emptyset. \tag{10}$$

The members of $\mathcal{C}_{(2,4)}$ are:

$$\phi, \begin{array}{|c|}\hline \ \\\hline\end{array}, \begin{array}{|c|c|}\hline \ & \ \\\hline\end{array}, \begin{array}{|c|c|c|}\hline \ & \ & \ \\\hline\end{array}, \begin{array}{|c|c|c|c|}\hline \ & \ & \ & \ \\\hline\end{array}, \begin{array}{|c|c|c|c|c|}\hline \ & \ & \ & \ & \ \\\hline\end{array}$$

The filling of the boxes of the row diagrams in $\mathcal{C}_{2,r}$ using numbers from the set $[2] = \{1,2\}$ is semi standard, that is, the numbers weakly increase from the left to the right. We denote the collection of all such fillings by $\mathcal{C}^2_{2,r}$ and call it the 2-filling set. For instance, The members of $\mathcal{C}^2_{(2,4)}$ are

$$\begin{array}{|c|}\hline \ \\\hline\end{array}, \begin{array}{|c|}\hline 1\\\hline\end{array}, \begin{array}{|c|}\hline 2\\\hline\end{array}, \begin{array}{|c|c|}\hline 1&1\\\hline\end{array}, \begin{array}{|c|c|}\hline 1&2\\\hline\end{array}, \begin{array}{|c|c|}\hline 2&2\\\hline\end{array}, \begin{array}{|c|c|c|}\hline 1&1&1\\\hline\end{array},$$

$$\begin{array}{|c|c|c|}\hline 1&1&2\\\hline\end{array}, \begin{array}{|c|c|c|}\hline 1&2&2\\\hline\end{array}, \begin{array}{|c|c|c|}\hline 2&2&2\\\hline\end{array}, \begin{array}{|c|c|c|c|}\hline 1&1&1&1\\\hline\end{array},$$

$$\begin{array}{|c|c|c|c|}\hline 1&1&1&2\\\hline\end{array}, \begin{array}{|c|c|c|c|}\hline 1&1&2&2\\\hline\end{array}, \begin{array}{|c|c|c|c|}\hline 1&2&2&2\\\hline\end{array}, \begin{array}{|c|c|c|c|}\hline 2&2&2&2\\\hline\end{array}$$

These 15 semi standard Young tableaux can be organized in terms of their defining one-row Young diagrams. It turns out that this arrangement can be expressed as a polynomial, given by $T_4(t) = 1 + 2t + 3t^2 + 4t^3 + 5t^4$. This is the triangular polynomial of degree 4 illustrated in the Figure 1.

(i.) The size $L^2(r)$ of the 2-filling set $\mathcal{C}^2_{2,r}$ is $\binom{r+2}{2}$ and the sequence $(L^2(r))_{r=0}^{\infty}$ of cardinalities as r grows is recorded by the generating function

