Journal of the Nigerian Mathematical Society
Vol. 42, Issue 1, pp. 1 - 18, 2023

BLOCK BI-BASIS COLLOCATION METHOD FOR DIRECT APPROXIMATION OF FOURTH-ORDER IVP

B. I. AKINNUKAWE[1] AND M. R. ODEKUNLE[2]

ABSTRACT. This study presents the derivation of the two-step fifth-order hybrid scheme based on the combination of Hermite and shifted Chebyshev polynomials as basis functions of the collocation technique. The technique was used to generate a set of hybrid schemes at selected grid and non-grid points and implemented as a block method. The derived block method (Block Bi-basis Collocation Method) was applied as a simultaneous integrator to linear and non-linear fourth-order initial value problems of ordinary differential equations. The zero stability, order, error constants, consistency, convergence and numerical results of the proposed block method are analyzed. The application of the block bi-basis collocation method to some fourth-order initial value problems demonstrated the effectiveness and accuracy of the method. The block bi-basis collocation method compared favorably with existing methods in the literature.

Keywords and phrases: Fourth-order initial value problems, Hermite polynomial, Shifted Chebyshev polynomial, Collocation method, Block method
2010 Mathematical Subject Classification: 65L05, 65L06

1. INTRODUCTION

The numerical solution of fourth-order initial value problems of the form

$$y^{(iv)} = f(t, y', y'', y'''), \quad t \in [t_0, t_N],$$

subject to the initial conditions

$$y(t_0) = y_0, \quad y'(t_0) = y'_0, \quad y''(t_0) = y''_0, \quad y'''(t_0) = y'''_0, \tag{1}$$

Received by the editors July 07, 2022; Revised: December 15, 2022; Accepted: January 25, 2023
www.nigerianmathematicalsociety.org; Journal available online at https://ojs.ictp.it/jnms/
www.nigerianmathematicalsociety.org; Journal available online at https://ojs.ictp.it/jnms/

are considered. The demand for the direct solution of higher order initial value problems of ordinary differential equations, which have applications in several fields of science, technology, engineering and management, is on the increase. Specifically, the fourth-order initial value problems of ordinary differential equations have its applications in neural networks (see [21]); beam- theory (see [14]); electric circuits (see [9]); fluid dynamics (see [5]); the dynamics of population in biology and several other areas. Most of these modeled fourth-order initial value problems cannot be solved analytically hence the need for numerical methods to profer direct approximation to the considered differential equation (1).

Researchers had, over the years, developed other schemes to solve 4th-order initial value problems such as the predictor-corrector schemes (see [7,8,15,16]) where incorporating the starting values led to lengthy computation time. The method of reduction of the order to system of first-order ordinary differential equations and applying numerical methods to solve the resulting system of first-order initial value problems have been extensively analyzed in [1,10,13,14,19]. This approach also leads to longer execution and computation time since the number of equations increased four times and more functions evaluation are carried out. Some scholars applied direct methods to solve (1) which include Kuboye et al. [17], Adeyeye and Omar [2], Areo and Omole [6], Mohammed [22] and Allogmany et al. [4] to mention few; but the accuracy of their methods in term of error can still be improved.

In this study, an efficient continuous two-step block bi-basis collocation method was developed for the direct integration of fourth-order initial value problems of the form (1) with the aim of ultimately reducing the error of the approximation. It is assumed that the solution of the fourth-order initial value problems (1) exists and is unique.

2. METHODOLOGY

This section considers the derivation of the two-step hybrid schemes of the form

$$\alpha_2 y_{n+2} = \alpha_0 y_n + \sum_{i=1}^{3} h^i \beta_{i0} y_n^{(i)} + h^4 \sum_{j=0}^{2} \phi_j f_{n+j} + h^4 \sum_{j=1}^{2} \phi_{v_j} f_{n+v_j} \quad (2)$$

where, $v_1 = \frac{1}{4}$ and $v_2 = \frac{5}{4}$ and α_0, α_2, $\beta_i, i = 1(1)3$, $\phi_j, j = 0(1)2$ and $\phi_{v_j}, j = 1(1)2$ are parameters to be determined uniquely. An approximate solution $y(t)$ to equation (1) is expressed in the form

$$y(t) = \sum_{r=0}^{p+1} a_r H_r(t) + \sum_{r=p+2}^{q} a_r T_r^*(t) \tag{3}$$

taking $q = 8$ and $p = \frac{q}{2}$ where $H_r(t)$ and $T_r^*(t)$ are orthogonal polynomials of Hermite and shifted Chebyshev respectively. To obtain the coefficients in (2),the following conditions are imposed:

$$y_{n+j} = y(t_{n+j}),\ \ j = 0,$$

$$y_{n+j}^{(i)} = y^{(i)}(t_{n+j}),\ \ j = 0,\ \ i = 1(1)3,$$

$$y_{n+j}^{(iv)} = f(t_{n+j}),\ \ j = 0, \frac{1}{4}, 1, \frac{5}{4}, 2, \tag{4}$$

which produces a system of 9 equations with 9 unknowns. Solving these 9 equations by Guass elimination method gives the corresponding coefficients of $a_r, r = 0(1)8$. Substituting the resulting coefficients $a_r, r = 0(1)8$ into equation (3) and its derivatives yields a continuous Adams type implicit scheme of the form,

