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STABILITY AND BOUNDEDNESS ANALYSIS OF A PREY-PREDATOR SYSTEM WITH PREDATOR CANNIBALISM

A. L. OLUTIMO, D. O. ADAMS AND A. A. ABDURASID

ABSTRACT. The prey-predator system with predator cannibalism is considered in this paper. We employ the Lyapunov's direct method for the prey-predator system and demonstrate its efficacy. This method is built upon theoretical Lyapunov's function that is constructed such that the scalar function and its derivative is positive and negative definite respectively to determine the dynamic behaviour of the system considered including stability and boundedness. The results show that the density functions describing the prey-predator system is better rapidly converging and have finite limits under certain sufficient conditions obtained by the Lyapunov functional. These improve on some earlier results in the literature. We give numeric example to support our findings.

Keywords and phrases: Prey-predator system, predator cannibalism, stability, boundedness, Lyapunov's direct method
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1. INTRODUCTION

In the relevant literature, several works have been done on the qualitative behaviour of prey-predator systems because of its important role in mathematical ecology and applied mathematics. (See Berryman [3], Gonzalez and Ramos [7] and Chen et al. [5]). Recent studies have also shown that cannibalism has a significant effect on the dynamic behaviour of the population as well. (See Smith and Reay [15], Rodriguez and Kang [14] and Deng et al

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[6]). Cannibalism is a nature that compels same species to consume one another to provide food sources or inability to capture prey. The effect of cannibalism on the model systems have been investigated by Magnusson [9], Marik and Pribylona [10] and the references cited therein. Basheer et al [2] studied the predator-prey model with prey non-linear cannibalism. They found that the prey-cannibalism alters dynamics of the predator-prey system which is stable in the absence of cannibalism while it is unstable with prey cannibalism under the same conditions. The qualitative analysis of prey-predator model incorporating nonlinear cannibalism is very difficult to analyze. For example, Zhang et al [17] investigated a diffusive predator-prey model with predator cannibalism which has both negative and positive effect on the dynamic behaviours of the system. Most studies mentioned above used simple mathematical analysis and numerical simulation to obtain their results.
The Lyapunov theory plays a very significant role in the analysis of stability and boundedness of models of natural phenomena. The analysis of prey-predator models show that Lyapunov's theory is rarely visible. For instance, very recently Olutimo et al [12] by constructing a Lyapunov functional related to the model established sufficient conditions for the stablity and boundedness of Lotka-Volterra prey-predator model with prey refuge and predator cannibalism. According to Qin et al [13], Lyapunov functional approach remains an excellent tool in the study of dynamical system. However, the construction of these Lyapunov functionals is indeed a general problem. (See Cartwright [4] and Yoshizawa [16]).

This paper is based on the following prey-predator system with predator cannibalism of the form

$$\begin{aligned} \dot{x}(t) &= x(\alpha - \beta x - \delta y) \\ \dot{y}(t) &= y(-\lambda + \mu + \upsilon x) - \rho\left(\frac{y^2}{\tau + y}\right)^2, \end{aligned} \qquad (1)$$

where x, y are the densities of the prey and predator respectively at time t, α and β are intrinsic growth rate and intraspecific of the prey, δ is the strength of intraspecific interaction between prey and predator, λ is the predator death rate, μ is the birth rate from the predator cannibalism and υ is the conversion efficiency of the ingested prey into a new predator. $\rho\left(\frac{y^2}{\tau+y}\right)^2$ is the cannibalism of the predator which incorporates continuous intra interaction between the predators while ρ is the rate of cannibalism and the constants

$\alpha, \beta, \delta, \lambda, \mu, \upsilon, \rho, \tau$ are positive. The Lyapunov functions used in some of the papers mentioned above do not posses a functional relationship to the original model system considered. In this paper, motivated by the results of Zhang et al [17] and Deng et al [6] where they discussed the stability and global stability of the equilibra of the systems investigated. Determining for more satisfactory results, we consider the global asymptotic stability with an extended region of stability and boundedness where the density functions are restricted to finite limits. Essentially, these results require the construction of a suitable complete Lyapunov function which possess a functional relationship to the same original model system under consideration and examine the global asymptotic stability and boundedness of the density functions describing the prey-predator system Eq. (1) with sufficient criteria to achieve our new results. The results obtained improve the dynamic behaviour of system Eq. (1). Also, we give numeric example and geometric arguments to support our findings on the dynamic behaviours of the system.

