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HYERS-ULAM STABILITY THEOREMS FOR SECOND ORDER NONLINEAR DAMPED DIFFERENTIAL EQUATIONS WITH FORCING TERM

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ABSTRACT. In this paper, Hyers-Ulam stability theorems of nonlinear second order damped differential equations with forcing term are considered. By using the Bihari inequality and Gronwall-Bellman-Bihari integral inequality, we obtain new sufficient conditions for the Hyers-Ulam stability of every nonlinear second order differential equation considered. Our results improve and extend some known results.

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1. INTRODUCTION

In this paper, we study the Hyers-Ulam stability of the following forced nonlinear second order differential equations with damping:

$$(r(t)K_1(u(t),u'(t)))'+p(t)K_2(u(t),u'(t))u'(t)+q(t)f(u(t)) = P(t,u(t),u'(t)), \tag{1.1}$$

$$u''(t) + np(t)Q(t,u(t))u'(t) + q(t)f(u(t)) = P(t,u(t),u'(t)), \tag{1.2}$$

for all $t > 0$, with initial conditions

$$u(t_0) = u'(t_0) = 0, \tag{1.3}$$

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where $r(t),\ p(t),\ q(t)\in C(\mathbf{R}_+)$, $f\in C(\mathbf{R})$, $K_1, K_2\in C(\mathbf{R}^2)$, $Q\in C(\mathbf{R}_+\times\mathbf{R})$, $P\in C(\mathbf{R}_+\times\mathbf{R}^2)$, $\mathbf{R}_+=[0,\infty)$ and $\mathbf{R}=(-\infty,\ \infty)$. The Hyers-Ulam stability of equations (1.1),(1.2) and their variants are considered by transforming the equations into integral inequalities for easy use of Bihari and Gronwall-Bellman-Bihari inequalities.

The study of stability problem for various functional equations originated from a famous talk of S.M. Ulam. In 1940, Ulam [32] posed a problem concerning the stability of functional equations: 'Give conditions in order for a linear function near an approximately linear function to exist'. Since then, this question has attracted the attention of many researchers. The solution to this question was given by Hyers [13] for additive functions defined on Banach spaces in 1941. Thereafter, the result by Hyers [13] was generalised by Rassias [26], Aoki [3] and Bourgin[5].

A generalisation of Ulam's problem was proposed by replacing functional equations with differential equations. Obloza [23] seems to be the first author to prove the Ulam stability of differential equations. Thereafter, the Hyers-Ulam stability of various linear differential equations were extensively studied. In 1998, Alsina and Ger [2] proved the following : Assume that a differentiable function $f:\mathbf{I}\to\mathbf{R}$ is a solution of the differential inequality $|u'(t)-u(t)|\leq\epsilon$, then there exists a solution $f_0:\mathbf{I}\to\mathbf{R}$ of the differential equation $u'(t)=u(t)$ such that $|f(t)-f_0(t)|\leq 3\epsilon$ for any $t\in\mathbf{I}$.
Following the same approach as in [2], Miura *et al.* [20], Miura [19], Takahasi *et al.* [30], and Miura *et al.*, [21] proved that the Hyers-Ulam stability holds true for the differential equation

$$u'(t)=\lambda u(t), \tag{1.4}$$

while Jung [17] proved a similar result for the differential equation

$$\psi(t)u'(t)=u(t). \tag{1.5}$$

Furthermore, the result of Hyers-Ulam for first-order linear differential equations was generalised by Miura *et al.*[22], by Takahasi *et al.* [30] and Jung [15]. They dealt with the nonhomogeneous linear differential equation of first order

$$u'(t)+p(t)u(t)+q(t)=0. \tag{1.6}$$

Recently, Jung [16] proved that the differential equations of the form

$$tu'(t)+\alpha u(t)+\beta t^r x_0=0 \tag{1.7}$$

satisfy the generalised Hyers-Ulam stability and then applied the result in the investigation of the Hyers-Ulam stability of the Euler(Cauchy) differential equation

$$t^2u''(t)+\alpha tu'(t)+\beta u(t)=0. \tag{1.8}$$

In their work, Li and Shen [29] proved that if the characteristic equation $\lambda^2+\alpha\lambda+\beta=0$ has two distinct positive roots, then the second order linear differential equation with constant coefficients

$$u''(t) + \alpha y'(t) + \beta y(t) = f(t) \tag{1.9}$$

has the Hyers-Ulam stability where $y \in C^2[a,b]$, $f \in C[a,b]$ and $-\infty < a, b < +\infty$. Ghaemi *et al.* [12] proved the Hyers-Ulam stability of the exact second order linear differential equation

$$p_0(t)u''(t) + p_1(t)u'(t) + p_2(t)u + f(t) = 0 \tag{1.10}$$

with $p_0''(t) - p_1(t)' + p_2(t) = 0$. Here, $p_0, p_1, p_2, f : (a,b) \to \mathbf{R}$ are continuous functions.

The following authors also discussed Hyers-Ulam stability of nonlinear differential equations: Rus [27, 28] Qarawani [24, 25], Algfiary and Jung [1], Fakunle and Arawomo [9, 10, 11].