$$y_{n+j} = y_n + \sum_{i=1}^{3} h^i \beta_{i0}(x) y_n^{(i)} + h^4 \sum_{j=0}^{2} \phi_j(x) f_{n+j} + h^4 \sum_{j=1}^{2} \phi_{v_j}(x) f_{n+v_j} \tag{5}$$

where, $x = \frac{t - t_{n+k-1}}{h}$, $\beta_{i0}, i = 1, 2, 3$ and $\phi_j, j = 0, \frac{1}{4}, 1, \frac{5}{4}, 2$ are written in matrix form as:

$$\begin{bmatrix} \beta_{10}(x) \\ \beta_{20}(x) \\ \beta_{30}(x) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{6} \end{bmatrix} \begin{bmatrix} x \\ x^2 \\ x^3 \end{bmatrix}, \tag{6}$$

$$\begin{bmatrix} \phi_0(x) \\ \phi_{\frac{1}{4}}(x) \\ \phi_1(x) \\ \phi_{\frac{5}{4}}(x) \\ \phi_2(x) \end{bmatrix} = \begin{bmatrix} \frac{1}{24} & -\frac{21}{400} & \frac{109}{3600} & -\frac{3}{350} & \frac{1}{1050} \\ 0 & \frac{4}{63} & -\frac{46}{945} & \frac{34}{2205} & -\frac{4}{2205} \\ 0 & -\frac{1}{36} & \frac{53}{1080} & -\frac{1}{45} & \frac{1}{315} \\ 0 & \frac{4}{225} & -\frac{22}{675} & \frac{26}{1575} & -\frac{4}{1575} \\ 0 & -\frac{1}{1008} & \frac{29}{15120} & -\frac{1}{882} & \frac{1}{4410} \end{bmatrix} \begin{bmatrix} x^4 \\ x^5 \\ x^6 \\ x^7 \\ x^8 \end{bmatrix}. \quad (7)$$

The coefficients of first and higher derivatives of (5) are

$$\begin{bmatrix} \beta'_{10}(x) \\ \beta'_{20}(x) \\ \beta'_{30}(x) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x^0 \\ x^1 \\ x^2 \end{bmatrix}, \quad (8)$$

$$\begin{bmatrix} \phi'_0(x) \\ \phi'_{\frac{1}{4}}(x) \\ \phi'_1(x) \\ \phi'_{\frac{5}{4}}(x) \\ \phi'_2(x) \end{bmatrix} = \begin{bmatrix} \frac{1}{6} & -\frac{21}{80} & \frac{109}{600} & -\frac{3}{50} & \frac{4}{525} \\ 0 & \frac{20}{63} & -\frac{92}{315} & \frac{34}{315} & -\frac{32}{2205} \\ 0 & -\frac{5}{36} & \frac{53}{180} & -\frac{7}{45} & \frac{8}{315} \\ 0 & \frac{4}{45} & -\frac{44}{225} & \frac{26}{225} & -\frac{32}{1575} \\ 0 & -\frac{5}{1008} & \frac{29}{2520} & -\frac{1}{216} & \frac{4}{2205} \end{bmatrix} \begin{bmatrix} x^3 \\ x^4 \\ x^5 \\ x^6 \\ x^7 \end{bmatrix}, \quad (9)$$

$$\begin{bmatrix} \beta''_{20}(x) \\ \beta''_{30}(x) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x^0 \\ x^1 \end{bmatrix}, \quad (10)$$

$$\begin{bmatrix} \phi_0''(x) \\ \phi_{\frac{1}{4}}''(x) \\ \phi_1''(x) \\ \phi_{\frac{5}{4}}''(x) \\ \phi_2''(x) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{21}{20} & \frac{109}{120} & -\frac{9}{25} & \frac{4}{75} \\ 0 & \frac{80}{63} & -\frac{92}{63} & \frac{68}{105} & -\frac{32}{315} \\ 0 & -\frac{5}{9} & \frac{53}{36} & -\frac{14}{15} & \frac{8}{45} \\ 0 & \frac{16}{45} & -\frac{44}{45} & \frac{52}{75} & -\frac{32}{225} \\ 0 & -\frac{5}{252} & \frac{29}{504} & -\frac{1}{21} & \frac{4}{315} \end{bmatrix} \begin{bmatrix} x^2 \\ x^3 \\ x^4 \\ x^5 \\ x^6 \end{bmatrix}, \tag{11}$$

$$\begin{bmatrix} \beta_{30}'''(x) \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix} \begin{bmatrix} x^0 \end{bmatrix}, \tag{12}$$

$$\begin{bmatrix} \phi_0'''(x) \\ \phi_{\frac{1}{4}}'''(x) \\ \phi_1'''(x) \\ \phi_{\frac{5}{4}}'''(x) \\ \phi_2'''(x) \end{bmatrix} = \begin{bmatrix} 1 & -\frac{63}{20} & \frac{109}{30} & -\frac{9}{5} & \frac{8}{25} \\ 0 & \frac{80}{21} & -\frac{368}{63} & \frac{68}{21} & -\frac{64}{105} \\ 0 & -\frac{5}{3} & \frac{53}{9} & -\frac{14}{3} & \frac{16}{15} \\ 0 & \frac{16}{15} & -\frac{176}{45} & \frac{52}{15} & -\frac{64}{75} \\ 0 & -\frac{5}{84} & \frac{29}{126} & -\frac{5}{21} & \frac{8}{105} \end{bmatrix} \begin{bmatrix} x \\ x^2 \\ x^3 \\ x^4 \\ x^5 \end{bmatrix}. \tag{13}$$

