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ALGEBRAIC POINTS OF DEGRRE AT MOST 14 ON THE FERMAT SEPTIC

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ABSTRACT. In this paper, we study the algebraic points of degree at most 14 over $\mathbb{Q}$ on the Fermat septic curve F_7 of projective equation $X^7 + Y^7 + Z^7 = 0$. Tzermias determined in 1998 in ([11]) all algebraic points of degree at most 5 over $\mathbb{Q}$ on F_7 and O. Sall improved the result of Tzermias by determining in 2003 in ([9]), the algebraic points of degree at most 10 over $\mathbb{Q}$. Using their results and Abel Jacobi's theorem, we extend their work by giving a geometric description of algebraic points of degree at most 14 over $\mathbb{Q}$ on F_7. .

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1. Introduction

Let $\mathcal{C}$ be a smooth projective plane curve of degree d defined over $\mathbb{Q}$. The degree of an algebraic point on $\mathcal{C}$ is the degree of its field of definition over $\mathbb{Q}$. A theorem of Debarre and Klassen ([5]) asserts that

(1) If $d \geq 7$, then the set of algebraic points on $\mathcal{C}$ of degree at most $d-2$ over $\mathbb{Q}$ is finite.
(2) If $d \geq 8$, then, with a finite number of exceptions, the set of algebraic points on $\mathcal{C}$ of degree at most $d-1$ over $\mathbb{Q}$ arise as the intersection of $\mathcal{C}$ with a rational line through a rational point of $\mathcal{C}$.

We denote by F_7 the Fermat septic, i.e., the smooth plane curve of degree 7 with projective equation

$$F_7 = \{(X, Y, Z) \in \mathbb{P}^2(\overline{\mathbb{Q}}) : X^7 + Y^7 + Z^7 = 0\}.$$

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We denote by J_7 the Jacobian of F_7 and its genus is 15. According to (1), the set of algebraic points on F_7 of degree at most 5 over $\mathbb{Q}$ is finite. Tzermias ([11]) has completely described this set. There are exactly five algebraic points of degree at most 5 on F_7, namely $a = (0, -1, 1)$, $b = (-1, 0, 1)$, $\infty = (-1, 1, 0)$, $P = (-\eta, -\overline{\eta}, 1)$ and $\overline{P} = (-\overline{\eta}, -\eta, 1)$ where η is a primitive 6−th root of unity in $\overline{\mathbb{Q}}$ and $\overline{\eta}$ is the complex conjugate of η.
Sall ([9], [10]) has pushad this description by determining the algebraic points on F_7 of degree at most 10 over $\mathbb{Q}$, and he has established the following theorem :

Theorem 1.

(1) The algebraic points on F_7 of degree 6 over $\mathbb{Q}$ are obtained as intersection of F_7 with a line defined over $\mathbb{Q}$ passing through a, b or ∞.
(2) The algebraic points on F_7 of degree 7 over $\mathbb{Q}$ are obtained as intersection of F_7 with a line defined over $\mathbb{Q}$.
(3) There are no algebraic points on F_7 of degree 8 or 9 over $\mathbb{Q}$.
(4) The algebraic points on F_7 of degree 10 over $\mathbb{Q}$ are obtained as residual intersection of F_7 with a conic $\mathcal{C}$ defined over $\mathbb{Q}$ having a contact point of order 2 at $\{a, b\}$ or $\{a, \infty\}$ or $\{b, \infty\}$.

In this note, we propose to extend this geometric description of algebraic points on F_7 of degree at most 14 over $\mathbb{Q}$.

2. Main Result

Our main result is the following theorem :

Theorem 2 : Let F_7 be the Fermat septic.

(1) The algebraic points on F_7 of degree 11 over $\mathbb{Q}$ are obtained as intersection of F_7 with a conic defined over $\mathbb{Q}$ passing through one of the points a, b, ∞ and tangent to one of the other two.
(2) The algebraic points on F_7 of degree 12 over $\mathbb{Q}$ are obtained either as intersection of F_7 with
   (a) a conic defined over $\mathbb{Q}$
      (i) passing through two of the points a, b, ∞ or through P and $\overline{P}$,
      (ii) tangent to F_7 at one of the points a, b, ∞,
   (b) a cubic defined over $\mathbb{Q}$ having a, b and ∞ as contact points of order 3 at each of its points,
   (c) a quartic defined over $\mathbb{Q}$ having P and $\overline{P}$ as contact points of order 8 at each of its points.
(3) The algebraic points on F_7 of degree 13 over $\mathbb{Q}$ are obtained either as intersection of F_7 with