2. PRELIMINARY

The following definitions, lemmas and theorems are necessary for our results

Definition 1. We say that equation (1.1) has the Hyers-Ulam stability, if there exists a constant $K_1^* \geq 0$ with the following property: for every $\epsilon > 0$, $u(t) \in C^2(\mathbf{R}_+)$, if

$$\begin{aligned} |(r(t)K_1(u(t), u'(t)))' + p(t)K_2(u(t), u'(t))u'(t) + q(t)f(u(t)) \\ -P(t, u(t), u'(t))| \leq \epsilon, \end{aligned} \tag{2.1}$$

then, there exists some $u_0(t) \in C^2(\mathbf{R}_+)$ such that

$$|u(t) - u_0(t)| \leq K_1^* \epsilon.$$

We call such K_1^* a Hyers-Ulam constant.

Definition 2. The differential equation (1.2) has the Hyers-Ulam stability with initial condition (1.3), if there exists a positive constant $K_2^* \geq 0$ with the following property: for every $\epsilon \geq 0$, $u(t) \in C^2(\mathbf{R}_+)$, which satisfies

$$|u''(t) + np(t)Q(t, u(t))(u'(t)) + q(t)f(u(t)) - P(t, u(t), u'(t))| \leq \epsilon, \tag{2.2}$$

then there exists a function $u_0(t) \in C^2(\mathbf{R}_+)$ satisfying (1.2) with initial condition (1.3) such that

$$|u(t) - u_0(t)| \leq K_2^* \epsilon,$$

We call such K_2^* Hyers-Ulam stability constant for the differential equation (1.2) with initial conditions (1.3).

Definition 3. A function $\omega : [0, \infty) \to [0, \infty)$ is said to belong to a class Ψ if

i $\omega(u)$ is nondecreasing and continuous for $u \geq 0$,

ii $(\frac{1}{v})\omega(u) \leq \omega(\frac{u}{v})$ for all u and $v \geq 1$, iii there exists a function ϕ, continuous on $[0,\infty)$ with $\omega(\alpha u) \leq \phi(\alpha)\omega(u)$ for $\alpha \geq 0$.

Lemma 1. [4] Let $u(t)$, $f(t)$ be positive continuous functions defined on $t_0 \leq t \leq b, (\leq \infty)$ and $K > 0$, $M \geq 0$, further let $\omega(u)$ be a nonnegative nondecreasing continuous function for $u \geq 0$, then the inequality

$$u(t) \leq K + M \int_{t_0}^{t} f(s)\omega(u(s))ds, \quad t_0 \leq t < b, \tag{2.3}$$

implies the inequality

$$u(t) \leq \Omega^{-1}\left(\Omega(k) + M\int_{t_0}^{t} f(s)ds\right), \quad t_0 \leq t \leq b' \leq b, \tag{2.4}$$

where

$$\Omega(u) = \int_{u_0}^{u} \frac{dt}{\omega(t)}, \quad 0 < u_0 < u. \tag{2.5}$$

In the case $\omega(0) > 0$ or $\Omega(0+)$ is finite, one may take $u_0 = 0$ and Ω^{-1} is the inverse function of Ω and t must be in the subinterval $[t_0, b']$ of $[t_0, b]$ such that

$$\Omega(k) + M\int_{t_0}^{t} f(s)ds \in Dom(\Omega^{-1}).$$

Theorem 1. [6] Let

i $u(t), r(t) : (0,\infty) \to (0,\infty)$ and continuous on $(0,\infty)$,
ii $\varpi \in \Psi$,
iii $n > 0$ be monotonic, nondecreasing and continuous on $(0,\infty)$,

if

$$u(t) \leq n(t) + \int_{0}^{t} f(s)\varpi(u(s))ds, \quad 0 < t < \infty, \tag{2.6}$$

then

$$u(t) \leq n(t)\Omega^{-1}\left(\Omega(1) + \int_{0}^{t} f(s)ds\right) \quad 0 < t \leq b, \tag{2.7}$$

for $(0,b) \subset (0,\infty)$, where $\Omega(u)$ is defined in (2.5), Ω^{-1} is the inverse of Ω and t is in the subinterval $(0,b)$ chosen so that

$$\Omega(1) + \int_{0}^{t} f(s)ds \in Dom(\Omega^{-1}).$$

Theorem 2. [18] If $f(t)$ and $g(t)$ are continuous in $[t_0, t] \subseteq \mathbf{I}$ and $f(t)$ does not change sign in the interval, then there is a point $\xi \in [t_0, t]$ such that $\displaystyle\int_{t_0}^{t} g(s)f(s)ds = g(\xi)\int_{t_0}^{t} f(s)ds$