Discrete schemes and its derivatives are derived by evaluating (5) as well as its derivatives at grid and non-grid points $\left(\frac{1}{4}, 1, \frac{5}{4}, 2\right)$. These are used to form the block method and its derivative methods as:

$$\begin{bmatrix} y_{n+\frac{1}{4}} \\ y_{n+1} \\ y_{n+\frac{5}{4}} \\ y_{n+2} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} y_n \end{bmatrix} + \begin{bmatrix} \frac{1}{4} \\ 1 \\ \frac{5}{4} \\ 2 \end{bmatrix} \begin{bmatrix} hy_n' \end{bmatrix} + \begin{bmatrix} \frac{1}{32} \\ \frac{1}{2} \\ \frac{25}{32} \\ 2 \end{bmatrix} \begin{bmatrix} h^2 y_n'' \end{bmatrix} + \begin{bmatrix} \frac{1}{384} \\ \frac{1}{6} \\ \frac{125}{384} \\ \frac{4}{3} \end{bmatrix} \begin{bmatrix} h^3 y_n''' \end{bmatrix}$$

$$+\begin{bmatrix} 0 & 0 & 0 & \frac{3491}{29491200} \\ 0 & 0 & 0 & \frac{149}{12600} \\ 0 & 0 & 0 & \frac{180125}{8257536} \\ 0 & 0 & 0 & \frac{16}{225} \end{bmatrix} \begin{bmatrix} h^4 f_{n-\frac{5}{4}} \\ h^4 f_{n-1} \\ h^4 f_{n-\frac{1}{4}} \\ h^4 f_n \end{bmatrix} + \begin{bmatrix} \frac{5531}{108380160} & -\frac{1019}{61931520} & \frac{803}{77414400} & -\frac{491}{867041280} \\ \frac{188}{6615} & \frac{17}{7560} & -\frac{4}{4725} & \frac{1}{52920} \\ \frac{1534375}{21676032} & \frac{190625}{12386304} & -\frac{2875}{442368} & \frac{40625}{173408256} \\ \frac{2816}{6615} & \frac{208}{945} & -\frac{256}{4725} & \frac{26}{6615} \end{bmatrix} \begin{bmatrix} h^4 f_{n+\frac{1}{4}} \\ h^4 f_{n+1} \\ h^4 f_{n+\frac{5}{4}} \\ h^4 f_{n+2} \end{bmatrix} \tag{14}$$

The first-, second- and third-derivative block methods are as in equations (15), (16) and (17) respectively.

$$\begin{bmatrix} hy'_{n+\frac{1}{4}} \\ hy'_{n+1} \\ hy'_{n+\frac{5}{4}} \\ hy'_{n+2} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} hy'_n \end{bmatrix} + \begin{bmatrix} \frac{1}{4} \\ 1 \\ \frac{5}{4} \\ 2 \end{bmatrix} \begin{bmatrix} h^2 y''_n \end{bmatrix} + \begin{bmatrix} \frac{1}{32} \\ \frac{1}{2} \\ \frac{25}{32} \\ 2 \end{bmatrix} \begin{bmatrix} h^3 y'''_n \end{bmatrix}$$

$$+\begin{bmatrix} 0 & 0 & 0 & \frac{1873}{1075200} \\ 0 & 0 & 0 & \frac{281}{8400} \\ 0 & 0 & 0 & \frac{125}{2688} \\ 0 & 0 & 0 & \frac{43}{525} \end{bmatrix} \begin{bmatrix} h^4 f_{n-\frac{5}{4}} \\ h^4 f_{n-1} \\ h^4 f_{n-\frac{1}{4}} \\ h^4 f_n \end{bmatrix} + \begin{bmatrix} \frac{4427}{4515840} & -\frac{47}{161280} & \frac{197}{1075200} & -\frac{1}{100352} \\ \frac{262}{2205} & \frac{8}{315} & -\frac{2}{175} & \frac{1}{2352} \\ \frac{68125}{301056} & \frac{625}{7168} & -\frac{4625}{129024} & \frac{625}{451584} \\ \frac{192}{245} & \frac{52}{105} & -\frac{64}{1575} & \frac{29}{2205} \end{bmatrix} \begin{bmatrix} h^4 f_{n+\frac{1}{4}} \\ h^4 f_{n+1} \\ h^4 f_{n+\frac{5}{4}} \\ h^4 f_{n+2} \end{bmatrix}. \tag{15}$$

$$\begin{bmatrix} h^2 y''_{n+\frac{1}{4}} \\ h^2 y''_{n+1} \\ h^2 y''_{n+\frac{5}{4}} \\ h^2 y''_{n+2} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} h^2 y''_n \end{bmatrix} + \begin{bmatrix} \frac{1}{4} \\ 1 \\ \frac{5}{4} \\ 2 \end{bmatrix} \begin{bmatrix} h^3 y'''_n \end{bmatrix}$$

