Journal of the Nigerian Mathematical Society
Vol. 42, Issue 2, pp. 67 - 95, 2023


# TWO-STEP SECOND-DERIVATIVE BLOCK HYBRID METHODS FOR THE INTEGRATION OF INITIAL VALUE PROBLEMS

DAUDA G. YAKUBU[1], ADELEGAN L. MOMOH[2], GEOFFREY M. KUMLENG[3] AND ALI SHOKRI[4]

ABSTRACT. One-step collocation and multistep collocation have recently emerged as powerful tools for the derivation of numerical methods for ordinary differential equations. The simplicity and the continuous nature of the collocation process have been the main attraction towards this development. In this paper we exploited some of these qualities of collocation to derive continuous block hybrid collocation methods based on collocation at some polynomial nodes inside the symmetric integration interval and the two end points of the interval for dense output and for application which favor continuous approximations, like stiff and highly oscillatory initial value problem in ordinary differential equations. The analysis of the block hybrid collocation methods show that they are convergent and provide dense output at all interior selected points of integration within the interval of choice. Preliminary numerical computation carried out is an evidence of better performance of the methods compare to some integrators with strong properties of algebraic stability. Many examples are used to illustrate these properties.
.

**Keywords and phrases:** Continuous scheme, Multistep collocation, Oscillatory system, Second derivative method, Stiff differential system.
2010 Mathematical Subject Classification: 34M10,35B05,35B35,65L05

Received by the editors July 12, 2022; Revised: April 05, 2023; Accepted: April 12, 2023
www.nigerianmathematicalsociety.org; Journal available online at https://ojs.ictp.it/jnms/

## 1. Introduction

We consider collocation using combined off-grid points and grid points to construct second-derivative block hybrid collocation methods for continuous integration of stiff and highly oscillatory system of initial value problem in ordinary differential equations (ODEs) of the form

$$y'(x) = f(x, y(x)),\ (x_0 \leq x \leq b), \tag{1}$$

with initial condition

$$y(x_0) \ = \ y_0,$$

which has a unique solution if $f : R \times R^N \to R$ is sufficiently differentiable in $y : R^N \to R^N$. Let $f(x, y)$ and $g(x, y)$ be first and second derivatives of two variables $(x, y)$. Also let $L_1$ and $L_2$ be two positive constants called the Lipschitz constants, so that $f(x, y)$ and $g(x, y)$ satisfy the Lipschitz condition with respect to $y$, that is,

$$|f(x, y_1) - f(x, y_2)| \leq L_1|y_1 - y_2|,$$

$$|g(x, y_1) - g(x, y_2)| \leq L_2|y_1 - y_2|,$$

for any $y_1$ and $y_2$. Thus, with these methods the numerical integration of stiff and highly oscillatory problems for which the evaluation of the second derivative $g(x, y)$ costs more than the first derivative $f(x, y)$ can be classified into two: the first consists of methods with constant coefficients which can be applied to problems with periodic solutions. The second having coefficients depending on the frequency of the problem, when a good estimate of the frequency is known in advance, it can be used to track the oscillations, [1, 2]. Methods based on the Lobatto points collocation are derived which are shown to be suitable for finding approximate solution of stiff systems with large Lipschitz constants. Lobatto integration methods are the useful templates for the calculation of stiff and highly oscillatory problems, as they are the most accurate methods, specifically including the two end points of the integration interval in addition to the interior collocation points, therefore minimize the number of internal function evaluations necessary to achieve a given order of accuracy [3]. Acceptable stability for stiff and highly oscillatory problem is retained by requiring the last stage of a step to be identical to the output value. This means that the last row of the coefficient of the dense matrix is identical to the vector of output value [4]. An improved efficiency may also be achieved by the numerical integration methods which take advantage of the second derivative terms while retaining the implicitness structure. In fact, the inclusion of the second derivative terms provide a strategy to derive high order methods which require considerably fewer function evaluations than are required in the conventional implicit methods [5, 6]. Further, our reason for adding the second derivative function $y''(x) = f'(x, y(x)) = g(x, y(x))$ first proposed by [7] into the collocation method is in contrast to Urabeâ2122s idea who added the second derivative term to get additional member to start his implicit method. In this paper the idea is to obtain high order methods with better stability properties than the conventional methods. The class of methods in the paper has a single continuous scheme, which on evaluation at both grid and off-grid points provides sufficiently accurate simultaneous block methods for dense output. The new class of methods computes a block of values simultaneously with each step of integration and copes effectively with linear and nonlinear differential equations. Extensive numerical tests which we report in this paper, show that the class of collocation methods achieve great accuracy for most problems than the standard implicit methods and require significantly less computational cost.

**Definition 1.1:** [8] *Let $Y_m$ and $F_m$ be vectors defined by*

$$Y_m = (y_n, y_{n+1}, ..., y_{n+r-1}),$$
$$F_m = (f_n, f_{n+1}, ..., f_{n+r-1}).$$

*Then a general k-block, r-point block method is a matrix of finite difference equation of the form*

$$Y_m = \sum_{i=1}^{k} A_i Y_{m-1} + \sum_{i=0}^{k} B_i F_{m-1} \tag{2}$$

*where all the $A_i^{'}s$* and *$B_i^{'}s$ are properly chosen $r \times r$ matrix coefficients and $m$ = 0, 1,2,3,..., represents the block number, $n = rm$ is the first step number of the form $m^{th}$ block and r is the proposed block size. If r = 1, then the method becomes the familiar classical k-step method. When* $B_0 = 0$ *obviously (2) is an explicit method. But when $B_0$ is not equal to zero then the method is implicit.*

## 2. The collocation Technique

For the solution of (1) based on Taylor series method of expansion we consider a polynomial of the form in Yakubu and Markus [9] given by,

$$y(x) = \sum_{j=0}^{t-1} \phi_j(x) y_{n+j} + h \sum_{j=0}^{m-1} \psi_j(x) f_{n+j} + h^2 \sum_{j=0}^{s-1} \gamma_j(x) g_{n+j} \tag{3}$$

where $m$ and $s$ denote distinct collocation points and $t$ denotes the interpolation point used, $h$ can be varied but for the sake of simplicity in this paper it is kept fixed (constant) and $\phi_j(x)$, $\psi_j(x)$ and $\gamma_j(x)$ are continuous coefficients. The values $y_{n+j}$, $f_{n+j}$ and $g_{n+j}$ are smooth real N-dimensional vector functions and are explicitly defined by,

$$y_{n+j} = y(x_n + jh), \qquad j \in \{0,1,2,...,t-1\}, \tag{4}$$
$$y'_{n+j} = f_{n+j} = f(x_n + jh, y(x_n + jh)), \qquad (j = 0,1,2,...,m-1), \tag{5}$$
$$y''_{n+j} = g_{n+j} = f_x + f_y y' = f_x + f f_y, \qquad (j = 0,1,2,...,s-1). \tag{6}$$

We can now specify the values of the continuous coefficients as assumed polynomials of the form

$$\phi_j(x) = \sum_{i=0}^{t+m+s-1} \phi_{j,i+1} x^i, \quad j \in \{0,1,2,...,t-1\}, \tag{7}$$
$$h\psi_j(x) = h \sum_{i=0}^{t+m+s-1} \psi_{j,i+1} x^i, \quad j = 0,1,2,...,m-1, \tag{8}$$
$$h^2\gamma_j(x) = h^2 \sum_{i=0}^{t+m+s-1} \gamma_{j,i+1} x^i, \quad j = 0,1,2,...,s-1, \tag{9}$$

which can be determined. The coefficients $\phi_{j,i+1}$,$\psi_{j,i+1}$ and $\gamma_{j,i+1}$ are selected so that accurate approximate solution of well-behaved problem can be obtained efficiently. Now we define the basis function $x^i$ in (7), (8) and (9) mathematically as

$$W_i(x) = \sum_{i=0}^{p-1} x^i \tag{10}$$

where $p = t + m + s$.
Inserting equation (7), (8), (9) into equation (3) to have

$$y(x) = \sum_{j=0}^{t-1}\sum_{i=0}^{p-1} \phi_{j,i+1} W_i(x) y_{n+j} + \sum_{j=0}^{m-1}\sum_{i=0}^{p-1} h\psi_{j,i+1} W_i(x) f_{n+j}$$
$$+ \sum_{j=0}^{s-1}\sum_{i=0}^{p-1} h^2\gamma_{j,i+1} W_i(x) g_{n+j} \tag{11}$$