Theorem 3. [8, 7] Suppose $u(t), r(t), h(t) \in C(\mathbf{I}, \mathbf{R}_+)$ and $\varpi(u), \beta(u) \in \Psi$ are nonnegative, monotonic, nondecreasing, continuous functions and $\omega(u)$ a submultiplicative function for $u > 0$. Let

$$u(t) \leq K + T\int_{t_0}^{t} r(s)\beta(u(s))ds + L\int_{t_0}^{t} h(s)\varpi(u(s))ds \tag{2.8}$$

for K, T and L positive constants, then

$$u(t) \leq \Omega^{-1}\left(\Omega(K) + L\int_{t_0}^{t} h(s)\varpi\left(F^{-1}\left(F(1) + T\int_{t_0}^{s} r(\alpha)d\alpha\right)\right)ds\right) F^{-1}\left(F(1) + T\int_{t_0}^{t} r(s)ds\right) \tag{2.9}$$

where $\beta(u) \neq \varpi(u)$, Ω is defined in equation (2.5) and $F(u)$ is defined as

$$F(u) = \int_{u_0}^{u} \frac{ds}{\beta(s)}, \quad 0 < u_0 \leq u, \tag{2.10}$$

F^{-1}, Ω^{-1} are the inverses of F, Ω respectively and t is in the subinterval $(0, b) \in \mathbf{I}$ so that

$$F(1) + T\int_{t_0}^{t} r(s)ds \in Dom(F^{-1})$$

and

$$\Omega(K) + L\int_{t_0}^{t} h(s)\varpi\left(F^{-1}\left(F(1) + T\int_{t_0}^{t} r(\alpha)d\alpha\right)\right)ds \in Dom(\Omega^{-1})$$

Corollary 1. [7, 8] Suppose $\rho(t)$ is a nonnegative, monotonic, nondecreasing continuous function on $\mathbf{R}_+$. Let

$$u(t) \leq \rho(t) + T\int_{t_0}^{t} r(s)\beta(u(s))ds + L\int_{t_0}^{t} h(s)\varpi(u(s))ds, \tag{2.11}$$

for T and L be positive constants, then

$$u(t) \leq \rho(t)\Omega^{-1}\left(\Omega(1) + L\int_{t_0}^{t} h(s)\varpi\left(F^{-1}\left(F(1) + T\int_{t_0}^{t} r(\alpha)d\alpha\right)\right)ds\right) F^{-1}\left(F(1) + T\int_{t_0}^{t} r(s)ds\right), \quad t \in \mathbf{I}, \tag{2.12}$$

where $\Omega(u)$ and $F(u)$ are defined as in (2.5) and (2.10) respectively.

Theorem 4. [8, 7] If $u(t), r(t), h(t), \rho(t), g(t) \in C(\mathbf{R}_+)$ and $\omega, f, \gamma \in \Psi$ be nonnegative, monotonic, nondecreasing continuous functions. Let γ

be a submultiplicative function. If

$$u(t) \leq \rho(t) + A\int_{t_0}^{t} r(s)\beta(u(s))ds + B\int_{t_0}^{t} h(s)\varpi(u(s))ds + L\int_{t_0}^{t} g(s)\gamma(u(s))ds \tag{2.13}$$

for $K, A, B, L > 0$, then

$$u(t) \leq \rho(t)\Upsilon^{-1} \left[\Upsilon(1) + L\int_{t_0}^{t} g(s)\gamma\left[\Omega^{-1}\left(\Omega(1) + B\int_{t_0}^{s} h(\alpha)\varpi\left(T(\alpha)\right)d\alpha\right)T(s)\right]ds\right] \Omega^{-1}\left(\Omega(1) + B\int_{t_0}^{t} h(s)\varpi\left(T(s)\right)ds\right)T(t) \tag{2.14}$$

where $T(t)$ is given as

$$T(t) = F^{-1}\left(F(1) + A\int_{t_0}^{t} r(s)ds\right) \tag{2.15}$$

and

$$\Upsilon(r) = \int_{t_0}^{t} \frac{ds}{\gamma(s)}, \quad 0 < r_0 \leq r, \tag{2.16}$$

and F^{-1}, Ω^{-1} and Υ^{-1} are the inverses of F, Ω, Υ respectively $t \in (0, b) \subset (I)$. So that

$$\Upsilon(1) + L\int_{t_0}^{t} g(s)\gamma\left[\Omega^{-1}\left(\Omega(1) + B\int_{t_0}^{s} h(\alpha)\varpi\left(T(\alpha)\right)d\alpha\right)T(s)\right]ds \in Dom(\Upsilon^{-1}$$

Lemma 2. [14] Let $r(t)$ be an integrable function then the n successive integration of r over the interval $[t_0, t]$ is given by

$$\int_{t_0}^{t} \cdots \int_{t_0}^{t} r(s)ds^n = \frac{1}{(n-1)!}\int_{t_0}^{t} (t-s)^{n-1} r(s)ds \tag{2.17}$$

3. MAIN RESULTS

In this section, we establish the Hyers-Ulam stability of the nonlinear differential equations (1.1) ,(1.2) and the case $P(t, u(t), u'(t)) = 0$. We shall also prove the Hyers-Ulam stability of the nonlinear differential equation (1.1) with initial conditions (1.3).