2. PRELIMINARY

We give the following basic concept to the boundedness and stability of solutions (see Adams and Omeike [1], Liapunoff [8], Ogundare [11], Yoshizawa [16]). Consider a first order non-autonomous differential equation of the form

$$\dot{x}= G(t, x), \tag{2}$$

where $G(t, x)$ is defined and continuous on $I \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$.

Definition 1.1. The solution $x(t) \equiv 0$ of Eq. (2) is stable, if for any $\epsilon > 0$ and any $t_0 \in I$, there exists a $\delta(t_0, \epsilon) > 0$ such that if $\|x_0\| < \delta(t_0, \epsilon)$, we have $\|x(t; t_0, x_0)\| < \epsilon$ for all $t \geq t_0$.

Definition 1.2. The zero solution of Eq. (2) is asymptotically stable, if $x(t) \equiv 0$ is stable and there exists a $\delta_0(t) > 0$ such that if $\|x_0\| < \delta_0(t)$, $x(t; t_0, x_0) \longrightarrow 0$ as $t \longrightarrow \infty$.

Definition 1.3. The solution $x(t)$ of Eq. (2) is said to be bounded if there exist a $q > 0$ and there exist a constant Q ($Q > 0$) such that $\|x(t; t_0, x_0)\| < Q$ whenever $\|x_0\| < q$, $t \geq t_0$.

Theorem 1.4. Suppose that there exist a Lyapunov function $W(t, x)$ defined on $I \times \mathbb{R}^n$ which satisfies the following conditions:

(i) $a(\|x\|) \leq W(t, x)$, where $a(r) \in CIP$ (Continuous Increasing positive Definite Function) and

(ii) $\dot{W}\ (t, x)_{(2)} \leq 0$.

Then, the solution (2) is stable.

Theorem 1.5. Under the above assumption in Theorem 1.4, if $\dot{W}\ (t,x) \leq -c(\|x\|)$, where $c(r)$ is continuous on $[0,\Phi]$, ($\Phi > 0$ is a constant) and positive definite, and if $G(t,x)$ is bounded, then the solution $x(t) \equiv 0$ of Eq. (2) is asymptotically stable.

3. STABILITY ANALYSIS

First, we consider the case where predator cannibalism is absent, that is $\rho\bigg(\frac{y^2}{\tau+y}\bigg)^2 = 0$ in Eq. (1).

Theorem 1: In addition to the basic assumptions on the constants $\alpha, \beta, \delta, \lambda, \mu, \upsilon, \rho, \tau,$ we assume that there exist positive constants $L, M, N,\ \epsilon,\ k_1, k_2$ and k_3 such that the following conditions hold

(i) $\alpha < \dfrac{\beta+\delta}{4\epsilon}$
(ii) $\mu + \alpha < \lambda$
(iii) $\mid \frac{k_1}{y} \mid \leq L,\ \mid \frac{k_2}{x} \mid \leq M$ and $\mid k_3 \mid \leq N$, for all x, y.

Then, the analyzed density functions $(x(t), y(t))$ describing the system Eq.(1) are globally asymptotically stable as $t \to \infty$.

Proof: Our main tool in the proof of the result is the Lyapunov function $V(x,y)$ defined by

$$2V(x,y) = \epsilon\upsilon^2x^2 + 2(\beta+\delta)\upsilon xy + (\beta+\delta)^2y^2, \tag{3}$$

where $\epsilon > 1$ will be determined later.

Clearly $V(0,0) = 0$ in Eq. (3). Also, $V(x,y)$ in Eq. (3) can be re-arranged as follows:

$$2V(x,y) = (\epsilon-1)\upsilon^2x^2 + \bigg(x\upsilon + (\beta+\delta)y\bigg)^2. \tag{4}$$