Factoring out we have the following representation

$$y(x) = \sum_{i=0}^{p-1} \left( \sum_{j=0}^{t-1} \phi_{j,i+1} y_{n+j} + \sum_{j=0}^{m-1} h\psi_{j,i+1} f_{n+j} + \sum_{j=0}^{s-1} h^2\gamma_{j,i+1} g_{n+j} \right) W_i(x) \tag{12}$$

Expanding(12) fully we get

$$\begin{aligned} y(x) &= \sum_{j=0}^{t-1} \phi_{j,t+m+s-1} y_{n+j} + \sum_{j=0}^{m-1} h\psi_{j,t+m+s-1} f_{n+j} + \\ &\sum_{j=0}^{s-1} h^2 \gamma_{j,t+m+s-1} g_{n+j} \left(1, x, x^2, ..., x^{t+m+s-1}\right). \end{aligned} \tag{13}$$

Finally, from the expansion of equation (13) we have the propose continuous scheme as,

$$\begin{aligned} y(x) &= (y_n, y_{n+1}, ..., y_{n+t-1}, f_n, f_{n+1}, ..., f_{n+m-1}, g_n, g_{n+1}, ..., g_{n+s-1}) \\ & C^T \times \left(1, x, x^2, ..., x^{t+m+s-1}\right)^T \end{aligned} \tag{14}$$

where T in equation (14) is the transpose of the basis function in (10) and the matrix C given as,

$$C = \begin{pmatrix} \phi_{1,0} & \cdots & \phi_{1,t-1} & h\psi_{1,0} & \cdots & h\psi_{1,m-1} & h^2\gamma_{1,0} & \cdots & h^2\gamma_{1,s-m} \\ \phi_{2,0} & \cdots & \phi_{2,t-1} & h\psi_{2,0} & \cdots & h\psi_{2,m-1} & h^2\gamma_{2,0} & \cdots & h^2\gamma_{2,s-m} \\ \phi_{3,0} & \cdots & \phi_{3,t-1} & h\psi_{3,0} & \cdots & h\psi_{3,m-1} & h^2\gamma_{3,0} & \cdots & h^2\gamma_{3,s-m} \\ \phi_{4,0} & \cdots & \phi_{4,t-1} & h\psi_{4,0} & \cdots & h\psi_{4,m-1} & h^2\gamma_{4,0} & \cdots & h^2\gamma_{4,s-m} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \phi_{p-1,0} & \cdots & \phi_{p-1,t-1} & h\psi_{p-1,0} & \cdots & h\psi_{p-1,m-1} & h^2\gamma_{p-1,0} & \cdots & h^2\gamma_{p-1,s-m} \end{pmatrix} \equiv D^{-1}. \tag{15}$$

We call D in (15) the multistep collocation and interpolation matrix which has a simple structure and of square dimension of the form $(t+m+s)\times(t+m+s)$ . Clearly from equation (15) the columns of C which give the continuous coefficients can be obtained from the corresponding columns of $D^{-1}$. More so the entries of C are the constant coefficients of the polynomial given in equation (3). Further, the matrix D is assumed to be non-singular for the existence of the inverse matrix C. An efficient algorithm to determine the elements of the inverse matrix C is found in [10] page 41. Based on the derivation techniques discussed above we now state the main result in the theorem below.

**Theorem 1.1:** *Let the collocation points $\dot{c}_j,\ j = 1, 2, ..., m-1,\ \ddot{c}_j,\ j = 1, 2, ..., s-1$, for the first and second derivatives respectively be distinct and consider the polynomial basis function $x^i$ defined in (10). Then the matrix D is non-singular, hence invertible.*

**Proof:** To proof that the matrix D is non-singular, we show that $C = D^{-1}$ as in equation (15), where the matrix D is defined as,

$$D = \begin{pmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & \cdots & x_n^{p-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n+r-1} & x_{n+r-1}^2 & x_{n+r-1}^3 & x_{n+r-1}^4 & \cdots & x_{n+r-1}^{p-1} \\ 0 & 1 & 2x_n & 3x_n^2 & 4x_n^3 & \cdots & D'x_n^{p-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & 2x_{n+s-1} & 3x_{n+s-1}^2 & 4x_{n+s-1}^3 & \cdots & D'x_{n+s-1}^{p-2} \\ 0 & 0 & 2 & 6x_n & 12x_n^2 & \cdots & D''x_n^{p-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 2 & 6x_{n+t-1} & 12x_{n+t-1}^2 & \cdots & D''x_{n+t-1}^{p-3} \end{pmatrix},$$

and the values $D' = (p-1)$ and $D'' = (p-1)(p-2)$ which appeared in the matrix D are the first and second differentials respectively. They correspond to the usual differentiation with respect to x. We also define

$$M = (y_n, ..., y_{n+t-1}, f_n, ..., f_{n+m-1}, g_n, ..., g_{n+s-1}).$$

Substituting equation (7)(8) and (9) into equation (3) to have

$$y(x) = \sum_{j=0}^{t-1}\sum_{i=0}^{p-1} \phi_{j,i+1} W_i(x) y_{n+j} + \sum_{j=0}^{m-1}\sum_{i=0}^{p-1} h\psi_{j,i+1} W_i(x) f_{n+j} + \sum_{j=0}^{s-1}\sum_{i=0}^{p-1} h^2 \gamma_{j,i+1} W_i(x) g_{n+j} \quad (16)$$

where p=t+m+s as defined in equation (10).
Factoring out we have the following representation

$$y(x) = \sum_{i=0}^{p-1} \left\{ \sum_{j=0}^{t-1} \phi_{j,i+1} y_{n+j} + \sum_{j=0}^{m-1} h\psi_{j,i+1} f_{n+j} + \sum_{j=0}^{s-1} h^2 \gamma_{j,i+1} g_{n+j} \right\} W_i(x). \quad (17)$$

If we let

$$F_i = \sum_{j=0}^{t-1} \phi_{j,i+1} y_{n+j} + \sum_{j=0}^{m-1} h\psi_{j,i+1} f_{n+j} + \sum_{j=0}^{s-1} h^2 \gamma_{j,i+1} g_{n+j} \quad (18)$$

then (17) becomes

$$y(x) = \sum_{i=0}^{p-1} F_i W_i(x). \quad (19)$$

Now consider (19) in the vector form as follows,

$$y(x) = (F_0, F_1, F_2, ..., F_{p-1})^T (W_0(x), W_1(x), W_2(x), ..., W_{p-1}(x))^T$$
$$= F_i^T (W_i(x))^T. \quad (20)$$

Inserting equations (4), (5) and (6) into equation (20) to have,

$$\sum_{i=0}^{p-1} \phi_i W_i(x_{n+j}) = y_{n+j} \; = \; y(x_n + jh), \; j \in \{0, 1, 2, ..., t-1\}, \quad (21)$$

$$\sum_{i=0}^{p-1} \psi_i W_i(x_{n+j}) \; = \; f_{n+j} = f(x_n + jh, y(x_n + jh)), \; (j = 0, 1, 2, ..., m-1), \quad (22)$$

$$\sum_{i=0}^{p-1} \gamma_i W_i(x_{n+j}) \; = \; g_{n+j} = f_x + f_y y' = f_x + f f_y, \; (j = 0, 1, 2, ..., s-1). \quad (23)$$

giving

$$DF = M. \quad (24)$$

Assuming that the matrix D is non-singular, then from equation (24) we have

$$F = D^{-1} M. \quad (25)$$

Putting equation (25) into equation (20) and recall that $p$=$t$+$m$+$s$ we get the propose continuous scheme of the multi-step collocation formula (3) written exactly as in equation (14)

$$y(x) = M^T (D^{-1})^T (W_i(x))^T$$

$$y(x) \; = \; (y_n, y_{n+1}, ..., y_{n+t-1}, f_n, f_{n+1}, ..., f_{n+m-1}, g_n, g_{n+1}, ..., g_{n+s-1})^T$$
$$C^T \times \left(1, x, x^2, ..., x^{t+m+s-1}\right)^T \quad (26)$$

Expanding (18) fully we get

$$F_i = (\phi_{i+1,0}, ..., \phi_{i+1,t-1}, h\psi_{i+1,0}, ..., h\psi_{i+1,m-1}, h^2\gamma_{i+1,0}, ..., h^2\gamma_{i+1,s-1}) \times$$
$$M, \; i = 0, 1, 2, ..., p-1. \quad (27)$$