   (a) a conic defined over $\mathbb{Q}$ passing through one of the points a, b, ∞,
   (b) a cubic defined over $\mathbb{Q}$ tangent to F_7 at one of the points a, b, ∞, and having a point of contact of order 3 with the other two.
(4) The algebraic points on F_7 of degree 14 over $\mathbb{Q}$ are obtained either as intersection of F_7 with
   (a) a conic defined over $\mathbb{Q}$,
   (b) a cubic defined over $\mathbb{Q}$
      (i) passing through one of the points a, b, ∞ and having a contact point of order 3 with other two,
      (ii) tangent to F_7 at two of the points a, b, ∞ and having a contact point of order 3 with the other,
   (c) a quartic defined over $\mathbb{Q}$ having P and $\overline{P}$ as contact points of order 7 at each of its points,
   (d) a quintic defined over $\mathbb{Q}$ having a contact point of order 5 at one of the points a, b, ∞ and of order 8 at each of its points P and $\overline{P}$,
   (e) a sextic defined over $\mathbb{Q}$ having two contact points of order 6 among the points a, b, ∞ and of order 8 at each of its points P and $\overline{P}$.

3. Preliminary

3.1. Linear systems.
Let D be a divisor on F_7. The vector space $\mathcal{L}(D)$ is defined to be the set of rational functions

$$\mathcal{L}(D) = \{f \in \overline{\mathbb{Q}}(F_7)^* : \operatorname{div}(f) \geq -D\} \cup \{0\}.$$

The dimension of $\mathcal{L}(D)$ as a $\overline{\mathbb{Q}}-$vector space is denoted by $l(D)$. Consider the rational functions x and y on F_7 given by

$$x(X, Y, Z) = \frac{X}{Z} \quad \text{and} \quad y(X, Y, Z) = \frac{Y}{Z}.$$

Let ε be a primitive $14-$th root of unity in $\overline{\mathbb{Q}}$. The cusps on F_7 are the points

$$a_j = (0, \varepsilon^{2j+1}, 1), \quad b_j = (\varepsilon^{2j+1}, 0, 1), \quad c_j = (\varepsilon^{2j+1}, 1, 0),$$

for $0 \leq j \leq 6$. Observe that $a = a_3$, $b = b_3$ and $\infty = c_3$.

Lemma 1 : [Rohrlich, [8]] We have :
(1) $\operatorname{div}(x) = (a_0 + \cdots + a_6) - (c_0 + \cdots + c_6)$
(2) $\operatorname{div}(x + y) = 7\infty - (c_0 + \cdots + c_6)$.

Lemma 2 : If $k \in \{4, 6\}$, the rational functions f_{rs} defined by

$$f_{rs}(x, y) = \frac{x^r}{(x+y)^s}, \quad \text{with} \quad 0 \leq r \leq s \leq k,$$

form a basis for the vector space $\mathcal{L}(7k\infty)$.

Proof :

(1) For $k = 4$:
According to Lemma 1, we have

$$\begin{aligned}\operatorname{div}(f_{rs}(x, y)) &= r\operatorname{div}(x) - s\operatorname{div}(x+y) \\ &= r(a_0 + \cdots + a_6) + (s-r)(c_0 + c_1 + c_2 + c_4 + c_5 \\ &\quad + c_6) - (6s + r)\infty.\end{aligned}$$

Since $0 \leq r \leq s \leq 4$, $6s + r \leq 28$, so $f_{rs}(x, y) \in \mathcal{L}(28\infty)$. Furthermore, if $6s + r = 6s' + r'$ with $0 \leq r \leq s \leq 4$ and $0 \leq r' \leq s' \leq 4$ then $r \equiv r' \pmod 6$. Since $0 \leq r, r' \leq 4$, then $r = r'$, which implies that $s = s'$. Therefore, the functions f_{rs} with $0 \leq r \leq s \leq 4$ are linearly independent. As the genus of F_7 is 15, 28∞ is a canonical divisor on F_7, hence $l(28\infty) = 15 = \#\{f_{rs},\ 0 \leq r \leq s \leq 4\}$.

(2) For $k = 6$:
As for $k = 4$, we show that the functions $f_{rs}(x, y)$ are in $\mathcal{L}(42\infty)$ and that they are linearly independent. Then, we compute the dimension of $\mathcal{L}(42\infty)$ using the Riemann-Roch theorem (see [4]) which says that $l(d\infty) = d - 14$ as soon as $d \geq 29$. Finally, we have $l(42\infty) = \#\{f_{rs},\ 0 \leq r \leq s \leq 6\}$.