$$+\begin{bmatrix} 0 & 0 & 0 & \frac{2773}{153600} \\ 0 & 0 & 0 & \frac{31}{600} \\ 0 & 0 & 0 & \frac{325}{6144} \\ 0 & 0 & 0 & \frac{2}{75} \end{bmatrix} \begin{bmatrix} h^4 f_{n-\frac{5}{4}} \\ h^4 f_{n-1} \\ h^4 f_{n-\frac{1}{4}} \\ h^4 f_n \end{bmatrix} + \begin{bmatrix} \frac{1189}{80640} & -\frac{35}{9216} & \frac{137}{57600} & -\frac{83}{645120} \\ \frac{16}{45} & \frac{29}{180} & -\frac{16}{225} & \frac{1}{360} \\ \frac{8125}{16128} & \frac{3125}{9216} & -\frac{275}{2304} & \frac{625}{129024} \\ \frac{64}{63} & \frac{28}{45} & \frac{64}{225} & \frac{16}{315} \end{bmatrix} \begin{bmatrix} h^4 f_{n+\frac{1}{4}} \\ h^4 f_{n+1} \\ h^4 f_{n+\frac{5}{4}} \\ h^4 f_{n+2} \end{bmatrix}. \tag{16}$$

$$\begin{bmatrix} h^3 y'''_{n+\frac{1}{4}} \\ h^3 y'''_{n+1} \\ h^3 y'''_{n+\frac{5}{4}} \\ h^3 y'''_{n+2} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} h^3 y'''_n \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & \frac{1981}{19200} \\ 0 & 0 & 0 & \frac{1}{300} \\ 0 & 0 & 0 & \frac{5}{768} \\ 0 & 0 & 0 & -\frac{7}{75} \end{bmatrix} \begin{bmatrix} h^4 f_{n-\frac{5}{4}} \\ h^4 f_{n-1} \\ h^4 f_{n-\frac{1}{4}} \\ h^4 f_n \end{bmatrix}$$

$$+\begin{bmatrix} \frac{3203}{20160} & -\frac{169}{5760} & \frac{263}{14400} & -\frac{79}{80640} \\ \frac{188}{315} & \frac{28}{45} & -\frac{52}{225} & \frac{11}{1260} \\ \frac{2375}{4032} & \frac{875}{1152} & -\frac{65}{576} & \frac{125}{16128} \\ \frac{256}{315} & -\frac{4}{45} & \frac{256}{225} & \frac{73}{315} \end{bmatrix} \begin{bmatrix} h^4 f_{n+\frac{1}{4}} \\ h^4 f_{n+1} \\ h^4 f_{n+\frac{5}{4}} \\ h^4 f_{n+2} \end{bmatrix}. \tag{17}$$

Equations (14) – (17) form the two-step block algorithm known as the Block Bi-basis Collocation Method (BBCM) developed for the direct approximation of linear and non-linear fourth-order initial value problems (1).

3. ANALYSIS OF THE METHOD

In this section, the properties of the derived method are examined.

3.1 Order and Error Constant of the Method

The Local truncation error associated with the derived methods can be defined as the linear difference operator [12]

$$L[y(t_n);h] = \sum_{j=0}^{2} \alpha_j y(t_n+jh) - \sum_{i=1}^{3} h^i \beta_{i0} y^{(i)}(t_n) - h^4 \sum_{j=0}^{2} \phi_j y^{(iv)}(t_n+jh)$$

$$-h^4\sum_{j=1}^{2}\phi_{v_j}y^{(iv)}(t_n+(v_j)h). \tag{18}$$

Assuming that $y(t_n)$ is sufficiently differentiable, then using Taylor series expansion on $y(t_n+jh)$,$y^{(i)}(t_n+jh), i=1(1)3$ and $y^{(iv)}(t_n+jh)$ about t_n, we have

$$y(t_n+jh)=\sum_{m=0}^{\infty}\frac{(jh)^m}{m!}y^{(m)}(t_n),$$

$$y^{(i)}(t_n+jh)=\sum_{m=0}^{\infty}\frac{(jh)^m}{m!}y^{(m+i)}(t_n),\ \ i=1(1)3,$$

$$y^{(iv)}(t_n+jh)=\sum_{m=0}^{\infty}\frac{(jh)^m}{m!}y^{(m+4)}(t_n).$$

Substituting $y(t_n+jh)$, $y^{(i)}(t_n+jh), i=1(1)3$ and $y^{(iv)}(t_n+jh)$ in equation (18) to obtain

$$L[y(t_n);h]=C_0y(t_n)+C_1hy'(t_n)+C_2h^2y''(t_n)+C_3h^3y'''(t_n)+...+C_{m+4}h^{m+4}y^{(m+4)}(t_n)+... \tag{19}$$

where, $C_m, m=0,1,2,...$ are constants given in terms of α_j, β_j and ϕ_j and are:

$$C_0=\sum_{j=0}^{2}\alpha_j+\sum_{j=1}^{2}\alpha_{v_j}$$

$$C_1=\left[\sum_{j=0}^{2}j\alpha_j+\sum_{j=1}^{2}v_j\alpha_{v_j}\right]-\beta_{10}$$