Comparing the right-hand side of (27) and (20) we have

$$F_i = C_{i+1} M. \quad (28)$$

Therefore

$$F^T = (CM)^T \; or \; (F = CM). \quad (29)$$

From (29) and (25)

$$C = D^{-1}. \tag{30}$$

This implies that the matrix D is invertible, which proves the Theorem. QED

## 3. The Specification of the Second Derivative Block Hybrid (SDBH)Collocation Methods

3.1. **Block hybrid method of order three with two interior collocation points.** Here we determine the coefficients of the block hybrid collocation method using the multistep collocation formula in (3). To do this, we consider the multistep collocation formula (3) for the construction of the type of continuous scheme in (14), for the second derivative block hybrid collocation method of order three. We introduce two off-grid points in between the usual grid points or integer points and denote the points by $u$ and $v$ obtained from the Chebychev polynomial of degree 2. The Chebychev polynomials were chosen because of their superior convergence rate and stiffly accurate characteristic properties in relation to the approximation of functions (see, [11, 12, 13]). The off-grid points were carefully chosen to guarantee the zero stability of the block hybrid collocation method (see, [14]) for accurate solution of linear and nonlinear stiff and highly oscillatory differential equations. Proceeding as is done in multistep implicit method for high-derivative evaluations (see, [15, 16]), from (3) we choose the specific cases of the values of the variables $t = 1,\ m = 2$ and $s = 2$, to obtain a polynomial of the form

$$y(x) = \phi_0(x)y_n + h\sum_{j=1}^{m-1}\psi_j(x)f_{n+i} + h^2\sum_{j=1}^{t-1}\gamma_j(x)g_{n+i},\ i = u, v \tag{31}$$

Expandng (31) fully, we have;

$$y(x) = \phi_0(x)y_n + h[\psi_0(x)f_{n+u} + \psi_1(x)f_{n+v}] + h^2[\gamma_0(x)g_{n+u} + \gamma_1(x)g_{n+v}] \tag{32}$$

Evaluating the proposed continuous scheme in (32) at the two zeros of Chebychev polynomial denoted by $x_{n+u}, x_{n+v}$ and at the only grid point $x_{n+2}$ we get the first block hybrid collocation method which the coefficients are presented in Table 1 below,

**Table 1:*Coefficients of the block hybrid method (32) of third order***

| | $\psi_0(x)$ | $\psi_1(x)$ | $\gamma_0(x)$ | $\gamma_1(x)$ |
|---|---|---|---|---|
| $y_{n+u}$ | $\frac{48-15\sqrt{2}}{96}$ | $\frac{48-33\sqrt{2}}{96}$ | $\frac{-11+4\sqrt{2}}{96}$ | $\frac{5-4\sqrt{2}}{96}$ |
| $y_{n+v}$ | $\frac{48+33\sqrt{2}}{96}$ | $\frac{48+15\sqrt{2}}{96}$ | $\frac{5+4\sqrt{2}}{96}$ | $\frac{-11-4\sqrt{2}}{96}$ |
| $y_{n+2}$ | $\frac{12}{12}$ | $\frac{12}{12}$ | $\frac{\sqrt{2}}{12}$ | $-\frac{\sqrt{2}}{12}$ |

3.2. **Block hybrid method of order five with collocation at three interior points. In the second block hybrid collocation method we consider a method with order slightly higher than the above method. To determine the coefficients of the method with second derivative evaluation using the multistep collocation formula in (3) we have to include equal number of collocation points for both the first and the second derivatives inside the symmetric interval with only one interpolation point at $x_n$. We consider the multistep collocation formula in (3) to obtain the type of continuous scheme in (14). We introduce two off-grid points $x_{n+u}, x_{n+v}$ and two grid points $x_{n+w}, x_{n+2}$ obtained from the zeros of Chebychev polynomial of degree 2 except the points $x_{n+w}$ and $x_{n+2}$. These are carefully chosen to guarantee the zero stability of the block hybrid collocation method (see, [14]) for accurate solution of linear and nonlinear stiff and highly oscillatory differential equations. From (3) the specific values of the limits for *t=1, m=3* and *s=3*, which will lead us to obtain a polynomial of the form**

$$y(x) = \phi_0(x)y_n + h\sum_{j=1}^{m-1}\psi_j(x)f_{n+i} + h\psi_2(x)f_{n+v}$$
$$+h^2\sum_{j=1}^{s-1}\gamma_j(x)g_{n+i} + h^2\gamma_2(x)g_{n+v}, \qquad i = u, w \tag{33}$$

**Expanding (33) fully, for the construction of the continuous scheme of the form in (14) or (26) so that we have;**

$$y(x) = \phi_0(x)y_n + h[\psi_0(x)f_{n+u} + \psi_1(x)f_{n+w} + \psi_2(x)f_{n+v}]$$
$$+h^2[\gamma_0(x)g_{n+u} + \gamma_1(x)g_{n+w} + \gamma_2(x)g_{n+v}] \tag{34}$$

**Evaluating the proposed continuous scheme in (34) at $x = x_{n+u}, x = x_{n+w}, x = x_{n+v}$ and $x = x_{n+2}$ to get the block hybrid collocation method of order five and display the coefficients in a Tabular form as follows:**

**Table 2:*****Coefficients of the block hybrid collocation method (34) of fifth-order***

| | $\psi_0(x)$ | $\psi_1(x)$ | $\psi_2(x)$ | $\gamma_0(x)$ | $\gamma_1(x)$ | $\gamma_2(x)$ |
|---|---|---|---|---|---|---|
| $y_{n+u}$ | $\frac{128-41\sqrt{2}}{480}$ | $\frac{224-128\sqrt{2}}{480}$ | $\frac{128-71\sqrt{2}}{480}$ | $\frac{-33+8\sqrt{2}}{480}$ | $-\frac{40}{480}$ | $\frac{-17+8\sqrt{2}}{480}$ |
| $y_{n+w}$ | $\frac{32+15\sqrt{2}}{120}$ | $\frac{56}{120}$ | $\frac{32-15\sqrt{2}}{120}$ | $\frac{-5-2\sqrt{2}}{120}$ | $-\frac{20}{120}$ | $\frac{-5+2\sqrt{2}}{120}$ |
| $y_{n+v}$ | $\frac{128+71\sqrt{2}}{480}$ | $\frac{224+128\sqrt{2}}{480}$ | $\frac{128+41\sqrt{2}}{480}$ | $\frac{-17-8\sqrt{2}}{480}$ | $-\frac{40}{480}$ | $\frac{-33+8\sqrt{2}}{480}$ |
| $y_{n+2}$ | $\frac{16}{30}$ | $\frac{28}{30}$ | $\frac{16}{30}$ | $-\frac{\sqrt{2}}{30}$ | $\mathbf{0}$ | $\frac{\sqrt{2}}{30}$ |

3.3. **Block hybrid method of order seven with Lobatto like collocation points. Here we include the two end points of the integration interval as collocation points in addition to the interior collocation points just like the Lobatto collocation approach. We consider the multistep collocation formula in (3) for the construction of the type of continuous scheme in (14) or (26). We introduce two off-grid points $x_{n+u}, x_{n+v}$ and two grid points $x_{n+w}, x_{n+2}$ obtained from the zeros of the Chebychev polynomial of degree 2 except the points $x_{n+w}$ and $x_{n+2}$. These are carefully chosen to guarantee the zero stability of the block hybrid collocation method, (see, [14]) for accurate solution of stiff and highly oscillatory differential equations. Thus, from (3) the specific values for $t = 1, m = 4$ and $s = 4$, which will yield a polynomial of the form**

$$y(x) = \phi_0(x)y_n + h\sum_{j=2}^{m-2}\psi_j(x)f_{n+i} + h\psi_3(x)f_{n+1}$$

$$+h^2\sum_{j=2}^{s-2}\gamma_j(x)g_{n+i} + h^2\gamma_3(x)g_{n+1}, \quad i = 0, u, v. \tag{35}$$

**Expanding (35) fully, for the construction of the continuous scheme of the form in (14) we get;**

$$y(x) = \phi_0(x)y_n + h[\psi_0(x)f_n + \psi_1(x)f_{n+u} + \psi_2(x)f_{n+v} + \psi_3(x)f_{n+1}]$$

$$+h^2[\gamma_0(x)g_n + \gamma_1(x)g_{n+u} + \gamma_2(x)g_{n+v} + \gamma_3(x)g_{n+1}] \tag{36}$$