Now it is obvious from Eq. (4) that the function $V(x,y)$ defined in Eq. (3) is a positive definite function. Hence, there is a positive constant ξ_1 small enough such that

$$V(x,y) \geq \xi_1(x^2+y^2). \tag{5}$$

Next, we differentiate the function $V(x,y)$ with respect to t along the trajectory Eq. (1) and after simplification we obtain,

$$\begin{aligned}\frac{dV(x,y)}{dt} = & - \upsilon^2\left[\frac{(\beta+\delta)}{2} - 2\alpha\epsilon\right]x^2 \\ & - \frac{(\beta+\delta)}{2}\left[2(\beta+\delta)(\lambda-\mu) + \upsilon(\lambda-\mu-\alpha)^2\right]y^2 \\ & + \frac{(\beta+\delta)}{2}\left[[x\upsilon - (\lambda-\mu-\alpha)y]^2 - \frac{2\alpha\upsilon^2\epsilon}{(\beta+\delta)}x^2\right] \\ & - \frac{(\beta+\delta)(1-\beta\upsilon) - \epsilon\upsilon^2\delta}{y}x^2y^2 - \frac{\beta\upsilon(\beta+\delta)}{x}x^2y^2 - \beta x\epsilon\upsilon^2x^2.\end{aligned}$$

$$\begin{aligned}\frac{dV(x,y)}{dt} \leq & - \upsilon^2\left[\frac{(\beta+\delta)}{2} - 2\alpha\epsilon\right]x^2 \\ & - \frac{(\beta+\delta)}{2}\left[2(\beta+\delta)(\lambda-\mu) + \upsilon(\lambda-\mu-\alpha)^2\right]y^2 \\ & + \frac{(\beta+\delta)}{2}\left[[x\upsilon - (\lambda-\mu-\alpha)y]^2 - \frac{2\alpha\upsilon^2\epsilon}{(\beta+\delta)}x^2\right] \\ & - \left|\frac{k_1}{y}\right| x^2y^2 - \left|\frac{k_2}{x}\right| x^2y^2 - |k_3|\,\epsilon\upsilon^2x^2,\end{aligned}$$

where $k_1 = (\beta+\delta)(1-\beta\upsilon) - \epsilon\upsilon^2\delta$, $k_2 = \beta\upsilon(\beta+\delta)$ and $k_3 = \beta x$. Using the condition (iii) of Theorem 1, we have that

$$\begin{aligned}\frac{dV(x,y)}{dt} \leq & - \upsilon^2\left[\frac{(\beta+\delta)}{2} - 2\alpha\epsilon\right]x^2 \\ & - \frac{(\beta+\delta)}{2}\left[2(\beta+\delta)(\lambda-\mu) + (\lambda-\mu-\alpha)^2\right]y^2 \\ & + \frac{(\beta+\delta)}{2}\left[[x\upsilon - (\lambda-\mu-\alpha)y]^2 - \frac{2\alpha\upsilon^2\epsilon}{(\beta+\delta)}x^2\right] \\ & - Lx^2y^2 - Mx^2y^2 - N\epsilon\upsilon^2x^2. \qquad (6)\end{aligned}$$

If we choose

$$\epsilon = \frac{[x\upsilon - (\lambda-\mu-\alpha)y]^2(\beta+\delta)}{2\alpha\upsilon^2x^2} > 1,$$

it follows that

$$\begin{aligned}\frac{dV(x,y)}{dt} \leq & - \upsilon^2\left[\frac{(\beta+\delta)}{2} - 2\alpha\epsilon\right]x^2 \\ & - \frac{(\beta+\delta)}{2}\left[2(\beta+\delta)(\lambda-\mu) + (\lambda-\mu-\alpha)^2\right]y^2.\end{aligned}$$

By the hypotheses (i) and (ii) of Theorem 1 and the fact that the last addends of Eq. (6) are certainly negative, we have that

$$\frac{dV(x,y)}{dt} \leq -\xi_2(x^2+y^2), \tag{7}$$

for some $\xi_2 > 0$, where $\xi_2 = min\left\{\frac{[(\beta+\delta)-4\alpha\epsilon)]}{2}\upsilon^2; (\beta+\delta)^2(\lambda - \mu) + \frac{(\beta+\delta)}{2}(\lambda-\mu-\alpha)^2\right\}$.

Thus,

$$\frac{dV(x,y)}{dt} < 0.$$

The proof of Theorem 1 is complete.

Thus, the prey-predator system (1) in the absence of cannibalism is globally asymptotically stable since the conditions in Theorem 1 are satisfied. This makes the dynamic behaviour of the system of the prey and predator to be persistent.