$$\vdots$$

$$C_{m+4}=\frac{1}{(m+4)!}\left[\sum_{j=0}^{2}j^{m+4}\alpha_j+\sum_{j=1}^{2}(v_j)^{m+4}\alpha_{v_j}\right]-\frac{1}{(m+3)!}\left[\sum_{j=0}^{2}j^{m+3}\beta_{1j}+\sum_{j=1}^{2}(v_j)^{m+3}\beta_{1v_j}\right]$$
$$-\frac{1}{(m+2)!}\left[\sum_{j=0}^{2}j^{m+2}\beta_{2j}+\sum_{j=1}^{2}(v_j)^{m+2}\beta_{2v_j}\right]-\frac{1}{(m+1)!}\left[\sum_{j=0}^{2}j^{m+1}\beta_{3j}+\sum_{j=1}^{2}(v_j)^{m+1}\beta_{3v_j}\right]$$
$$-\frac{1}{m!}\left[\sum_{j=0}^{2}j^{m}\phi_j+\sum_{j=1}^{2}(v_j)^{m}\phi_{v_j}\right]$$

The order and error constants of BBCM are shown in Table 1.

Table 1 Table of Order and Error Constants of BBCM

S/N	Scheme	Order(m)	Error Constant (C_{m+4})
1	$y_{n+\frac{1}{4}}$	5	$\frac{11381}{475634073600}$
2	y_{n+1}	5	$\frac{1}{3628800}$
3	$y_{n+\frac{5}{4}}$	5	$-\frac{104375}{19025362944}$
4	y_{n+2}	5	$-\frac{11}{113400}$
5	$y'_{n+\frac{1}{4}}$	5	$\frac{5531}{13212057600}$
6	y'_{n+1}	5	$-\frac{17}{1612800}$
7	$y'_{n+\frac{5}{4}}$	5	$-\frac{2875}{75497472}$
8	y'_{n+2}	5	$-\frac{13}{50400}$
9	$y''_{n+\frac{1}{4}}$	5	$\frac{4427}{825753600}$
10	y''_{n+1}	5	$-\frac{1}{12600}$
11	$y''_{n+\frac{5}{4}}$	5	$-\frac{4625}{33030144}$
12	y''_{n+2}	5	$-\frac{29}{50400}$
13	$y'''_{n+\frac{1}{4}}$	5	$\frac{1189}{29491200}$
14	y'''_{n+1}	5	$-\frac{29}{115200}$
15	$y'''_{n+\frac{5}{4}}$	5	$-\frac{275}{1179648}$
16	y'''_{n+2}	5	$-\frac{1}{900}$

3.2 Zero Stability

A block method is said to be zero-stable if the roots R_s, $s = 1, 2, \ldots, 16$ of the first characteristic polynomial $\rho(R)$ satisfy $|R_s| \leq 1, s = 1, \ldots, 16$ multiplicity not exceeding the order of the differential equation. The first characteristic polynomial $\rho(R) = 0$ of the derived method is calculated as

$$\rho(R) = \det(RA_{(1)} - A_{(0)})$$

where,

$$A_{(1)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$A_{(0)} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & \frac{1}{32} & 0 & 0 & 0 & \frac{1}{384} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{6} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{5}{4} & 0 & 0 & 0 & \frac{25}{32} & 0 & 0 & 0 & \frac{125}{384} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & \frac{4}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & \frac{1}{32} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{5}{4} & 0 & 0 & 0 & \frac{25}{32} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{5}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

According to [19], $\rho(R) = R^{z-u}(R-1)^u$ where u is the order of the differential equation and z is the order of the matrices $A_{(1)}$ and $A_{(0)}$. The BBCM is found to be zero-stable since $\rho(R) = R^{12}(R-1)^4$ satisfies $|R_s| \leq 1, s = 1(1)16$.

3.3 Consistency

The developed method is concluded to be consistent since according to Lambert [20], the necessary and sufficient condition for a numerical scheme to be consistent is for it to have order of at least one ($m \geq 1$) and the derived method is of order 5.

3.4 Convergence

A numerical method converges if it is consistent and zero-stable (see [3]). This implies that BBCM converges since the method is of order $m = 5 > 1$ and it satisfies the conditions for zero-stability.

4. NUMERICAL RESULTS

The following test problems shall be considered in order to examine the accuracy and computational efficiency of the derived method (BBCM). The numerical results obtained with constant step size are compared with existing methods and the exact solution of the considered initial value problems. All computations are carried out using MATHEMATICA 9.0.