**Evaluating the proposed continuous scheme in (36) at $x_{n+u}, x_{n+w}, x_{n+v}$ and $x_{n+2}$ to get the order seven block hybrid collocation method and the coefficients are compactly displayed in Table 3:**

**Table 3:** ***Coefficients of the block hybrid collocation method(36) of seventh-order***

| | $\psi_0(x)$ | $\psi_1(x)$ | $\psi_2(x)$ | $\psi_3(x)$ |
|---|---|---|---|---|
| $y_{n+u}$ | $\frac{2844-1408\sqrt{2}}{6720}$ | $\frac{1448-245\sqrt{2}}{6720}$ | $\frac{424-299\sqrt{2}}{6720}$ | $\frac{2004-1408\sqrt{2}}{6720}$ |
| $y_{n+w}$ | $\frac{134}{420}$ | $\frac{76+52\sqrt{2}}{420}$ | $\frac{76-52\sqrt{2}}{420}$ | $\frac{134}{420}$ |
| $y_{n+v}$ | $\frac{2844+1408\sqrt{2}}{6720}$ | $\frac{424+299\sqrt{2}}{6720}$ | $\frac{1448+245\sqrt{2}}{6720}$ | $\frac{2004+1408\sqrt{2}}{6720}$ |
| $y_{n+2}$ | $\frac{242}{105}$ | $\frac{-32-44\sqrt{2}}{105}$ | $\frac{-32+44\sqrt{2}}{105}$ | $\frac{32}{105}$ |

| | $\gamma_0(x)$ | $\gamma_1(x)$ | $\gamma_2(x)$ | $\gamma_3(x)$ |
|---|---|---|---|---|
| $g_{n+u}$ | $\frac{214-128\sqrt{2}}{6720}$ | $\frac{-109+22\sqrt{2}}{6720}$ | $\frac{755-534\sqrt{2}}{6720}$ | $\frac{358-256\sqrt{2}}{6720}$ |
| $g_{n+w}$ | $\frac{9}{420}$ | $\frac{18+13\sqrt{2}}{420}$ | $\frac{18-13\sqrt{2}}{420}$ | $-\frac{17}{420}$ |
| $g_{n+v}$ | $\frac{214+128\sqrt{2}}{6720}$ | $\frac{755+534\sqrt{2}}{6720}$ | $\frac{-109-22\sqrt{2}}{6720}$ | $\frac{358+256\sqrt{2}}{6720}$ |
| | $\frac{22}{105}$ | $\frac{44+24\sqrt{2}}{105}$ | $\frac{44-24\sqrt{2}}{105}$ | $\frac{44}{105}$ |

## 4. Analysis of the Properties of SDBH Collocation Methods

### 4.1. Zero-stability, order of accuracy, consistency and convergence of the methods.

**Writing the methods in block form using the matrix difference equation we have:**

$$U^{(1)}Y_\mu = U^{(0)}Y_{\mu-1}+h\left(V^{(0)}F_{\mu-1}+V^{(1)}F_\mu\right)+h^2\left(W^{(0)}G_{\mu-1}+W^{(1)}G_\mu\right) \tag{37}$$

**where**

$$Y_\mu = (y_{n+u}, y_{n+w}, y_{n+v}, y_{n+1})^T,\ Y_{\mu-1} = (y_{n-u}, y_{n-w}, y_{n-v}, y_n)^T$$

$$F_\mu = (f_{n+u}, f_{n+w}, f_{n+v}, f_{n+1})^T,\ F_{\mu-1} = (f_{n-u}, f_{n-w}, f_{n-v}, f_n)^T$$

$$G_\mu = (g_{n+u}, g_{n+w}, g_{n+v}, g_{n+1})^T,\ G_{\mu-1} = (g_{n-u}, g_{n-w}, g_{n-v}, g_n)^T$$

**and the matrices $U^{(1)}$,$U^{(0)}$,$V^{(1)}$,$V^{(0)}$, $W^{(1)}$ and $W^{(0)}$ are square matrices with their corresponding entries given by the coefficients of the methods in (32), (34) and (36) as indicated. In particular the matrices are given as below where $U^{(1)}$ is the matrix identity and of dimension 3. Thus, from method (32) of order three we obtain the following,**

$$U^{(1)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad U^{(0)} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix},$$

$$V^{(0)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad V^{(1)} = \begin{bmatrix} \frac{48-15\sqrt{2}}{96} & \frac{48-33\sqrt{2}}{96} & 0 \\ \frac{48+33\sqrt{2}}{96} & \frac{48+15\sqrt{2}}{96} & 0 \\ \frac{12}{12} & \frac{12}{12} & 0 \end{bmatrix},$$

$$W^{(0)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad W^{(1)} = \begin{bmatrix} \frac{-11+4\sqrt{2}}{96} & \frac{5-4\sqrt{2}}{96} & 0 \\ \frac{5+4\sqrt{2}}{96} & \frac{-11-4\sqrt{2}}{96} & 0 \\ \frac{\sqrt{2}}{12} & \frac{-\sqrt{2}}{12} & 0 \end{bmatrix}.$$

**For the implicit block hybrid collocation method (36) with coefficients in Table 3 we have the matrices $U^{(1)}$,$U^{(0)}$,$V^{(1)}$,$V^{(0)}$, $W^{(1)}$ and $W^{(0)}$ as follows:**

$$U^{(1)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad U^{(0)} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{38}$$

$$V^{(0)} = \begin{bmatrix} 0 & 0 & 0 & \frac{2844-1408\sqrt{2}}{6720} \\ 0 & 0 & 0 & \frac{134}{420} \\ 0 & 0 & 0 & \frac{2844+1408\sqrt{2}}{6720} \\ 0 & 0 & 0 & \frac{242}{105} \end{bmatrix}$$

,

$$V^{(1)} = \begin{bmatrix} \frac{1448-245\sqrt{2}}{6720} & \frac{424-299\sqrt{2}}{6720} & \frac{2004-1408\sqrt{2}}{6720} & 0 \\ \frac{76+52\sqrt{2}}{420} & \frac{76-52\sqrt{2}}{420} & \frac{134}{420} & 0 \\ \frac{424+299\sqrt{2}}{6720} & \frac{1448+245\sqrt{2}}{6720} & \frac{2004+1408\sqrt{2}}{6720} & 0 \\ \frac{-32-44\sqrt{2}}{105} & \frac{-32+44\sqrt{2}}{105} & \frac{32}{105} & 0 \end{bmatrix},$$

$$W^{(0)} = \begin{bmatrix} 0 & 0 & 0 & \frac{214-128\sqrt{2}}{6720} \\ 0 & 0 & 0 & \frac{3}{140} \\ 0 & 0 & 0 & \frac{214+128\sqrt{2}}{6720} \\ 0 & 0 & 0 & \frac{22}{105} \end{bmatrix},$$

$$W^{(1)} = \begin{bmatrix} \frac{-109+22\sqrt{2}}{6720} & \frac{755-534\sqrt{2}}{6720} & \frac{358-256\sqrt{2}}{6720} & 0 \\ \frac{18+13\sqrt{2}}{420} & \frac{18-13\sqrt{2}}{420} & -\frac{17}{420} & 0 \\ \frac{755+534\sqrt{2}}{6720} & \frac{-109-22\sqrt{2}}{6720} & \frac{358+256\sqrt{2}}{6720} & 0 \\ \frac{44+24\sqrt{2}}{105} & \frac{44-24\sqrt{2}}{105} & \frac{44}{105} & 0 \end{bmatrix}.$$

**We note that the zero-stability is concerned with the stability of difference system in limit as $h$ tends to zero. Hence, as $h \to 0$, the method in (32) with coefficients in Table 1 tends to the difference system**

$$U^{(1)}Y_{\mu} - U^{(0)}Y_{\mu-1} \;=\; 0$$

**with the first characteristic polynomial $\rho(R)$ of the form**

$$\rho(R) = det\left(RU^{(1)} - U^{(0)}\right) \tag{39}$$

$$\rho(R) = \left( R \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \right) \tag{40}$$

$$\rho(R) = R^2(R-1) = 0 \quad R = (0,0,1) \tag{41}$$

**Since $R = (0,0,1)$ the method in (32) is zero-stable.**

**We can similarly calculate the zero-stability of method (34). Now let calculate the zero-stability of method (36) with coefficients displayed in Table 3 as follows. From equation (38) we have,**

$$\rho(R) = \left( R \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) \tag{42}$$

$$\rho(R) = R^3(R-1) = 0 \quad R = (0,0,0,1) \tag{43}$$

**Since $R = (0,0,0,1)$ the method in (36) is zero stable.**

**Definition 4.1:[17]** ***(Zero-stable) A family of methods in (3) are said to be zero-stable if the roots condition***

$$\rho(R) = det\left[\sum_{i=0}^{k} A^i \lambda^{k-1}\right] = 0 \tag{44}$$

**satisfies the inequality $|\lambda_j| \leq 1,\ j = 1,2,3,...,k.$** ***But when the inequality reduces to $|\lambda_j| = 1$, then the multiplicity should not be greater than two.***