3. BOUNDEDNESS ANALYSIS

As in Theorem 1, the boundedness analysis depend on the differentiable function V defined in Eq. (3). Here, we consider the case where there is a presence of predator cannibalism incorporating intra interaction, that is, $\rho\left(\frac{y^2}{\tau+y}\right)^2 \neq 0$ in Eq. (1).

Theorem 2: Let all the conditions of Theorem 1 be satisfied and suppose further that

(iv) $\mid \rho\left(\frac{y^2}{\tau+y}\right)^2 \mid \leq \phi(t)$ for all t≥ 0.

Then, there exist a constant $\sqrt{B_2}$ such that the density functions $x(t), y(t)$ describing the system Eq. (1) are uniformly satisfying

$$\mid x(t) \mid \leq \sqrt{B_2}, \quad \mid y(t) \mid \leq \sqrt{B_2},$$

where the magnitude of $\sqrt{B_2}$ depends only on $\alpha, \beta, \delta, \lambda, \mu, \upsilon, \rho, \tau$.

Proof: In view of Eq. (7), we have that

$$\frac{dV(x,y)}{dt} \leq -\xi_2(x^2+y^2) + (\beta+\delta)\Big[x\upsilon + (\beta+\delta)y\Big] \mid \rho\left(\frac{y^2}{\tau+y}\right)^2 \mid .$$

Then

$$\frac{dV(x,y)}{dt} \leq -\xi_2(x^2+y^2) + \eta_1(\mid x\mid + \mid y \mid) \mid \rho \mid \left(\frac{y^2}{\tau+y}\right)^2,$$

where $\eta_1 = max\{(\beta+\delta)\upsilon; (\beta+\delta)^2\}$.
Using the condition (iv) of Theorem 2, we have

$$\frac{dV(x,y)}{dt} \le -\xi_2(x^2+y^2) + \eta_1(\mid x \mid + \mid y \mid)\phi(t).$$

Making use of the inequalities $\mid x \mid < 1+x^2$, $\mid y \mid < 1+y^2$, it is clear that

$$\frac{dV(x,y)}{dt} \le \eta_1(2+x^2+y^2)\phi(t).$$

By Eq. (5), we have that $(x^2+y^2) \le \xi_1^{-1}V(x,y)$. Thus, it follows that

$$\frac{dV(x,y)}{dt} \le \eta_1(2+\xi_1^{-1}V(x,y))\phi(t).$$

This implies,

$$\frac{dV(x,y)}{dt} - \frac{\eta_1}{\xi_1}\phi(t)V(x,y) \le 2\eta_1\phi(t). \tag{8}$$

Multiplying both sides of Eq. (8) by the integrating factor $exp(-\frac{\eta_1}{\xi_1}\int_0^t \phi(s)ds)$ gives

$$\frac{d}{dt}\left(V(x,y)exp(-\frac{\eta_1}{\xi_1}\int_0^t \phi(s)ds)\right) \le 2\eta_1\phi(t)exp(-\frac{\eta_1}{\xi_1}\int_0^t \phi(s)ds). \tag{9}$$

Integrating both sides of inequality (9) from 0 to t yields

$$V(x,y)exp\left(-\frac{\eta_1}{\xi_1}\int_0^t \phi(s)ds\right) \le V(0)+2\eta_1\int_0^t \phi(s)exp\left(-\frac{\eta_1}{\xi_1}\int_0^t \phi(s)ds\right), \tag{10}$$

where $V(0) = V(0,0)$.
By Gronwall - Reid inequality Eq. (10) becomes

$$V(x,y) \le \left(V(0)+2\eta_1\phi(t)\right)\exp\left(\frac{\eta_1}{\xi_1}\int_0^t \phi(s)ds\right).$$

If we choose $\int_0^t \phi(s)ds \le A$, we have that

$$V(x,y) \le \left(V(0)+2\eta_1 A\right)\exp\left(\frac{\eta_1}{\xi_1}A\right) = B_1 < \infty, \tag{11}$$

where $B_1 > 0$.
In view of Eq. (5) such that $x^2+y^2 \le \xi_1^{-1}V(x,y) \le B_2$, where $B_2 = B_1\xi_1^{-1}$. Therefore Eq. (11) implies that

$$\mid x(t) \mid \le \sqrt{B_2}, \quad \mid y(t) \mid \le \sqrt{B_2} \ \ for \ \ all \ \ t \ge t_o > 0.$$

The proof of Theorem 2 is complete.