Theorem 5. Assume the following conditions:

i $P(t, u(t), u'(t)) = \phi(t)g(u(t))h((u'(t))^4)$
ii $K_1(u(t), u'(t)) = \gamma(u(t))b(u'(t))u'(t)^n$, where $n \in \mathbf{N}$
iii $K_2(u(t), u'(t)) = \omega(u(t))(u'(t))^2$

iv $\lim_{t_0\to\infty}\displaystyle\int_{t_0}^{t}|u'(s)|ds = L,$ where $L>0,$

v $\lim_{t_0\to\infty}\displaystyle\int_{t_0}^{t}\phi(s)ds \leq n_1 < \infty,$ where $n_1>0,$

vi $\lim_{t_0\to\infty}\displaystyle\int_{t_0}^{t}p(s)ds \leq n_2 < \infty,$ where $n_2>0,$

vii $\lim_{t_0\to\infty}\displaystyle\int_{t_0}^{t}r(s)ds \leq n_3 < \infty,$ where $n_3>0,$

viii $q(t)|F(u(t))| \geq |u(t)|,$

are satisfied and $\phi,\gamma,\omega,g,h,b \in C(\mathbf{R}_+)$. In addition, if $\varpi(u(t)) \in \Psi$ is continuous, nondecreasing and monotonic, then equation (1.1) has the Hyers-Ulam stability with Hyers-Ulam constant given by

$$\begin{aligned} K_1^* = L\Upsilon^{-1}\left[\Upsilon(1) + h(|u'(\delta)|^4)|u'(\delta)|\right.\\ n_3 g\left[\Omega^{-1}\left(\Omega(1) + (|u'(\rho)|^4 n_2\omega\left(T^*\right)\right)T^*\right]\Big]\\ \Omega^{-1}\left(\Omega(1) + (|u'(\rho)|)^4 n_2\omega\left(T^*\right)\right)T^*. \end{aligned} \tag{3.1}$$

Proof. Using inequality (2.1) and multiplying both sides by $u'(t)$ we have

$$\begin{aligned} -\epsilon u'(t) \leq (r(t)K_1(u(t),u'(t))'u'(t)\\ +p(t)K_2(u(t),u'(t))(u'(t))^2 + q(t)f(u(t))u'(t) - P(t,u(t),u'(t))u'(t) \leq \epsilon u'(t), \end{aligned} \tag{3.2}$$

Considering (3.2) in the form

$$\begin{aligned} (r(t)K_1(u(t),u'(t))'u'(t) + p(t)K_2(u(t),u'(t))(u'(t))^2 + q(t)f(u(t))u'(t)\\ -P(t,u(t),u'(t))u'(t) \leq \epsilon u'(t). \end{aligned} \tag{3.3}$$

Integrate both sides of (3.3) twice and apply Lemma 2, to obtain

$$\begin{aligned} \int_{t_0}^{t}\int_{t_0}^{t}(r(s)K_1(u(s),u'(s))'u'(s)dsds + t\int_{t_0}^{t}p(s)K_2(u(s),u'(s))(u'(s))^2 ds\\ +t\int_{t_0}^{t}q(s)f(u(s))u'(s)ds - t\int_{t_0}^{t}P(s,u(s),u'(s))u'(s)ds \leq t\epsilon\int_{t_0}^{t}u'(s)ds. \end{aligned} \tag{3.4}$$

Set

$$F(u(t)) = \int_{u_0}^{u(t)} f(s)ds, \tag{3.5}$$

apply equation (3.5) in inequality (3.4) and integrate to get

$$\int_{t_0}^{t} r(s)K_1(u(s),u'(s))u'(s)ds + t\int_{t_0}^{t} p(s)K_2(u(s),u'(s))(u'(s))^2ds$$
$$+tq(t)F(u(t)) - t\int_{t_0}^{t} P(s,u(s),u'(s))u'(s)ds \leq t\epsilon\int_{t_0}^{t} u'(s)ds. \tag{3.6}$$

Using conditions (i-iii) we obtain

$$\int_{t_0}^{t} r(s)\gamma(u(s))b(u'(s))u'(s)^{n+1}ds + t\int_{t_0}^{t} p(s)\omega(u(s))(u'(s))^4ds$$
$$+tq(t)F(u(t)) - t\int_{t_0}^{t} \phi(s)g(u(s))h((u'(s))^4)u'(s)ds \leq t\epsilon\int_{t_0}^{t} u'(s)ds. \tag{3.7}$$

The application of Theorem 2 implies there exists $\xi, \rho, \delta \in [t_0, t]$ such that

$$b(u'(\xi))u'(\xi)^{n+1}\int_{t_0}^{t} (r(s)\gamma(u(s))ds + t(u'(\rho))^4\int_{t_0}^{t} p(s)\omega(u(s))ds$$
$$+tq(t)F(u(t)) - th((u'(\delta))^4)u'(\delta)\int_{t_0}^{t} \phi(s)g(u(s))ds \leq t\epsilon\int_{t_0}^{t} u'(s)ds. \tag{3.8}$$