**Based on the definition in 4.1, the newly derived second derivative block hybrid collocation methods are zero-stable.**

**From our calculations the orders and error constants for the constructed block hybrid collocation methods are presented in Table 4. It is clear from the Table that the second derivative block hybrid collocation methods are of high-order and hence**

**more accurate than some of the collocation methods in existence.**

**Table 1:Order and Error Constants of the SDBH Collocation Methods**

| Method | Order | Error constant |
|---|---|---|
| **Method (32)** | **(i)** $y_{n+2}$, **p = 3** | $C_4 = 4.8611 \times 10^{-3}$ |
| | **(ii)** $y_{n+u}$, **p = 3** | $C_4 = 9.3274 \times 10^{-4}$ |
| | **(iii)** $y_{n+v}$, **p = 3** | $C_4 = 8.7894 \times 10^{-3}$ |
| **Method (34)** | **(i)** $y_{n+2}$, **p = 5** | $C_6 = 3.6375 \times 10^{-5}$ |
| | **(ii)** $y_{n+u}$, **p = 5** | $C_6 = 2.7022 \times 10^{-5}$ |
| | **(iii)** $y_{n+w}$, **p = 5** | $C_6 = 4.5728 \times 10^{-5}$ |
| | **(iv)** $y_{n+v}$, **p = 5** | $C_6 = 7.2751 \times 10^{-5}$ |
| **Method (36)** | **(i)** $y_{n+2}$, **p = 7** | $C_8 = 6.8893 \times 10^{-8}$ |
| | **(ii)** $y_{n+u}$, **p = 7** | $C_8 = 3.2110 \times 10^{-9}$ |
| | **(iii)** $y_{n+w}$, **p = 7** | $C_8 = 3.9293 \times 10^{-7}$ |
| | **(iv)** $y_{n+v}$, **p = 7** | $C_8 = 2.2046 \times 10^{-6}$ |

**Definition 4.2:(Consistency)** ***The family of methods in (3) are consistent if the order of the method satisfy the inequality*** $p \geq 1$ **(see [17, 19]).**
**(i)**$\rho(1) = 0$ **and**
**(ii)**$\rho'(1) = \sigma(1)$**,** ***where*** $\rho(z)$ **and** $\sigma(z)$***are respectively the 1st and 2nd characteristic polynomials which are assumed to have no common factor.***

**From Table 4 and definition 4.1, we can attest that the second derivative block hybrid collocation methods are consistent.**

**Definition 4.3:[18](A**$(\alpha)$**-stable)** ***A family of methods in (3) are said to be*** $A(\alpha)$***-stable*** **with** $0 < \alpha \leq \pi/2$ ***and the angular sector***

$$S_\alpha = \{z \in C : |arg(-z)| < \alpha, z \neq 0\}$$

***is contained in the stability domain*** $\Re$.
***An interval [a,b] of the real line is said to be an interval of absolute stability if the method is absolutely stable for all*** $\bar{h} \in [a, b]$***.*** ***When the interval of absolute stability is consists of the negative real axis, the method is said to be*** $A_0$***-stable***

**Definition 4.4:[18]** ***The family of methods given in(3) are said to be normalized if the continuous coefficients*** $\phi_j(x)$ ***satisfy the following conditions***
**(i)**$\phi_j(x) = 1$ **and**

**(ii)**

$$|\phi_0| + \sum_{j=0}^{s-1} |\phi_{j,0}| \neq 0, \;\; j = 0, 1, 2, ..., s-1.$$

**These are some of the requirements for a new method to satisfy. Since the second derivative block hybrid collocation methods are consistent, zero-stable, hence they are convergent.**

**Theorem 2:[18]** ***The family of methods given in (3) which are for the numerical solution of the initial value problem (1) are convergent.***

**Proof: Since the methods are zero-stable and consistent by definitions 4.1 and 4.2, they are convergent.**

4.2. **Regions of absolute stability of the SDBH methods. Linear stability is necessary in the design of algorithm for the numerical integration of ordinary differential equations. For this reason, for the block hybrid collocation methods we consider the test equation,**

$$\frac{dy(t)}{dt} = \lambda y(t), \qquad \lambda \in C \; and \; \Re\lambda < 0, \tag{45}$$

**with a fixed step size** $h > 0$ **[20]. Since the block hybrid collocation methods contained the second derivative *g(x,y)*, it is natural to suppose that** $g(x, y) = \lambda^2 y$**. Therefore, to study the stability properties of the block hybrid collocation methods with second derivative evaluations we reformulate (32), (34) and (36) as general linear methods, (see [21]). Hence, we use the notations introduced in [22] where a general linear method is represented by the partitioned** $(s+r)\times(s+r)$ **matrices (containing *A, U, B and V*). Here for the sake of convenience, we replace the matrices U with C and V with D. These matrices are denoted in a simplified form as,**

$$A = (a_{ij})s, s, \;\; C = (c_{ij})s, r, \;\; B = (b_{ij})r, s, \;\; D = (d_{ij})r, r. \tag{46}$$

**The methods for the stage values and the output values using Kronecker product notations for an N-dimensional problem, gives**

$$\begin{aligned} Y &= e \oplus y_n + h(A \oplus I_N)F(Y) + h^2(\hat{A} \oplus I_N)G(Y), \\ y_{n+1} &= y_n + h(b^T \oplus I_N)F(Y) + h^2(\hat{b}^T \oplus I_N)G(Y), \end{aligned} \tag{47}$$

**where**

$$e = [1]_{s\times 1}, \;\; A = [a_{ij}]_{s\times s}, \;\; \hat{A} = [\hat{a}_{ij}]_{s\times s}, \;\; b = [b_i]_{s\times 1}, \;\; \hat{b} = [\hat{b}_i]_{s\times 1}$$

**and I is the identity matrix of size equal to the differential equation system to be solved and N is the dimension of the system. The block vectors in $R^{sN}$ are defined by**

$$Y = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_s \end{bmatrix}, \qquad F(Y) = \begin{bmatrix} f(Y_1) \\ f(Y_2) \\ \vdots \\ f(Y_s) \end{bmatrix}, \qquad G(Y) = \begin{bmatrix} g(Y_1) \\ g(Y_2) \\ \vdots \\ g(Y_s) \end{bmatrix}.$$

**The coefficients of these matrices A, C, B and D indicate the relationship between the various numerical quantities which arise in the computation of the region of absolute stability (RAS). The elements of the matrices A, C, B and D are substituted into the stability matrix, which yields the recurrent relation**

$$y^{[n-1]} = M(z)y^{[n]}, \qquad n \ = \ 1, 2, ..., N-1, \ z \ = \ \lambda h, \tag{48}$$

**where the stability matrix $M(z)$ is defined by**

$$M(z) = D + zB(1 - zA)^{-1}C.$$

**The stability polynomial $\rho(\eta, z)$ is also defined by the relation**

$$\rho(\eta, z) = det(\eta I - M(z)) \ = \ det(r(A - Cz - DIz^2) - B) \tag{49}$$

**where we obtain the stability polynomial (function) of the collocation method of order three with only two interior collocation points as**

$$R(z) = \frac{18 + 8z + z^2}{18 - 8z + z^2}$$

**and for z in the left-half complex plane we have $|R(z)| \leq 1$ showing that the method is *A-stable.* The following and the subsequent cases follow a study similar as the above. For the order five block hybrid collocation method the stability function is,**

$$R(z) = \frac{2880 - 2880z - 408z^2 + 312z^3 + 38z^4 - 10z^5 - z^6}{2880 + 2880z + 408z^2 - 312z^3 + 38z^4 + 10z^5 - z^6}$$

**Similarly, from the order seven block hybrid collocation method we have,**

$$R(z) = \frac{P(z)}{Q(z)}$$

**where**

$P(z) = 80640 - 100800z - 187824z^2 - 31944z^3 + 15624z^4 + 1278z^5 - 621z^6 + 22z^7 + 2z^8$