$$P(\mathcal{C}^2_{(2,r)}, z) = \frac{1}{(1-z)^3}$$

[fill] (1,1) circle [radius=0.075]; [fill] (2,0) circle [radius=0.075]; [fill] (3,-1) circle [radius=0.075]; [fill] (-1,-1) circle [radius=0.075]; [fill] (1,-1) circle [radius=0.075]; [fill] (-2,-2) circle [radius=0.075]; [fill] (-0.08,-2) circle [radius=0.075]; [fill] (2.08,-2) circle [radius=0.075]; [fill] (4,-2) circle [radius=0.075]; [fill] (0,0) circle [radius=0.075]; [fill] (-3,-3) circle [radius=0.075]; [fill] (-1.5,-3) circle [radius=0.075]; [fill] (1,-3) circle [radius=0.075]; [fill] (3.5,-3) circle [radius=0.075]; [fill] (5,-3) circle [radius=0.075]; [-¿] (-2,-2)–(-3,-3)–(-1.5,-3)–(1,-3)–(3.5,-3)–(5,-3)–(4,-2); [-¿] (0,0)–(1,1)–(-1,-1)–(-2,-2); [-¿] (1,1)–(2,0)–(3,-1)–(4,-2); [-¿] (-2,-2)–(-0.08,-2)–(2.08,-2)–(4,-2); (3,-1)–(1,-1)–(-1,-1); (0,0)–(2,0); [left] at (0,0) $\begin{array}{|c|}\hline 1\\ \hline\end{array}$; [] at (1,1) $\begin{array}{|c|}\hline \\ \hline\end{array}$; [right] at (2,0) $\begin{array}{|c|}\hline 2\\ \hline\end{array}$; [left] at (-1,-1) $\begin{array}{|c|c|}\hline 1&1\\ \hline\end{array}$; [] at (1,-1) $\begin{array}{|c|c|}\hline 1&2\\ \hline\end{array}$; [right] at (3,-1) $\begin{array}{|c|c|}\hline 2&2\\ \hline\end{array}$; [left] at (-2,-2) $\begin{array}{|c|c|c|}\hline 1&1&1\\ \hline\end{array}$; [] at (2.08, -2) $\begin{array}{|c|c|c|}\hline 1&2&2\\ \hline\end{array}$; [] at (-0.08,-2) $\begin{array}{|c|c|c|}\hline 1&1&2\\ \hline\end{array}$; [right] at (4,-2) $\begin{array}{|c|c|c|}\hline 2&2&2\\ \hline\end{array}$; [left] at (-3,-3) $\begin{array}{|c|c|c|c|}\hline 1&1&1&1\\ \hline\end{array}$; [] at (-1.5,-3) $\begin{array}{|c|c|c|c|}\hline 1&1&1&2\\ \hline\end{array}$; [] at (1,-3) $\begin{array}{|c|c|c|c|}\hline 1&1&2&2\\ \hline\end{array}$; [] at (3.5,-3) $\begin{array}{|c|c|c|c|}\hline 1&2&2&2\\ \hline\end{array}$; [right] at (5,-3) $\begin{array}{|c|c|c|c|}\hline 2&2&2&2\\ \hline\end{array}$;

FIGURE 1. $T_4(t) = 1 + 2t + 3t^2 + 4t^3 + 5t^4$

(ii.) More is true, there is a graded counting polynomial of the semi standard tableaux in $\mathcal{C}^2_{2,r}$ given by

$$T^2_r(t) = \sum_{k=0}^{r} (k+1)t^k$$

that is, a k-box row diagram gives $(k+1)$ semi standard Young tableaux. This has a generating function

$$G(t, z) = \frac{z}{(1-z)(1-tz)^2}.$$

Proof. (i) $(L^2(r))_{r=0}^{\infty}$ is a well known sequence whose terms are triangular numbers and hence the general term $\binom{r+2}{2}$. Its generating function follows from Ehrhart polynomial $f(r) = \frac{1}{2}r^2 + \frac{3}{2}r + 1$.
(ii) The row diagrams are given by the partitions $\lambda = (k),\ 0 \le k \le r$. The number of semi standard fillings of each of the row diagram with shape $\lambda = (k)$ using the elements of the set $\{1, 2\}$ is given by

$$\prod_{1\le i<j\le 2} \frac{\lambda_i - \lambda_j + j - i}{j - i}$$

this is precisely $k+1$. Lastly, notice that

$$T_r(t) = T_{r-1}(t) + (r+1)t^r \quad \text{and} \quad \sum_{r\geq 0}(r+1)z^r = \frac{1}{(1-z)^2}.$$

$$G(t,z) = \sum_{r\geq 0} T_r(t)z^r = \sum_{r\geq 0}\left[T_{r-1}(t) + (r+1)t^{r-1}\right]z^r.$$

$$G(t,z) = zG(t,z) + \sum_{r\geq 1}\left[(r+1)t^{r-1}\right]z^r, \quad \text{and so}$$

$$G(t,z) = \frac{z}{(1-z)(1-tz)^2}.$$

□

There is a bijection between the set $\mathcal{C}^2_{2,r}$ and the set $r\Delta_2\cap^2_{\geq 0}$ of the lattice points of the dilation $r\Delta_2$. Furthermore, the triangular polynomial $T_r(t)$ is precisely the slicing of the dilation $r\Delta_2$ with lines $y+x = c,\ 0 \leq c \leq r$.