Multiplying by $\frac{1}{t}$, $t \neq 0$ we obtain

$$\begin{aligned} q(t)F(u(t)) &\leq \epsilon\int_{t_0}^{t} u'(s)ds - b(u'(\xi))u'(\xi)^{n+1}\int_{t_0}^{t} r(s)\gamma(u(s))ds \\ -(u'(\rho))^4&\int_{t_0}^{t} p(s)\omega(u(s))ds + h((u'(\delta))^4)u'(\delta)\int_{t_0}^{t} \phi(s)g(u(s))ds. \end{aligned} \tag{3.9}$$

By conditions (iv) and (viii) we have

$$|u(t)| \leq \epsilon L + b(|u'(\xi)|)|u'(\xi)|^{n+1}\int_{t_0}^{t} r(s)\gamma(|u(s)|)ds$$
$$+(|u'(\rho)|)^4\int_{t_0}^{t} p(s)\omega(|u(s)|)ds + h((|u'(\delta)|)^4)|u'(\delta)|\int_{t_0}^{t} \phi(s)g(|u(s)|)ds \tag{3.10}$$

and the application of Theorem 4 gives

$$\begin{aligned} |u(t)| \le \epsilon L \Upsilon^{-1} \left[\Upsilon(1) + h(|u'(\delta)|^4)|u'(\delta)| \right. \\ \int_{t_0}^{t} \phi(s) g \left[\Omega^{-1} \left(\Omega(1) + (|u'(\rho)|^4 \int_{t_0}^{s} p(\alpha)\omega\left(T(\alpha)\right) d\alpha \right) T(s) \right] ds \Bigg] \\ \Omega^{-1} \left(\Omega(1) + (|u'(\rho)|)^4 \int_{t_0}^{t} p(s)\omega\left(T(s)\right) ds \right) T(t) \end{aligned} \tag{3.11}$$

for

$$T(t) = F^{-1} \left(F(1) + b(|u'(\xi)|)|u'(\xi)|^{n+1} \int_{t_0}^{t} r(s) ds \right). \tag{3.12}$$

Applying conditions (vi)- (vii), we arrive at

$$\begin{aligned} |u(t)| \le \epsilon L \Upsilon^{-1} \left[\Upsilon(1) + h(|u'(\delta)|^4)|u'(\delta)| \right. \\ n_3 g \left[\Omega^{-1} \left(\Omega(1) + (|u'(\rho)|^4 n_2 \omega\left(T^*\right) \right) T^* \right] \Big] \\ \Omega^{-1} \left(\Omega(1) + (|u'(\rho)|)^4 n_2 \omega\left(T^*\right) \right) T^*, \end{aligned} \tag{3.13}$$

where T^* is defined by

$$T^* = F^{-1} \left(F(1) + b(|u'(\xi)|)|u'(\xi)|^{n+1} n_3 \right). \tag{3.14}$$

Hence,

$$|u(t) - u(t_0)| \le |u(t)| \le K_1^* \epsilon$$

with

$$\begin{aligned} K_1^* = L \Upsilon^{-1} \left[\Upsilon(1) + h(|u'(\delta)|^4)|u'(\delta)| \right. \\ n_3 g \left[\Omega^{-1} \left(\Omega(1) + (|u'(\rho)|^4 n_2 \omega\left(T^*\right) \right) T^* \right] \Big] \\ \Omega^{-1} \left(\Omega(1) + (|u'(\rho)|)^4 n_2 \omega\left(T^*\right) \right) T^*. \end{aligned}$$

□

Theorem 6. Suppose that the conditions of Theorem 5 are satisfied. In addition, let

i' $Q(t, u(t)) = v(t)\alpha(u(t))$ where $v(t)$ a continuous function on $\mathbf{R}_+$

ii' $\lim_{t_0 \to \infty} \int_{t_0}^{t} p(s)v(s) ds \le k_4 < \infty$, where $k_4 > 0$,

hold true for a function $\alpha(u(t)) \in \Psi$ continuous, nondecreasing and monotonic, then equation (1.2) has the Hyers-Ulam stability with Hyers-Ulam constant given by

$$\begin{aligned} K_2^* = \left(L + u''(\xi) L \right) \Omega^{-1} \left(\Omega(1) + h(\lambda)^4 \right) \lambda \\ k_1 g \left(F^{-1} \left(F(1) + n(\lambda)^2 n_4 \right) \right) \\ F^{-1} \left(F(1) + n\lambda^2 n_4 \right). \end{aligned} \tag{3.15}$$

Proof. Evaluate inequality (2.2) and multiply both sides by $u'(t)$, we get

$$\begin{aligned} -u'(t)\epsilon \leq u''(t)u'(t) + np(t)Q(t,u(t))(u'(t))u'(t) \\ +q(t)f(u(t))u'(t) - P(t,u(t),u'(t))u'(t) \leq u'(t)\epsilon. \end{aligned} \tag{3.16}$$

Integrate inequality (3.16) twice, apply Lemma 2 and equation (3.5) to obtain

$$\begin{aligned} -\epsilon t\int_{t_0}^{t} u'(s)ds \leq t\int_{t_0}^{t} u'(s)u''(s)ds + nt\int_{t_0}^{t} p(s)Q(s,u(s))(u'(s))^2 ds \\ +t\int_{t_0}^{t} q(s)\frac{d}{ds}F(u(s)ds - t\int_{t_0}^{t} P(s,u(s),u'(s))u'(s)ds \leq \epsilon t\int_{t_0}^{t} u'(s)ds. \end{aligned} \tag{3.17}$$