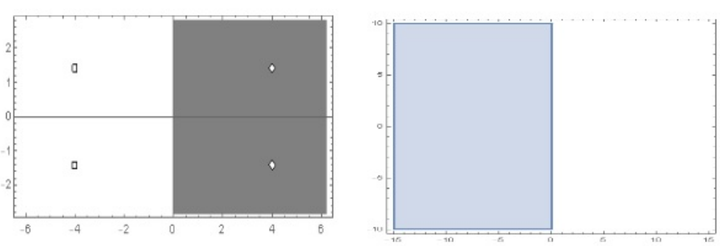
**and**

$Q(z) = 80640 + 60480z + 8400z^2 - 1704z^3 - 168z^4 + 30z^5 - z^6$

**The region of absolute stability (RAS) $\Re$ of the methods are given by**

$$\Re = x \in C : \rho(\eta, z) = 1 \rightarrow |\eta| \leq 1$$

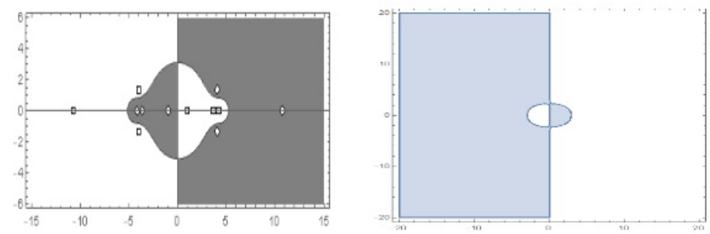
**which are plotted to produce the required graphs of RAS of the block hybrid collocation methods as shown in Figures 1, 2 and 3 respectively**



(A) Stability region with poles

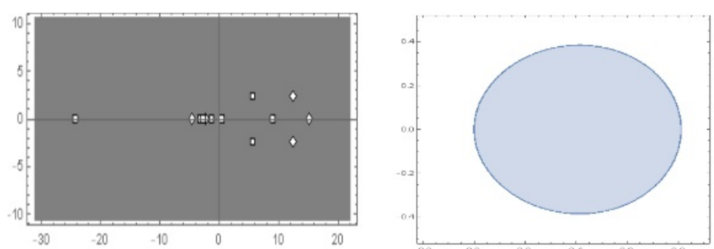
(B) Stability region without poles

FIGURE 1. Region of absolute stability of method (32) with poles inclusive.



(A) Stability region with poles (B) Stability region without poles

FIGURE 2. Region of absolute stability of method (34) with poles inclusive.



(A) Stability region with poles (B) Stability region without poles

FIGURE 3. Region of absolute stability of method (36) with poles inclusive.

**Remarks 1:** ***The graphical plots of the Regions of Absolute Stability for the methods obtained from collocation at the polynomial nodes are displayed in Figures 1, 2 and 3 respectively. Method (36) is not A-stable going by the stability regions of the collocation methods. Method (32) and (34) are A-stable since the regions contain the complex plane outside the enclosed Figures.***

## 5. Application of the algorithms

**In the present section we illustrate the accuracy and efficiency of the two-step second derivative block hybrid collocation methods by solving some examples using the constructed methods in section 3 of the paper. We consider here seven challenging examples, each with its own peculiarity for the experimentation with the derived block hybrid collocation methods of section 3. The continuous solutions were evaluated at some**

**equidistant points. Numerical results are presented in terms of maximum absolute errors on some selected grid points within the interval of interest. We used constant step size for the experimentation and compare the obtained results with the exact values side by side in Tables and the solution curves are displayed in Figures. The value *nfe* stands for the number of function evaluations in each computation and *Ext* denotes the exact values in each evaluation.**

**Example 1: The first example is a highly linear stiff differential equation with the initial conditions as well as the exact value for comparison purposes, given by,**

$$\begin{aligned} Dy_1(t) &= y_2(t), & y_1(0) &= 1, \\ Dy_2(t) &= -1000y_1(t) - 1001y_2(t), & y_2(0) &= -1. \end{aligned}$$

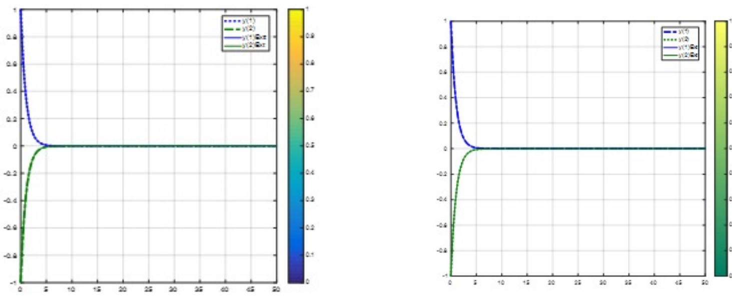
**The exact solution is**

$$\begin{aligned} y_1(t) &= exp(-t), \\ y_2(t) &= -exp(-1000t). \end{aligned}$$

**We consider the solution of the stiff problem in the interval [0, 50] with value of $h = 0.1$. The computed solutions are compared with the exact solutions side by side in the below Table, while the solution curves are displayed in Figure 4.**

**Table 5: Absolute errors in the numerical integration of example 1**

| t | $y_i$ | **Method(34)**$\lvert y(t) - y_n(t)\rvert$ | **Method(36)**$\lvert y(t) - y_n(t)\rvert$ |
|---|---|---|---|
| **5** | $y_1$ | $9.132250511356688 \times 10^{-12}$ | $1.210143096841421 \times 10^{-14}$ |
| | $y_2$ | $6.697829306520181 \times 10^{-01}$ | $6.697829306428695 \times 10^{-01}$ |
| **50** | $y_1$ | $1.231566931769734 \times 10^{-12}$ | $1.656660919557851 \times 10^{-15}$ |
| | $y_2$ | $7.373818159834551 \times 10^{-03}$ | $7.373818158601255 \times 10^{-03}$ |
| **250** | $y_1$ | $1.239264868503327 \times 10^{-20}$ | $1.670193979018203 \times 10^{-23}$ |
| | $y_2$ | $1.460145698426518 \times 10^{-11}$ | $1.460145697185569 \times 10^{-11}$ |
| **500** | $y_1$ | $3.280528962170302 \times 10^{-31}$ | $4.413511089597361 \times 10^{-34}$ |
| | $y_2$ | $1.928749851244447 \times 10^{-22}$ | $1.928749847959486 \times 10^{-22}$ |

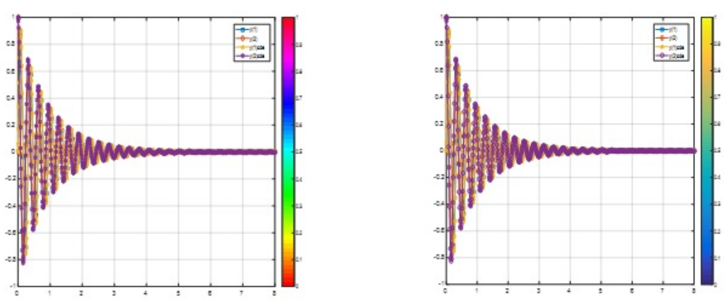


(A) *Solution curve of method (34)* (B) *Solution curve of method(36)*

FIGURE 4. Graphical plots of example 1 using the block hybrid methods with $nfe = 500$

**Example 2: [23] The problem considered here has a highly oscillatory component with asymptotic values of zero which is very good for testing new derived methods like the presented methods. Hence comparison is made with the ode MatLab solver. As depicted in Figure 5 below the solution curves obtained with the block hybrid collocation methods coincide with the solution from the ode code solvers showing the efficiency and effectiveness of the derived collocation methods.**

$$Dy_1(t) = y_1(t) - y_1(t)y_2(t)^2 + 20y_2(t), \qquad y_1(0) = 0,$$
$$Dy_2(t) = -y_2(t) - y_1(t)^2y_2(t) - 20y_1(t), \qquad y_2(0) = 1.$$



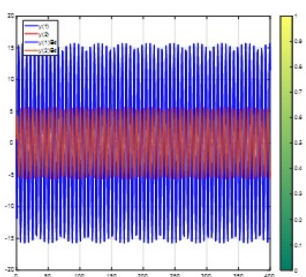
(A) *Solution curve of method (32)* (B) *Solution curve of method(34)*

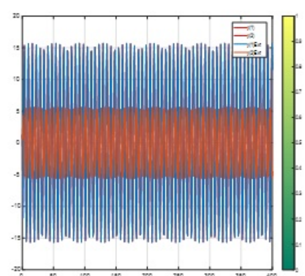
FIGURE 5. Graphical plots of example 2 using the block hybrid method and ode solver from MatLab.