Test Problem 1 [22]

$$y^{(iv)}(t) - t = 0, \;\; h = 0.1;$$
$$y(0) = 0, \;\; y'(0) = 1, \;\; y''(0) = y'''(0) = 0.$$

Exact solution is

$$y(t) = \frac{t^5}{120} + t.$$

Test Problem 2 [6]

$$y^{(iv)}(t) - y(t) = 0, \;\; h = \frac{1}{320};$$
$$y(0) = 1, \;\; y'(0) = 0, \;\; y''(0) = -2,$$
$$y'''(0) = 0.$$

Exact solution is

$$y(t) = -\frac{1}{4}exp(t) - \frac{1}{4}exp(-t) + \frac{3}{2}cos(t).$$

Test Problem 3 [17]

$$y^{(iv)}(t) - (y')^2 + yy''' + 4x^2 - e^x(1 - 4x + x^2) = 0, \;\; h = \frac{0.1}{32};$$
$$y(0) = 1, \;\; y'(0) = 1, \;\; y''(0) = 3,$$
$$y'''(0) = 1.$$

The theoretical solution is

$$y(t) = t^2 + e^t.$$

Test Problem 4 [17]

$$y^{(iv)}(t) + y''(t) = 0, \;\; 0 \leq t \leq \frac{\pi}{2};$$
$$y(0) = 0,$$
$$y'(0) = -\frac{1.1}{72 - 50\pi},$$
$$y''(0) = \frac{1}{144 - 100\pi},$$
$$y'''(0) = \frac{1.2}{144 - 100\pi},$$

Exact solution is

$$y(t) = \frac{1 - t - cos(t) - 1.2sin(t)}{144 - 100\pi}.$$

Test Problem 5 [17]

$$y^{(iv)}(t) - y''' - y'' - y' - 2y = 0, \;\; t \in [0, 2]\,,$$
$$y(0) = y'(0) = y''(0) = 0,$$
$$y'''(0) = 30,$$

Exact solution is

$$y(t) = 2exp(2t) - 5exp(-t) + 3cos(t) - 9sin(t).$$

Test Problem 6 [17]

Consider the Ship Dynamic Problem

$$y^{(iv)}(t) + 3y'' + y(2 + \rho cos(wt)) = 0, \;\; t > 0,$$
$$y(0) = y'(0) = y''(0) = y'''(0) = 0.$$

When $\rho = 0$, the theoretical solution is

$$y(t) = 2cos(t) - cos(\sqrt{2}t).$$

These Test Problems are chosen to aid comparison with other existing methods in literature. The notations used in representing the existing methods and the derived method in the result Tables are:

h	step size
$EIFBM$	Error of order 7 in First Block Method in [17]
EIM	Error in [22]
$EIFO$	Error in [11]
$EIOK$	Error in [18]
$EIAO$	Error in [6]
$EIBBCM$	Error in Block Bi-basis Collocation Method

where,

$$Error^{(i)} = |y^{(i)}_{exact} - y^{(i)}_{appro}|$$

Table 2 Numerical Results for Test Problem 1

t	Exact	BBCM	EIBBCM ($h = 0.1$)
0.1	0.1	0.1	1.38778×10^{-17}
0.2	0.2000026666666667	0.20000266666666666	2.77556×10^{-17}
0.3	0.30002025000000004	0.30002025	5.55112×10^{-17}
0.4	0.40008533333333335	0.4000853333333334	5.55112×10^{-17}
0.5	0.5002604166666667	0.5002604166666667	0
0.6	0.6006480000000001	0.6006480000000001	0
0.7	0.7014005833333334	0.7014005833333334	0
0.8	0.8027306666666667	0.8027306666666667	0
0.9	0.90492075	0.90492075	0
1.0	1.0083333333333333	1.0083333333333335	2.22045×10^{-16}

Table 3 Comparison of Errors for Test Problem 1

t	EIFBM [17]	EIM [22]	EIOK [18]	EIBBCM
0.1	$0.00000E+00$	$7.000024E-10$	$1.002087E-12$	$1.38778E-17$
0.2	$0.00000E+00$	$8.9999912E-10$	$0.000000E+00$	$2.77556E-17$
0.3	$0.00000E+00$	$2.599993E-09$	$0.00000E+00$	$5.55112E-17$
0.4	$5.5511151E-17$	$5.100033E-09$	$0.00000E+00$	$5.55112E-17$
0.5	$1.1102230E-16$	$7.799979E-09$	$1.002087E-12$	$0.00000E+00$
0.6	$1.1102230E-16$	$1.180009E-08$	$2.755907E-12$	$0.00000E+00$
0.7	$2.220446E-16$	$1.180009E-08$	$3.597306E-12$	$0.00000E+00$
0.8	$0.00000E+00$	$1.410006E-08$	$3.597306E-12$	$0.00000E+00$
0.9	$1.110223E-16$	$1.880000E-08$	$4.175549E-12$	$0.00000E+00$
1.0	$2.220446E-16$	$1.008335E-08$	$4.759970E-12$	$2.22045E-17$

Table 4 Comparison of Errors for Test Problem 2

t	EIAO [6]	EIFBM [17]	EIBBCM
0.003125	$4.440892E-16$	$2.2204460E-16$	$1.11022E-16$
0.006250	$2.176037E-14$	$0.0000000E+00$	$1.11022E-16$
0.009375	$7.771916E-13$	$2.2204460E-16$	$1.11022E-16$
0.012500	$7.666090E-13$	$4.4408921E-16$	$1.11022E-16$
0.015625	$2.367773E-12$	$0.0000000E+00$	$1.11022E-16$
0.018750	$5.932477E-12$	$4.4408921E-16$	$1.11022E-16$
0.021875	$1.287681E-11$	$2.2204460E-16$	$2.22045E-16$
0.025000	$2.517841E-11$	$4.4408921E-16$	$1.11022E-16$
0.028125	$4.546752E-11$	$4.4408921E-16$	$2.22045E-16$
0.031250	$7.712331E-11$	$0.0000000E+00$	$0.00000E+00$