3.2. Mordell-Weil Group.

We denote by $J_7(\mathbb{Q})$ the Mordell-Weil group of rational points of the jacobian J_7 of the curve F_7. For an integer s with $1 \leq s \leq 5$, C_s denotes the affine equation curve $v^7 = u(1-u)^s$ and J_s its jacobian. Consider the rational map defined by

$$f_s : F_7 \longrightarrow C_s, \quad (x, y) \longmapsto (-x^7, (-1)^{s+1}xy^s).$$

This map induces a morphism (also denoted by f_s) $f_s : J_7 \longrightarrow J_s$ and its dual $f_s^* : J_s \longrightarrow J_7$.
Let A and B be the automorphisms of F_7 given by

$$A(X, Y, Z) = (\zeta X, Y, Z) \quad \text{and} \quad B(X, Y, Z) = (X, \zeta Y, Z),$$

where ζ is a primitive $7-$th root of unity such that $\varepsilon^2 = \zeta$. Since $f_s : F_7 \longrightarrow C_s$ is Galois covering whose Galois group is generated by $A^{-s}B$, then for a divisor D of degree zero on F_7, we have

$$f_s^* \circ f_s(D) = \sum_{j=0}^{6} (A^{-s}B)^j(D)$$

on J_7 (see [1]).
It is well-known (see [2], [3], [6]) that these maps induce an isogeny defined over $\mathbb{Q}$

$$f = \prod_{s=1}^{5} f_s : J_7 \longrightarrow \prod_{s=1}^{5} J_s$$

and the dual isogeny

$$f^* = \sum_{s=1}^{5} f_s^* : \prod_{s=1}^{5} J_s \longrightarrow J_7$$

such that $f^* \circ f = 7$ on J_7.

Lemma 3 : $J_7(\mathbb{Q})$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2 \times (\mathbb{Z}/7\mathbb{Z})^2$.

Proof : According to Faddeev ([2], [3]), $J_7(\mathbb{Q})$ is finite. In [11], Tzermias concludes that the following facts :

(1) For a prime $l \neq 2, 7$, the group $J_7[l^\infty](\mathbb{Q})$ is trivial.
(2) The group $J_7[7^\infty](\mathbb{Q})$ is isomorphic to $(\mathbb{Z}/7\mathbb{Z})^2$ and is generated by $[a - \infty]$ and $[b - \infty]$.

It remains to compute the $2-$primary part of $J_7(\mathbb{Q})$. Since there exists an isogeny $f : J_7 \longrightarrow \prod_{s=1}^{5} J_s$, then this amounts to computing the $2-$primary part of each $J_s(\mathbb{Q})$. But now a result of Gross and Rohrlich in [1] states that

$$J_s(\mathbb{Q})_{\text{tors}} \cong \begin{cases} \mathbb{Z}/7\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} & \text{if } 1 \equiv s^3 \equiv (6-s)^3 \pmod 7 \\ \mathbb{Z}/7\mathbb{Z} & \text{otherwise,} \end{cases}$$

but the only $s \leq 5$ such that $1 \equiv s^3 \equiv (6-s)^3 \pmod 7$ are $s = 2$ or $s = 4$, which gives two copies of $\mathbb{Z}/2\mathbb{Z}$ and so

$$J_7[2^\infty](\mathbb{Q}) \cong (\mathbb{Z}/2\mathbb{Z})^2.$$

Now put $x_0 = -P - \overline{P} + 2\infty$, $x_1 = f_4^* \circ f_4(x_0)$ and $x_2 = f_2^* \circ f_2(x_0)$. Referring to the works of Gross and Rohrlich ([1]) and of Sall ([9], [10]), we have the properties given by the following proposition:

Proposition 1 :

(P1) The disivor x_0 is of order 14. We show that $-2x_1$ is the divisor of $y^3 + x + x^3 y$. Thus, x_1 is a point of order 2 on $J_7(\mathbb{Q})$.
(P2) $f_4^* \circ f_4(x_1) = 7x_1$ and $f_2^* \circ f_2(x_2) = 7x_2$.
(P3) $(\mathbb{Z}/2\mathbb{Z})^2 =< x_1, x_2 >$.
(P4) $f_s(a - \infty)$ and $f_s(b - \infty)$ are of order 7.
(P5) $J_s(\mathbb{Q})_{\text{tors}} \subseteq \ker(f_s^*)$.
(P6) $\ker(f_s^* \circ f_s) \subseteq J_7[7]$.