Proof. The bijection is given by $T \mapsto v(T)$. To each semi standard tableau $T \in \mathcal{C}^2_{(2,r)}$ there exists a unique exponent vector $v(T) = (v(T)_1, v(T)_2)$ in which the coordinate $v(T)_j$ is the number of appearances of j in T, $1 \leq j \leq 2$. For a fixed point $\mathbf{s} = (1,1)$ and non-negative integers c, consider

$$L^s_{r\Delta_2}(c,r) = \#\{\mathbf{a} \in r\Delta_2\cap^2_{\geq 0} : \mathbf{s}\cdot\mathbf{a} = c,\ 0 \leq c \leq r\}$$

Notice that $L^s_{r\Delta_2}(c,r) = (c+1),\ 0 \leq c \leq r$. Therefore, the triangular polynomial, viewed now as $T_r(t) = \sum_{c=0}^r (c+1)t^c$ counts the number of lattice points $(a_1,a_2) \in r\Delta_2$ on each of the lines $y+x = c,\ 0 \leq c \leq r$. □

The triangular polynomial $T_r(t)$ is a refinement of the Ehrhart polynomial. The triangular polynomial $T_r(t) = \sum_{c=0}^r (c+1)t^c$ specialises at $t = 1$ to the Ehrhart polynomial $\binom{r+2}{2}$.

The triangular polynomial associated to the dilation $4\Delta_2$ is illustrated below

3. Grassmannian Monomials

For every lattice point $\mathbf{a} \in r\Delta_2\cap^2_{\geq 0}$, there is a corresponding monomial $\mathbf{t}^{\mathbf{a}}$ in the polynomial ring $[t_1,t_2]$ given by $\mathbf{t}^{\mathbf{a}} := t_1^{a_1}t_2^{a_2}$. We call these monomials Grassmannian and denote their collection by W_2^r,that is,

$$W_2^r = \{t_1^{a_1}t_2^{a_2} \in [t_1,t_2] : (a_1,a_2) \in r\Delta_2\cap^2_{\geq 0}\} \tag{11}$$

To every monomial $\mathbf{t}^{\mathbf{a}} \in W_2^r$ we associate a weight $w_{\mathbf{a}}$ defined by

$$w_{\mathbf{a}} = \sum_{k=1}^{d} ka_k \tag{12}$$

(0,0)–(1,0)–(2,0)–(3,0)–(4,0); (0,0)–(0,1)–(0,2)–(0,3)–(0,4); (0,4)–(1,3)–(2,2)–(3,1)–(4,0); [ultra thick,red] (1,0)–(0,1); [ultra thick,red] (0,2)–(1,1)–(2,0); [ultra thick,red] (0,3)–(1,2)–(2,1)–(3,0); [fill] (1,3) circle [radius=0.075]; [fill] (2,2) circle [radius=0.075]; [fill] (3,1) circle [radius=0.075]; [fill] (4,0) circle [radius=0.075]; [fill] (0,4) circle [radius=0.075]; [fill] (1,0) circle [radius=0.075]; [fill] (0,0) circle [radius=0.075]; [fill] (3,0) circle [radius=0.075]; [fill] (2,0) circle [radius=0.075]; [fill] (1,1) circle [radius=0.075]; [fill] (0,3) circle [radius=0.075]; [fill] (2,1) circle [radius=0.075]; [fill] (1,2) circle [radius=0.075]; [fill] (0,2) circle [radius=0.075]; [fill] (0,1) circle [radius=0.075];

FIGURE 2. $T_4(t) = 1 + 2t + 3t^2 + 4t^3 + 5t^4$

It turns out that $w_{\mathbf{a}}$ admits two important partitions $\lambda, \lambda^* \vdash w_{\mathbf{a}}$ which can be identified with the monomial $\mathbf{t}^{\mathbf{a}}$. These partitions, λ and λ^* are called α-partition and β-partition respectively. A partition $\lambda \vdash w_{\mathbf{a}}$ is said to be the α-partition of the monomial $t_1^{a_1} t_2^{a_2} \in W_2^r$ if the number of parts of size i in λ is a_i, $1 \leq i \leq 2$, while the β partition $\lambda^* = (\lambda_1^*, \lambda_2^*)$ of $w_{\mathbf{a}}$ is such that $\lambda_k^* = \sum_{i \geq k}^{2} a_i$, $1 \leq k \leq 2$. For instance, for the monomial $t_1^2 t_2^3 \in W_2^5$, the corresponding α-partition λ and β-partition λ^* are (2,2,2,1,1) and (5,3) respectively.

Let λ be the α-partition associated with the monomial $\mathbf{t}^{\mathbf{a}} = t_1^{a_1} t_2^{a_2} \in [t_1, t_2]$. Then its corresponding β-partition λ^* is its conjugate.