Table 5 Comparison of Errors for Test Problem 3

t	EIFBM [17]	EIFO [11]	EIBBCM
0.103125	$1.8149238E-10$	$9.02145880E-10$	$4.44089E-16$
0.206250	$1.1543254E-08$	$1.21681228E-09$	$1.35003E-13$
0.309375	$1.2194148E-07$	$1.21681228E-09$	$1.13887E-12$
0.412500	$6.5296082E-07$	$1.71379609E-09$	$4.53237E-12$
0.515625	$2.3972196E-06$	$1.48197092E-08$	$1.24527E-12$
0.618750	$6.7092614E-06$	$3.05833850E-08$	$2.77791E-11$
0.721875	$1.6438756E-05$	$4.94185815E-08$	$5.39191E-11$
0.825000	$3.5549856E-05$	$7.12867908E-08$	$9.49667E-11$
0.928125	$6.9845227E-05$	$1.05877308E-07$	$1.55466E-10$
1.031250	$1.2716790E-04$	$1.44552007E-07$	$2.40636E-10$

Table 6 Numerical Results for Test Problem 4

t	Exact	BBCM	EIBBCM ($h = 0.01$)
0.01	0.00012899562284403668	0.0001289956228439637	$7.29668E-17$
0.02	0.0002573965432101358	0.0002573965432102199	$8.40799E-17$
0.03	0.00038519579791147405	0.00038519579791140846	$6.55942E-17$
0.04	0.000512386483927295	0.000512386483927374	$7.89299E-17$
0.05	0.0006389617590932021	0.0006389617590932817	$7.95804E-17$
0.06	0.0007649148427853702	0.0007649148427855135	$1.43223E-16$
0.07	0.0008902390165986053	0.0008902390165986818	$7.64363E-16$
0.08	0.0010149276250181782	0.0010149276250184247	$2.46548E-16$
0.09	0.0011389740760853638	0.0011389740760856526	$2.88831E-16$
0.10	0.0012623718420566414	0.0012623718420570196	$3.78170E-16$

Table 7 Numerical Results for Test Problem 5

t	Exact	BBCM	EIBBCM ($h = 0.01$)
0.2	0.04217138626080574	0.04217138626080914	$3.40060E-15$
0.4	0.35789952803753966	0.3578995280376137	$7.40519E-14$
0.6	1.2904002491766824	1.2904002491771174	$4.34985E-13$
0.8	3.2933353381499177	3.2933353381515276	$1.60982E-12$
1.0	6.98638304633745	6.986383046342062	$4.61142E-12$
1.2	13.239103191447201	13.23910319145849	$1.12887E-11$
1.4	23.2971625812907	23.297162581315575	$2.48761E-11$
1.6	38.971816809968104	38.971816810019014	$5.09100E-11$
1.8	62.92373948426520	62.92373948436379	$9.85949E-11$
2.0	99.08750629903315	99.08750629921636	$1.83206E-10$

Table 8 Numerical Results for Test Problem 6

t	Exact	BBCM	EIBBCM ($h = 0.01$)
3	-1.527323135908539	-1.5273230429245035‘	$9.29840E-08$
6	2.510535059206013	2.5105347506092786	$3.08597E-07$
9	-2.8092394453688807	-2.8092390262763756	$4.19093E-07$
12	1.9910488550789986	1.9910487618540555	$9.32249E-08$
15	-0.8070186485444338	-0.8070191456233216	$4.97079E-07$

5. DISCUSSION OF RESULTS

In Tables 2 and 3, exact and computed solutions of BBCM for solving Test Problem 1 are shown, and the resulting errors are compared with EIFBM [17], EIM [22] and EIOK [18]. Table 3 reveals the superiority of the derived method over EIM [22] and EIOK [18] but compares favourably with EIFBM [17]. Furthermore, the numerical results of EIBBCM in solving Test Problem 2 are compared to those of EIAO [6] and EIFBM [17] in Table 4 and it shows that EIBBCM has better accuracy than EIAO [6] but compared favorably with EIFBM. In addition, the performance of BBCM in solving Test Problem 3 proved superior to that of EIFBM [17] and EIFO [11] as shown in Table 5. Also, the capability of BBCM in solving nonlinear equations is established in Table 5. The exact and computed solutions of BBCM for solving Test Problems 4 - 6 are shown in Tables 6 to 8 and the displayed errors show that the method can integrate fourth-order initial value problems directly. Finally, the numerical results of the Test Problems displayed in Tables 2 to 8

reveal that the new block method (BBCM) is an effective and efficient direct integrator for linear and non-linear fourth-order initial value problems of ordinary differential equations.

6. CONCLUSION

Two-step fifth-order block method for direct approximation of linear and nonlinear fourth-order initial value problems of ordinary differential equations via multistep collocation was developed. The use of the combination of Hermite and shifted Chebyshev polynomials as basis functions and their fourth derivatives as collocating equations were considered. The properties of BBCM were also discussed. The efficiency of the BBCM was shown by applying the method to six Test Problems (linear and non-linear) as seen in Tables 2 to 8 and the numerical results from the method were compared to the exact solutions and the numerical results of some existing numerical methods in literature. BBCM has proven effective in the direct integration of fourth-order initial value problems of ordinary differential equations.