Divide through by $t \neq 0$ and integrate, for $q(t)$ nondecreasing, $q'(t) \geq 0$ then the application of Theorem 2 implies that there exists $\xi \in [t_0, t]$ such that

$$\begin{aligned} q(t)F(u(t)) \leq \epsilon\int_{t_0}^{t} u'(s)ds - u''(\xi)\int_{t_0}^{t} u'(s)ds \\ -n\int_{t_0}^{t} p(s)Q(s,u(s))(u'(s))^2 ds + \int_{t_0}^{t} P(s,u(s),u'(s))u'(s)ds. \end{aligned} \tag{3.18}$$

We use condition (i') of Theorem 6 and condition (ii) of Theorem 5 to get

$$\begin{aligned} q(t)|F(u(t))| \leq \epsilon\left(\int_{t_0}^{t} |u'(s)|ds + u''(\xi)\int_{t_0}^{t} |u'(s)|ds\right) \\ +n\int_{t_0}^{t} p(s)v(s)\alpha(|u(s)|)(|u'(s)|)^2 ds + \int_{t_0}^{t} \phi(s)g(|u(s)|)h(|(u'(s)|)^4)(|u(s)|)ds. \end{aligned} \tag{3.19}$$

For $t > 0$, the use of condition (viii) in Theorem 5 gives

$$\begin{aligned} |u(t)| \leq \epsilon\left(\int_{t_0}^{t} |u'(s)|ds + |u''(\xi)|\int_{t_0}^{t} |u'(s)|ds\right) \\ +n(|u'(t)|)^2\int_{t_0}^{t} p(s)v(s)\alpha(|u(s)|)ds + h((|u'(t)|)^4)(|u'(t)|)\int_{t_0}^{t} \phi(s)g(|u(s)|)ds. \end{aligned} \tag{3.20}$$

Applying Corollary 4 we obtain

$$\begin{aligned} |u(t)| \leq \epsilon \Bigg(\int_{t_0}^{t} |u'(s)|ds + u''(\xi) \int_{t_0}^{t} |u'(s)|ds \\ \Omega^{-1} \left(\Omega(1) + h((|u'(t)|)^4)(|u'(t)|) \right. \\ \int_{t_0}^{t} \phi(s) g \left(F^{-1} \left(F(1) + n(|u'(t)|)^2 \int_{t_0}^{t} p(\alpha)v(\alpha)d\alpha \right) \right) ds \Bigg) \\ F^{-1} \left(F(1) + n(|u'(t)|)^2 \int_{t_0}^{s} p(s)v(s)ds \right), \quad t \in \mathbf{I}. \end{aligned} \tag{3.21}$$

Setting $|u'(t)| \leq \lambda$ and applying conditions (vi), (v) of Theorem 5 and (ii') of Theorem 6, we arrive at

$$\begin{aligned} |u(t)| \leq \epsilon \left(L + u''(\xi)L \right) \Omega^{-1} \left(\Omega(1) + h(\lambda)^4)(\lambda) \right. \\ n_1 g \left(F^{-1} \left(F(1) + n\lambda^2 n_4 \right) \right) \\ F^{-1} \left(F(1) + n\lambda^2 n_4 \right). \end{aligned} \tag{3.22}$$

Hence,

$$|u(t) - u(t_0)| \leq |u(t)| \leq K_2^* \epsilon,$$

where

$$\begin{aligned} K_2^* = \left(L + u''(\xi)L \right) \Omega^{-1} \left(\Omega(1) + h(\lambda)^4)\lambda \right. \\ n_1 g \left(F^{-1} \left(F(1) + n\lambda^2 n_4 \right) \right) \\ F^{-1} \left(F(1) + n\lambda^2 n_4 \right). \end{aligned}$$

□

For $P(t, u(t), u'(t)) = 0$ in equations (1.1) and (1.2) the results are given in the following theorems:

Theorem 7. Suppose that all the conditions of Theorem 5 remain valid. Then for $P(t, u(t), u'(t)) = 0$ in equation (1.1), the equation

$$(r(t)K_1(u(t), u'(t)))'u'(t) + p(t)K_2(u(t), u'(t))(u'(t))^2 + q(t)f(u(t))u'(t) = 0, \tag{3.23}$$

has Hyers-Ulam stability with the Hyers-Ulam constant given by

$$\begin{aligned} K_3^* = L\Omega^{-1} \left(\Omega(1) + (|u'(\rho)|)^4 n_2 \varpi \left(F^{-1} \left(F(1) \right. \right. \right. \\ \left. \left. +b(|u(\xi)|)|u'(\xi)|^{n+1} n_3 \right) \right) \\ F^{-1} \left(F(1) + b(|u'(\xi)|)|u'(\xi)|^{n+1} n_3 \right). \end{aligned} \tag{3.24}$$