**Example 3: Stiff differential equations are ubiquitous in astrochemical kinetics, many in control systems and electronics, and also many in non-industrial areas such as weather prediction and biology. In order to illustrate the new constructed algorithms (methods) we solve the stiff initial value problem, given here as an example problem of system of initial-value problems:**

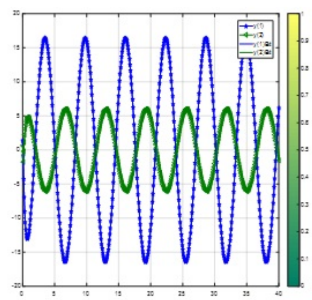
$$Dy_1(t) = 9y_1(t) + 24y_2(t) + 5cost - \frac{1}{3}sint, \qquad y_1(0) = \frac{4}{3},$$
$$Dy_2(t) = -24y_1(t) - 51y_2(t) - 95cost + \frac{1}{3}sint. \qquad y_2(0) = \frac{2}{3}.$$

**The transient term $e^{-\lambda t}$ which occur in the solution causes this system to be stiff ($\lambda = 39$). Here the graphical output of the system portrait an oscillatory picture as depicted below in Figure 6(a, b). But if the integration interval is increased we obtain the curves in Figure 6(c, d).**

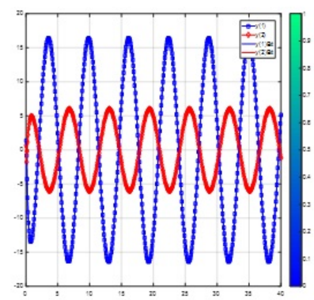




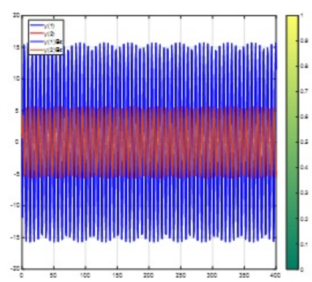
Example 4: In this example we consider system of nonlinear differential equation together with the initial conditions for better test of the new derived methods. The graphical output of the computation is displayed in Figure 7 below.



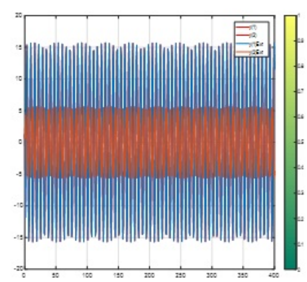
(A)



(B)



(C)



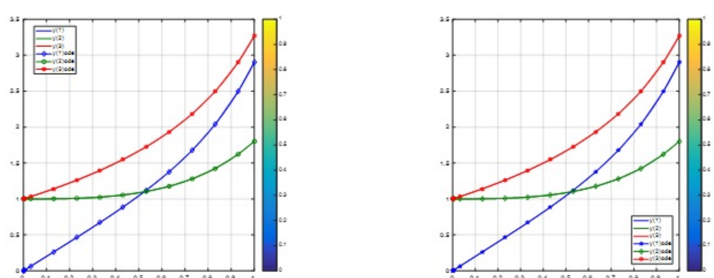
(D)

FIGURE 6. Graphical plots of example 3 using the block hybrid method(32,a,c), method(36,b,d) and ode solver from MatLab.

**Example 4: In this example we consider system of non-linear differential equation together with the initial conditions for better test of the new derived methods. The graphical output of the computation is displayed in Figure 7 below.**

$$\begin{aligned} Dy_1(t) &= 2y_2(t)^2, & y_1(0) &= 0, \\ Dy_2(t) &= ty_1(t), & y_2(0) &= 1, \\ Dy_3(t) &= y_2(t)y_3(t), & y_3(0) &= 1. \end{aligned}$$

**In this example we compare the graphical plots obtained with the results of the MatLab code solver. We observe that the graphs of the approximated solutions and the graphs of the solutions from the ode solver coincide with each other, see Figure 7.**



(A) *Solution curve of method (32)*

(B) *Solution curve of method(34)*

FIGURE 7. Graphical plots of example 4 using the block hybrid method and ode solver from MatLab.

**Example 5: This example is a good problem for testing new derived methods taken from Lambert [19]. The eigenvalues of the system are complex which shows that the solutions oscillate in the interval of integration. The eigenvalues are $\lambda_1 = -50$, $\lambda_2 = 1-8i$ and $\lambda_3 = 1+8i$ , showing that the system has one real and two complex conjugate pair which indicates that the system is very difficult to handle with method of limited stability region. The solution of this problem is sinusoidal with a very**

**slowly increasing amplitude as shown in the curves of Figure 8.**

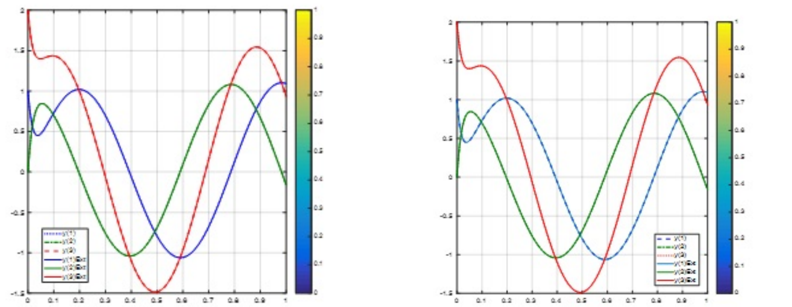
$$\begin{aligned} Dy_1(t) &= 42.2y_1(t) + 50.1y_2(t) - 42.1y_3(t), \qquad & y_1(0) = 1, \\ Dy_2(t) &= -66.1y_1(t) - 58y_2(t) + 58.1y_3(t), \qquad & y_2(0) = 0, \\ Dy_3(t) &= 26.1y_1(t) + 42.1y_2(t) - 34y_3(t), \qquad & y_3(0) = 2, \end{aligned}$$

**The exact solution is**

$$\begin{aligned} y_1(t) &= exp(-0.1t)sin(8t) + exp(-50t), \\ y_2(t) &= exp(-0.1t)cos(8t) - exp(-50t), \\ y_3(t) &= exp(-0.1t)(cos(8t) + sin(8t)) + exp(-50t). \end{aligned}$$

**Table 6: Absolute errors in the numerical integration of example 5**

| $t$ | $y_i$ | **Method(34)**$\lvert y(t) - y_n(t)\rvert$ | **Method(36)**$\lvert y(t) - y_n(t)\rvert$ |
|---|---|---|---|
| | $y_1$ | $9.131917444449300 \times 10^{-12}$ | $1.110223024625157 \times 10^{-16}$ |
| **5** | $y_2$ | $9.131861933298069 \times 10^{-12}$ | $2.775557561562891 \times 10^{-16}$ |
| | $y_3$ | $9.131806422146838 \times 10^{-12}$ | **0** |
| | $y_1$ | $1.232569601938849 \times 10^{-12}$ | $4.440892098500626 \times 10^{-16}$ |
| **50** | $y_2$ | $1.232125512728999 \times 10^{-12}$ | $3.330669073875470 \times 10^{-16}$ |
| | $y_3$ | $1.231681423519149 \times 10^{-12}$ | $2.220446049250313 \times 10^{-16}$ |
| | $y_1$ | $3.330669073875470 \times 10^{-15}$ | $1.665334536937735 \times 10^{-15}$ |
| **250** | $y_2$ | $3.108624468950438 \times 10^{-15}$ | $1.332267629550188 \times 10^{-15}$ |
| | $y_3$ | $4.440892098500626 \times 10^{-16}$ | $2.886579864025407 \times 10^{-15}$ |
| | $y_1$ | $1.110223024625157 \times 10^{-15}$ | $8.215650382226158 \times 10^{-15}$ |
| **500** | $y_2$ | $9.242606680004428 \times 10^{-14}$ | $2.886579864025407 \times 10^{-15}$ |
| | $y_3$ | $7.993605777301127 \times 10^{-15}$ | $4.773959005888173 \times 10^{-15}$ |

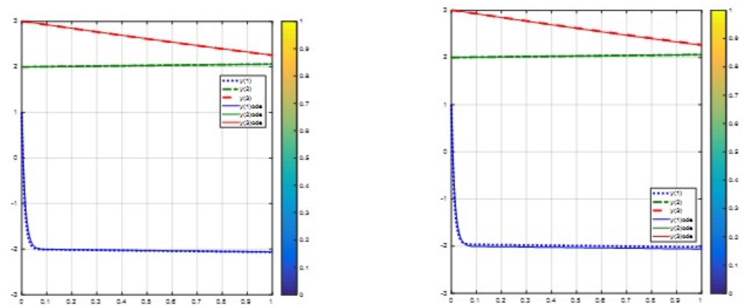


(A) *Solution curve of method (34)* (B) *Solution curve of method(36)*

FIGURE 8. Graphical plots of example 5 using the block hybrid method with *nfe*= 500.