REFERENCES

[1] A. O. Adesanya, A. A. Momoh, M. A. Alkali, A. Tahir, *Five steps block method for the solution of fourth-order ODEs.* International Journal of Engineering Research and Application. 2 (5) 991 - 998, 2012.

[2] O. Adeyeye and Z. Omar, *Solving fourth-order linear initial and boundary value problems using an implicit block method.* In the proceedings of the Third International Conference on Computing, Mathematics and Statistics (iCMS2017) (2019) 167 - 177 Springer, Singapore 2019.

[3] B. I. Akinnukawe, K. O. Muka, *L-stable block hybrid numerical algorithm for first-order ordinary differential equations.* Journal of the Nigerian Society of Physical Sciences 2, 160 - 165, 2020.

[4] R. Allogmany, F. Ismail, Z. Abdul Majid, Z. Bibi Ibrahim, *Implicit two-point block method for solving fourth-order IVPs directly with application.* Mathematical Problems in Enginneering, Article ID 6351279, 13 pages, 2020.

[5] A. K. Alomari, N. Ratib-Anakira, A. S. Bataineh, I. Hashim, *Approximate solution of non-linear system of bvp arising in fluid flow problem*, Mathematical problems in Engineering. 136043, 1 - 7, 2013. https://doi.org/10.1155/2013/136043.

[6] E. A. Areo, E. O. Omole, *Half-step symmetric continuous hybrid block method for numerical solution of fourth-order ODEs*, Archieves of Applied Science Research 7, 39, 2015.

[7] D. O. Awoyemi, *Algorithmic collocation approach for direct solution of fourth-order initial value problems of ordinary differential equations*, International Journal of Computer Mathematics. 82(3) 321 - 329, 2005. http://doi.org/10.1080/00207160412331296634.

[8] D. O. Awoyemi, *A P-stable linear multistep method for solving general third-order ordinary differential equations.* Intern. J. Comp. Math., 80 (8) 987 - 993, 2003. http://doi.org/10.1080/0020716031000079572.

[9] A. Boutayeb and A. Chetouani,*A mini-review of numerical methods for high order problems*, International Journal of Computer Mathematics. 84 (4) 563 -579, 2007. https://doi.org/10.1080/00207160701242250

[10] J. C. Butcher, *Numerial methods for ordinary differential equations.* Second Edition, John Wiley and sons Ltd, England, 2008.

[11] A. B. Familua and E. O. Omole, *Five points mono hybrid linear multistep method for solving nth-order ODEs using power series function.* Asian Research Journal of Mathematics (Science Domian International) 3 (2017) 1, 2017.

[12] S. O. Fatunla,*A class of block methods for second-order IVPs.* Int. J. Comp. Math., 55 (1994) 119 - 133, 1994.

[13] E. Hairer, S. P. Norsett and G. Wanner, *Solving ordinary differential equations 1: Non-stiff problems.* Springer- Verlag, Berlin, 1993. htps://doi.org/10.1007/978-3-540-78862-1.

[14] S. N. Jator, *Numerical integrator for fourth-order initial and boundary value problems*, International Journal of pure and applied Mathematics. 47(4) 563-576, 2008.

[15] S. J. Kayode, *An order six zero-stable method for direct solution of fourth-order ordinary differential equations.* American Journal of Applied Sciences. 5 (11) 1461 - 1466, 2008. http://doi.org/10.3844/ajassp.2008.1461.1466.

[16] S. J. Kayode, *An efficient zero-stable method for fourth-order differential equations*, International Journal for Mathematics and Mathematical Sciences. 2008 (2008) 1 - 10. https://doi.org/10.1155/2008/364021.

[17] J. O. Kuboye, O. R. Elusakin, O. F. Quadri, *Numerical algorithm for direct solution of fourth-order ODEs.* Journal of the Nigerian Society of Physical Sciences 2, 218 - 227, 2020.

[18] J. O. Kuboye and Z. Omar, *New Zero-stable block method for direct solution of fourth-order ODEs*, Indian Journal of Science and Technology. 8 , 1, 2015.

[19] J. D. Lambert, *Numerical methods for ordinary differential systems*, Wiley, Chichester, 1991.

[20] J. D. Lambert, *Computationall methods for ordinary differential equations*, John Wiley, New York, 1973.

[21] A. Malek and R. S. Beidokhti, *Numerical solution of high order differential equations using a hybrid neural network-optimization method*, Applied Mathematics and Computation. 183 (1) 260 - 271, 2006. https://doi.org/10.1016/j.amc.2006.05.068

[22] U. Mohammed, *A Six step block method for solution of fourth-order ODEs.* The Pacific Journal of Science and Technology, 11, 259, 2010.

[1]DEPARTMENT OF MATHEMATICS, UNIVERSITY OF LAGOS, LAGOS STATE, NIGERIA

E-mail address: iakinnukawe@unilag.edu.ng

[2]DEPARTMENT OF MATHEMATICS, MODIBBO ADAMA UNIVERSITY, YOLA, NIGERIA

E-mail addresses: odekunle@mautech.edu.ng