Proof. By definition, if we denote the α-partition by $\lambda = (\lambda_1, \ldots, \lambda_{a_1+a_2})$ and the β partition by $\lambda^* = (\lambda_1^*, \lambda_2^*)$ then it is obvious that the following identity is satisfied

$$\sum_{k=1}^{a_1+a_2} (2k-1)\lambda_k = \sum_{k=1}^{2} \lambda_k^{*2}$$

□

Every β-partition λ^* identified with each of the monomials $\mathbf{t}^{\mathbf{a}} \in W_2^r$ fits into the $2 \times r$ rectangle $\square_{2\times r}$.

Proof. It is sufficient to establish that the parts of λ^* cannot exceed r and the length $\ell(\lambda^*)$ of λ^* is 2. Notice that the exponent vector $\mathbf{a}$ is a lattice point of $r\Delta_2$ and by definition $a_1 + a_2 \leq r$. Therefore each part λ_k^* of λ^* is at most r and length $\ell(\lambda^*)$ is 2 by the definition of λ^*. □

The set of β-partitions λ^* identified with monomials in W_2^r index the Schubert varieties in the Grassmannian $\mathrm{Gr}(2, 2+r)$.

Let λ^* be the β-partition identified with the monomial $\mathbf{t}^{\mathbf{a}} \in W_2^r$ then the length $\ell(w(\lambda^*))$ of the Grassmannian permutation $w(\lambda^*)$ is the weight $w_{\mathbf{a}}$ of the exponent vector $\mathbf{a} \in r\Delta_2 \cap^2_{\geq 0}$. The weight $w_{\mathbf{a}}$ defined in the equation (3.2) gives another refinement $P^h_{r\Delta_2}(z)$ of the Ehrhart polynomial of $r\Delta_2$ with respect to a fixed point $h = (1, 2)$.

$$P^h_{r\Delta_2}(z) = \sum_{m=0}^{2r} A_m z^m. \tag{13}$$

where $A_m = \#\{\mathbf{a} \in r\Delta_2 \cap^2_{\geq 0} : \mathbf{a} \cdot h = m,\ 0 \leq m \leq 2r\}$. That is, A_m is the number of exponent vectors $\mathbf{a}$ which share the weight m. We call $P^h_{r\Delta_2}(z)$ the weighted polynomial associated with the dilation $r\Delta_2$.
The polynomial $P^h_{r\Delta_2}(z) = \sum_{m=0}^{2r} A_m z^m$ specializes at $z = 1$ to the Ehrhart polynomial $L_{\Delta_2}(r)$. There is an interesting geometric description of the two refinements of the Ehrhart polynomial $L_{\Delta_2}(r)$ we have introduced. While the triangular polynomial $T_r(t)$ describes the set of r translations of the line $y + x = 0$ controlled by the lattice points of the dilation $r\Delta_2$, the polynomial $P^h_{r\Delta_2}(z)$ gives the list of $2r$ translations of the line $x + 2y = 0$, controlled by the same lattice points.

Let $P^h_{r\Delta_2}(z)$ be the weighted polynomial of the lattice points of the dilation $r\Delta_2$. Then the Poincaré polynomial $P(Gr(2, 2 + r), t)$ of the Grassmannian $Gr(2, 2+r)$ coincides with the weighted polynomial $P^h_{r\Delta_2}(z)$.

Proof. It is known from the Borel presentation of the cohomology ring $H^*(Gr(2, 2 + r),)$ of the Grassmannian $\mathrm{Gr}(2, 2 + r)$ that the Poincaré polynomial $\mathrm{P}(\mathrm{Gr}(2, 2 + r), t)$ is given by the following Gaussian polynomial

$$\frac{(1-t)(1-t^2)\cdots(1-t^{2+r})}{(1-t)(1-t^2)(1-t)\cdots(1-t^r)}.$$

This is combinatorially simplified as

$$\sum_{\lambda \subseteq \Box_{2\times r}} t^{|\lambda|}$$

where $|\lambda|$ is the number of boxes in the Young diagram of shape λ. The size $|\lambda|$ coincides with the length $\ell(w(\lambda))$ (the number of inversions) of the Grassmannian permutation $w(\lambda)$ identified with λ in the equation (2.2). Notice that $|\lambda| \leq 2r$, therefore, It follows from the Corollary 3.4 that $|\lambda|$ is the weight $w_{\mathbf{a}}$ of the monomial $t^{\mathbf{a}} \in W_2^r$, $a \in r\Delta \cap^2_{\geq 0}$, therefore, $\sum_{\lambda \subseteq \Box_{2\times r}} t^{|\lambda|}$ is precisely the polynomial $\sum_{m=0}^{2r} A_m z^m$. $\Box$

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