Proof. Using inequality (2.1) and multiplying both sides of the equation by $u'(t)$ we have

$$\begin{aligned} -\epsilon u'(t) \leq (r(t)K_1(u(t), u'(t)))'u'(t) \\ +p(t)K_2(u(t), u'(t))(u'(t))^2 + q(t)f(u(t))u'(t) \leq \epsilon u'(t). \end{aligned} \tag{3.25}$$

It is clear that

$$\begin{aligned}(r(t)K_1(u(t),u'(t)))'u'(t)\\+p(t)K_2(u(t),u'(t))(u'(t))^2+q(t)f(u(t))u'(t)\leq \epsilon u'(t).\end{aligned} \tag{3.26}$$

Integrating the inequality (3.26) and applying Lemma 2, we obtain

$$\begin{aligned}\int_{t_0}^{t}\int_{t_0}^{t}(r(s)K_1(u(s),u'(s)))'u'(s)dsds+t\int_{t_0}^{t}p(s)K_2(u(s),u'(s))(u'(s))^2ds\\+t\int_{t_0}^{t}q(s)f(u(s))u'(s)\leq t\epsilon\int_{t_0}^{t}u'(s)ds.\end{aligned} \tag{3.27}$$

If we apply equation (3.5) to inequality (3.27) and integrate for $q'(t)\geq 0$ since $q(t)$ is nondecreasing, we get

$$\begin{aligned}\int_{t_0}^{t}(r(s)K_1(u(s),u'(s))u'(s)ds+t\int_{t_0}^{t}p(s)K_2(u(s),u'(s))(u'(s))^2ds\\+tq(t)F(u(t))\leq t\epsilon\int_{t_0}^{t}u'(s)ds.\end{aligned} \tag{3.28}$$

Using conditions (ii) and (iii) of Theorem 5, we obtain

$$\begin{aligned}\int_{t_0}^{t}r(s)\gamma(u(s))b(u'(s))u'(s)^{n+1}ds+t\int_{t_0}^{t}p(s)\omega(u(s))(u'(s))^4ds\\+tq(t)F(u(t))\leq t\epsilon\int_{t_0}^{t}u'(s)ds.\end{aligned} \tag{3.29}$$

Applying Theorem 2 implies there exists $\xi,\rho\in[t_0,t]$ such that

$$\begin{aligned}b(u'(\xi))u'(\xi)^{n+1}\int_{t_0}^{t}r(s)\gamma(u(s))ds+t(u'(\rho))^4\int_{t_0}^{t}p(s)\omega(u(s))ds\\+tq(t)F(u(t))\leq t\epsilon\int_{t_0}^{t}u'(s)ds.\end{aligned} \tag{3.30}$$

Let inequality (3.30) be multiplying by $\frac{1}{t},\quad t\neq 0$ to get

$$\begin{aligned}q(t)F(u(t))\leq \epsilon\int_{t_0}^{t}u'(s)ds-b(u'(\xi))u'(\xi)^{n+1}\int_{t_0}^{t}r(s)\gamma(u(s))ds\\-(u'(\rho))^4\int_{t_0}^{t}p(s)\omega(u(s))ds.\end{aligned} \tag{3.31}$$

By condition (iv) of Theorem 5 and for $q(t)|F(u(t))| \geq |u(t)|$ we have

$$\begin{aligned} |u(t))| \leq \epsilon L + b(|u'(\xi)|)|u'(\xi)|^{n+1} \int_{t_0}^{t} r(s)\gamma(|u(s)|)ds \\ +(|u'(\rho)|)^4 \int_{t_0}^{t} p(s)\omega(|u(s)|)ds. \end{aligned} \tag{3.32}$$

Apply Corollary 1 to get

$$\begin{aligned} |u(t)| \leq \epsilon L\Omega^{-1} \bigg(\Omega(1) + (|u'(\rho)|)^4 \int_{t_0}^{t} p(s)\varpi \left(F^{-1}\left(F(1)\right.\right. \\ \left.\left. +b(|u(\xi)|)|u'(\xi)|^{n+1} \int_{t_0}^{t} r(\alpha)d\alpha\right)\right) ds\bigg) \\ F^{-1}\left(F(1) + b(|u'(\xi)|)|u'(\xi)|^{n+1} \int_{t_0}^{t} r(s)ds\right), \quad t \in \mathbf{I}. \end{aligned} \tag{3.33}$$

Further application of conditions (vi)- (vii) of Theorem 5, gives

$$\begin{aligned} |u(t)| \leq \epsilon L\Omega^{-1} \left(\Omega(1) + (|u'(\rho)|)^4 n_2 \varpi \left(F^{-1}\left(F(1)\right.\right.\right. \\ \left.\left.\left. +b(|u(\xi)|)|u'(\xi)|^{n+1} n_3\right)\right) ds\right) \\ F^{-1}\left(F(1) + b(|u'(\xi)|)|u'(\xi)|^{n+1} n_3\right). \end{aligned} \tag{3.34}$$