**Example 6: The famous chemical reaction with initial limit cycle in three dimensions is the âœOregonatorâ reaction between HBrO2, Br- and Ce(IV). The Oregonator model describes the chemical reaction of bromous acid, bromide ion and cerium ion which oscillates with changes rapidly in color and structure over many orders of magnitude. The chemical reaction was modeled by the stiff system of 3 nonlinear ordinary differential equations with corresponding initial conditions, as follows**

$$
\begin{aligned}
Dy_1(t) &= 77.27(y_2(t)+y_1(t)(1-8.375\times10^{-6}y_1(t)-y_2(t))), \qquad && y_1(0)=1,\\
Dy_2(t) &= \frac{1}{77.27}(y_3(t)-(1+y_1(t))y_2(t)), && y_2(0)=2,\\
Dy_3(t) &= 0.16(y_1(t)-y_3(t)), && y_3(0)=3,
\end{aligned}
$$



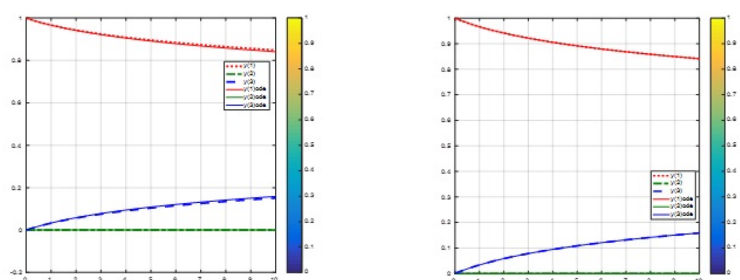
(A) *Solution curve of method (32)*

(B) *Solution curve of method(34)*

FIGURE 9. Graphical plots of example 6 using the block hybrid method and ode solver from MatLab.

**Example 7: This classical problem that models the kinetics of a chemical reaction consists of a system of three equations given by**

$$Dy_1(t) = -0.04y_1(t) + 10^4 y_2(t)y_3(t),\ \ y_1(0) = 1,$$
$$Dy_2(t) = 0.04y_1(t) - 10^4 y_2(t)y_3(t) - 3\times 10^7 y_2(t)^2, \qquad y_2(0) = 0,$$
$$Dy_3(t) = 3\times 10^7 y_2(t)^2,\ \ y_3(0) = 0.$$



(A) *Solution curve of method (32)*

(B) *Solution curve of method(36)*

FIGURE 10. Graphical plots of example 7 using the block hybrid method and ode solver from MatLab.

## 6. Concluding Remarks

**In the present study, effective block hybrid collocation methods to deal with linear and nonlinear stiff and highly oscillatory problems have been proposed. The proposed collocation methods provide efficient technique for obtaining accurate approximate solutions to linear and nonlinear stiff and highly oscillatory differential equations. The performance in the preliminary numerical experiments confirm that the collocation methods offer better accuracy and convergence compared to other derived methods for stiff and oscillatory problems in literature. In our next research paper, we will apply some new derived methods to modeled problems of differential equations, such as enzyme kinetics, cardiac electrophysiology, drug magnetic nanoparticles transport, etc.**

**Acknowledgement:** ***The authors are grateful to the esteemed reviewers for their constructive suggestions, based on which the present manuscript is revised.***

## References

[1] **S. O. Fatunla, Numerical integrators for stiff and highly oscillatory differential equations,** ***J. Math. Comp.,34 (150) 373–390, 1980.***

[2] ***L. R. Petzold, L.O.Jay and J. Yen, Numerical solution of highly oscillatory ordinary differential equations, Acta Numerica, 61 437 –483, 1997.***

[3] ***S. D. Capper, J. R. Cash and D. R. Moore, Lobatto-Obrechkoff formulae for 2nd , order two-points boundary value problems,*****J. Numer. Anal. Indus. and Appl. Math.** ***(JNAIAM), 314 13 – 25, 2006.***

[4] ***J. C. Butcher, D. J. L. Chen, A new type of singly-implicit Runge-Kutta method,*** **Appl. Numer. Math.*****, 34 (2-3) 179 – 188, 2000.***

[5] ***A. Shokri, A.A. Shokri, Implicit one-step L-stable generalized hybrid methods for the numerical solution of first order initial value problems,*** **Iranian J. Math. Chem.*****, 4 (2) 201 – 212, 2013.***

[6] ***J. R. Cash, High order methods for numerical integration of ordinary differential equations,*** **Numer. Math.*****, 30 385 – 409, 1978.***

[7] ***M. Urabe, An implicit one-step method of high-order accuracy for the numerical integration of ordinary differential equations, J. Numer. Math. 15 151 – 164, 1970.***

[8] *M. T. Chu, H. Hamilton, Parallel solution of ordinary differential equations by multi-block methods, SIAM J.Sci. Stat. Comput. 8 (3) 342 – 353, 1987.*

[9] *D. G. Yakubu, S. Markus, Second derivative of high order accuracy methods for the numerical integration of stiff initial value problems, Afrika Math. , 27 (5-6) 963–977, 2016.*

[10] *P. Onumanyi, D. O. Awoyemi, S. N. Jator, U. W. Sirisena, New linear multi-step methods with continuous coefficients for first order initial value problems, J. Nig. Math. Soc. 13 37 – 51, 1994.*

[11] *J. Vigo-Aguiar, H. Ramos, A family of A-stable Runge-Kutta collocation methods of high order for initial value problems, IMA J. Numer. Anal. , 27 (4) 798–817, 2007.*

[12] *F. Costabile, A. Napoli, Stability of Chebychev collocation methods, Comput. Math. Appl. 47 659 – 666, 2004.*

[13] *F. Costabile, A. Napoli, A class of collocation methods for numerical integration of initial value problems, Comput. Math. Appl. 62 3221 – 3235, 2011.*

[14] *S. O. Fatunla, Block methods for second order ODEs, Inter. J. Comput. Math. 41 55 – 63, 1991.*

[15] *D. G. Yakubu, S. Markus, The efficiency of second derivative multistep methods for the numerical integration of stiff systems, J. Nig. Math. Soc., 35 (2) 107–127, 2016.*

[16] *A. Shokri, S. Hosein, High phase-lag order trigonometrically fitted two-step Obrechkoff methods for the numerical solution of periodic initial value problems,* **Numer. Algor.,** *68 337 – 354, 2015.*

[17] *J. D. Lambert, Computational Methods in Ordinary Differential Equations, John Wiley and Sons, Ltd, 1973.*

[18] *H. Ramos, M. F. Patracio, Some new implicit two-step multi-derivative methods for solving special second-orderIVPs,* **Appl, Math, Comp.,** *239 227 – 241, 2014.*

[19] *J. D. Lambert, Numerical Methods for Ordinary Differential Equations. The initial Value Problem, John Wiley and Sons, Ltd New York 1991.*

[20] *G. Dahlquist, 'A special stability problem for linear multistep methods',*BIT J. Numer. Math. *3, 27 – 43, 1963.*

[21] *K. Burrage, J. C. Butcher, Nonlinear stability for a general class of differential equations, BIT Numer. Math. 20 185 – 203, 1980.*

[22] *J.C. Butcher, Numerical Methods for Ordinary Differential Equations, Second Edition, John Wiley and Sons, Ltd, 2008.*

[23] *J. D. Lawson, Generalized Runge-Kutta processes for stable system with large Lipschitz constants, SIAM J. Numer. Anal. 4 (3) 372 – 380, 1967.*

[1]Department of Mathematical Sciences,
Abubakar Tafawa Balewa University, Bauchi, Nigeria
email: dgyakubu@atbu.edu.ng

[2]Department of Mathematical Sciences,
Federal University of Technology, Akure, Nigeria
email: almomoh@futa.edu.ng

[3]Department of Mathematics,
University of Jos, Jos, Nigeria
email: gkumleng@gmail.com

[4]Department of Mathematics,
Faculty of Science,
University of Maragheh, Maragheh, Iran
email: higra@usal.es