Hence, the density functions x, y describing the prey-predator system (1) in the presence of cannibalism is bounded with finite limits since the conditions in Theorem 1 and Theorem 2 are satisfied. This makes the dynamic behaviour of the system of the prey and predator to be persistent despite the presence of predator cannibalism incorporating continuous intra interaction between the predators.

4. NUMERICAL EXAMPLE

Consider Eq. (1) in the form

$$\begin{aligned} \dot{x}(t) &= x(2 - 20x - 12y) \\ \dot{y}(t) &= y(-10 + 1.1 + 1.6x) - 3\left(\frac{y^2}{1.3 + y}\right)^2, \end{aligned} \tag{12}$$

where $\alpha = 2$, $\beta = 20$, $\delta = 12$, $\lambda = 10$, $\mu = 1.1$, $\upsilon = 1.6$, $\rho = 3$ and $\tau = 1.3$
First, since $\epsilon > 1$ for all x, y in,

$$\epsilon = \frac{[x\upsilon - (\lambda - \mu - \alpha)y]^2(\beta + \delta)}{2\alpha\upsilon^2 x^2} > 1,$$

we pick $\epsilon = 2 > 1$, so it is easy to check that the hypothesis in Theorem 1 are satisfied since $\alpha = 2 < 4$, $\lambda > 2 + 1.1 = 3.2$ and $k_1, k_2, k_3 > 0$ for all x, y. Hence, this shows that all the conditions of Theorem 1 and Theorem 2 are satisfied. Thus, we conclude that analyzed density functions $(x(t), y(t))$ describing the system (12) are globally asymptotically stable and bounded.

5. CONCLUSION

The application of Lyapunov theory to analysis of prey-predator system is rarely scarce. The Lyapunov's second or direct method allow to predict and characterize the dynamic behaviour of prey-predator system with cannibalism. In this paper, the global asymptotic stability and boundedness analysis of the prey-predator system with predator cannibalism incorporating continuous intra interaction between the predators were carried out by the Lyapunov's direct method. By constructing a suitable Lyapunov function that possess a functional relationship with system considered, we obtained sufficient criteria for the global asymptotic stability and boundedness of the density functions $x(t), y(t)$ describing the prey-predator system. Numerical example was given to support our findings. The results show that the conditions obtained serve as a stabilizing mechanism in the dynamic behaviour of the prey-predator system. The global asymptotic stability obtained has its region of stability extended. A suitable value of $\epsilon > 1$ can be found to give the largest stability boundary of the system under investigation. The boundedness result obtained showed that the density functions x, y must have finite limits which played a very significant role in characterizing the density functions and ensure that the two species are persistent despite the presence of predator cannibalism with 'intra interactions' which can increase the rate of cannibalism ρ if is fight or decrease if it is affinity.

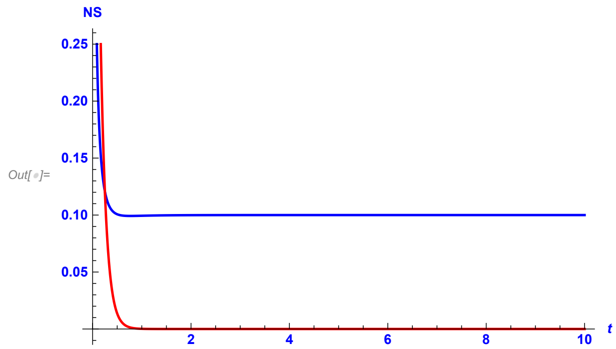

FIGURE 1. The population densities $x(t), y(t)$ of prey (in blue) and predator in (red) respectively describing system (12) are asymptotically stable as $t \to \infty$. ($\alpha = 2, \beta = 20, \delta = 12, \lambda = 10, \mu = 1.1, \upsilon = 1.6$.)