Therefore,

$$\begin{aligned} K_3^* = L\Omega^{-1} \left(\Omega(1) + (|u'(\rho)|)^4 n_2 \varpi \left(F^{-1}\left(F(1)\right.\right.\right. \\ \left.\left.\left. +b(|u(\xi)|)|u'(\xi)|^{n+1} n_3\right)\right)\right) \\ F^{-1}\left(F(1) + b(|u'(\xi)|)|u'(\xi)|^{n+1} n_3\right). \end{aligned}$$

□

Theorem 8. Suppose all the conditions of Theorem 5 and those of Theorem 6 hold true. Then, the equation

$$u''(t) + np(t)Q(t, u(t))(u'(t)) + q(t)f(u(t)) = 0, \tag{3.35}$$

has the Hyers-Ulam stability and Hyers-Ulam constant:

$$K_4^* = \left(L + u''(\xi)L\right)\Omega^{-1}\left(\Omega(1) + n\lambda^2 n_4\right). \tag{3.36}$$

Proof. Put $P(t, u(t), u'(t)) = 0$, in inequality (2.2), multiply by $u'(t)$, integrate twice and apply Lemma 2 to obtain

$$\begin{aligned} -u'(t)\epsilon \leq u''(t)u'(t) + np(t)Q(t, u(t))(u'(t))u'(t) \\ +q(t)f(u(t))u'(t) \leq u'(t)\epsilon. \end{aligned} \tag{3.37}$$

Integrate inequality (3.37) twice, apply Lemma 2 and equation (3.5) to get

$$-\epsilon t\int_{t_0}^{t}u'(s)ds\le t\int_{t_0}^{t}u'(s)u''(s)ds+nt\int_{t_0}^{t}p(s)Q(s,u(s))(u'(s))^2ds$$
$$+t\int_{t_0}^{t}q(s)\frac{d}{ds}F(u(s))ds\le\epsilon t\int_{t_0}^{t}u'(s)ds. \tag{3.38}$$

Divide through by t, integrate and note that for $q(t)$ nondecreasing, $q'(t)\ge 0$
the application of Theorem 2 implies there exists $\xi\in[t_0,t]$ such that

$$\begin{aligned}q(t)F(u(t))\le\epsilon\int_{t_0}^{t}u'(s)ds-u''(\xi)\int_{t_0}^{t}u'(s)ds\\ -n\int_{t_0}^{t}p(s)Q(s,u(s))(u'(s))^2ds\end{aligned} \tag{3.39}$$

and using condition (i') of Theorem 6 together with condition (ii) of Theorem 5 gives

$$\begin{aligned}q(t)|F(u(t))|\le\epsilon\left(\int_{t_0}^{t}|u'(s)|ds+u''(\xi)\int_{t_0}^{t}|u'(s)|ds\right)\\ +n\int_{t_0}^{t}p(s)v(s)\alpha(|u(s)|)(|u'(s)|)^2ds.\end{aligned} \tag{3.40}$$

For $t>0$, use condition (viii) of Theorem 5 to get

$$\begin{aligned}|u(t)|\le\epsilon\left(\int_{t_0}^{t}|u'(s)|ds+u''(\xi)\int_{t_0}^{t}|u'(s)|ds\right)\\ +n(|u'(t)|)^2\int_{t_0}^{t}p(s)v(s)\alpha(|u(s)|)ds.\end{aligned} \tag{3.41}$$

Applying Theorem 1, we obtain

$$\begin{aligned}|u(t)|\le\epsilon\left(\int_{t_0}^{t}|u'(s)|ds+u''(\xi)\int_{t_0}^{t}|u'(s)|ds\right)\\ \Omega^{-1}\left(\Omega(1)+n(|u'(t)|)^2\int_{0}^{t}p(s)v(s)ds\right),\ 0<t\le b.\end{aligned} \tag{3.42}$$

Setting $|u'(t)|\le\lambda$ and applying condition (ii') of Theorem 6, we arrive at

$$|u(t)|\le\epsilon\left(L+u''(\xi)L\right)\Omega^{-1}\left(\Omega(1)+n\lambda n_4\right). \tag{3.43}$$

Hence,

$$|u(t)-u(t_0)|\le|u(t)|\le K_4^*\epsilon$$

with

$$K_4^*=\left(L+u''(\xi)L\right)\Omega^{-1}\left(\Omega(1)+n\lambda^2n_4\right).$$

□

Example 1. Consider the following equation

$$\left(\frac{1}{t^4}u^4(t)(u'(t))^2\right) + \frac{1}{t^2}u^2(t)(u'(t))^4 + t^2u^2(t) = \frac{1}{t^6}u^2(t)(u'(t))^8, \ \ t > 0,$$

where $K_1(u(t), u'(t)) = u^4(t)(u'(t))^2$, $K_2(u(t), u'(t)) = u^2(t)(u'(t))^3$, $q(t)f(u(t)) = t^2u^2(t)$, $P(t, u(t), u'(t)) = \frac{1}{t^6}u^2(t)(u'(t))^8$, $r(t) = \frac{1}{t^4}$, $p(t) = \frac{1}{t^2}$, $\phi(t) = \frac{1}{t^6}$. By the criteria of Theorem 5, we arrive at the result.

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