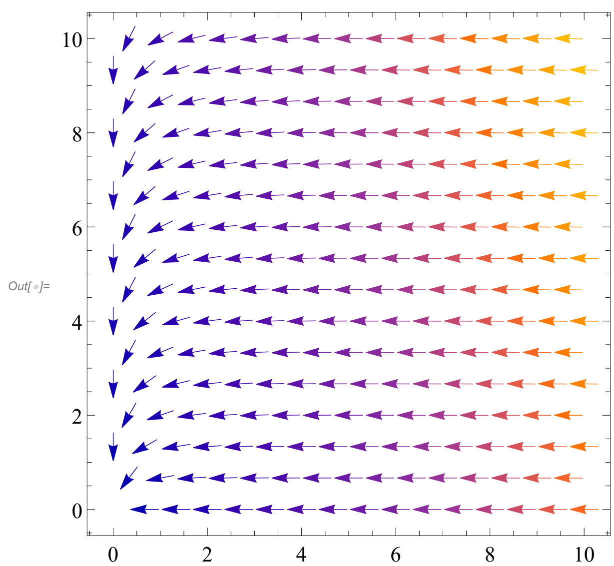

FIGURE 2. Visualizing the trajectories of system (12) satisfying the conditions of Theorem 1 tend to $(0, 0)$ as $t \to \infty$. ($\alpha = 2, \beta = 20, \delta = 12, \lambda = 10, \mu = 1.1, \upsilon = 1.6$.)

Fig. 1 and Fig. 2 show that the density functions are globally asymptotically stable with rapid convergence to $(0, 0)$ and in Fig. 3 and Fig. 4 are bounded which ensure that the two species will be persistent and in the long run will not lead to predator extinction and vice-versa.

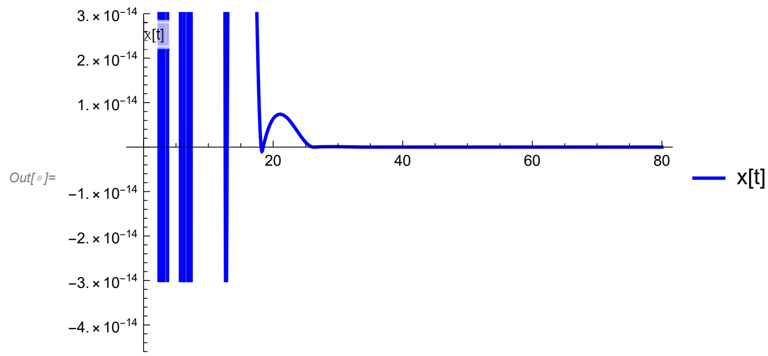

FIGURE 3. A graph of population density of prey $x(t)$ satisfying the conditions of Theorem 1 and 2 is bounded by a single constant as $t \to \infty$. ($\alpha = 2, \beta = 20, \delta = 12, \lambda = 10, \mu = 1.1, \upsilon = 1.6, \rho = 3, \tau = 1.3$.)

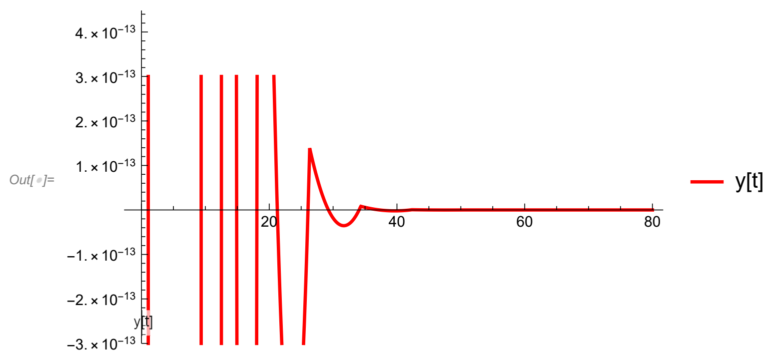

FIGURE 4. A graph of population density of prey $y(t)$ satisfying the conditions of Theorem 1 and 2 is bounded by a single constant as $t \to \infty$. ($\alpha = 2, \beta = 20, \delta = 12, \lambda = 10, \mu = 1.1, \upsilon = 1.6, \rho = 3, \tau = 1.3$.)

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DEPARTMENT OF MATHEMATICS, LAGOS STATE UNIVERSITY, OJO, NIGERIA
E-mail address: akinwale.olutimo@lasu.edu.ng
DEPARTMENT OF MATHEMATICS, FEDERAL UNIVERSITY OF AGRICULTURE, ABEOKUTA, NIGERIA.
E-mail addresses: danielogic2008@yahoo.com
DEPARTMENT OF MATHEMATICS AND STATISTICS, LAGOS STATE UNIVERSITY OF SCIENCE AND TECHNOLOGY, IKORODU, NIGERIA.
E-mail addresses: abdurasid.abdullai@yahoo.com