Corollary : We have

$$J_7(\mathbb{Q}) = \Big\{m[\infty - a] + n[\infty - b] + kx_0 + lx_2,\ \text{with } 0 \leq m, n \leq 6 \\ \text{and } 0 \leq k, l \leq 1\Big\},\ \text{or}$$

$$J_7(\mathbb{Q}) = \Big\{m[\infty - a] + n[\infty - b] + kx_0 + lx_1,\ \text{with } 0 \leq m, n \leq 6 \\ \text{and } 0 \leq k, l \leq 1\Big\}$$

Proof : Combining Lemma 3 and (P3), we have

$$J_7(\mathbb{Q}) = \Big\{m_1[\infty - a] + n_1[\infty - b] + kx_1 + lx_2,\ \text{with } 0 \leq m_1, n_1 \leq 6 \\ \text{and } 0 \leq k, l \leq 1\Big\}.$$

Then (P1) and (P2) give $f_4^* \circ f_4(x_1) = x_1 = f_4^* \circ f_4(x_0)$. Thus, according to (P6), we have $x_1 - x_0 \in \ker(f_4^* \circ f_4) \subseteq J_7[7]$, so $x_1 - x_0 = m_2[\infty - a] + n_2[\infty - b]$ with $0 \leq m_2, n_2 \leq 6$. Therefore,

$$J_7(\mathbb{Q}) = \Big\{m[\infty - a] + n[\infty - b] + kx_0 + lx_2,\ \text{with } 0 \leq m, n \leq 6 \\ \text{and } 0 \leq k, l \leq 1\Big\}.$$

Similarly, using $f_2^* \circ f_2$, we find the other expression of $J_7(\mathbb{Q})$.

3.3. Geometric lemmas.

Lemma 4 : Let L_a, L_b and L_∞ be the tangent lines to F_7 at a, b and ∞, respectively.

(1) The lines L_a, L_b and L_∞ have a point of contact of order 7 with F_7 at a, b and ∞, respectively.
(2) If a plane algebraic curve Γ of degree ≤ 6 has a contact point of order $> \deg(\Gamma)$ with F_7 at a, b or ∞, then Γ is reducible and contains L_a, L_b or L_∞, respectively.

Proof :

(1) In affine, we have $F_7 : x^7 + y^7 + 1 = 0$. The tangent line to F_7 at a is $L_a : y + 1 = 0$. It is clear that a is the only point of intersection of the line L_a and the curve F_7. Thus, by Bezout's theorem, we have

$$L_a.F_7 = (\deg L_a \times \deg F_7)a = 7a = \text{mult}_a(L_a \cap F_7)a.$$

We show the same for L_b and L_∞.

(2) Let H, G and F be plane curves. Assume that H is irreducible and is neither a component of G nor of F. Let $\mathcal{O}$ be a non-singular point of H. Then, according to Lemma 2.3.2 in [7], we have:

$$\min\{\mathrm{mult}_{\mathcal{O}}(H \cap F), \mathrm{mult}_{\mathcal{O}}(H \cap G)\} \leq \mathrm{mult}_{\mathcal{O}}(F \cap G).$$

Thus, to obtain the desired result, it suffices to take $\mathcal{O} \in \{a, b, \infty\}$, $H = L_{\mathcal{O}}$, $G = \Gamma$ and $F = F_7$ taking into account (1).

Lemma 5 : Let L be the line with equation $X + Y + Z = 0$. Then $L.F_7 = a + b + \infty + 2P + 2\overline{P}$.

Proof : In affine, $L : x + y + 1 = 0$ and $F_7 : x^7 + y^7 + 1 = 0$. According to Gross and Rohrlich ([1]), we have $L \cap F_7 = \{a, b, \infty, P, \overline{P}\}$; then there exist strictly positive integers n_1, n_2, n_3, n_4, n_5 such that $L.F_7 = n_1 a + n_2 b + n_3 \infty + n_4 P + n_5 \overline{P}$ with $n_1 + n_2 + n_3 + n_4 + n_5 = 7$. Since P and $\overline{P}$ are conjugates, so $n_4 = n_5$. The tangent line to F_7 at P is $T_P : x+y+1 = 0$, hence $n_4 \geq 2$; thus $n_1 = n_2 = n_3 = 1$ and $n_4 = n_5 = 2$.

4. PROOF OF THE MAIN RESULT

Let R be an algebraic point on F_7 of degree $d \leq 14$ over $\mathbb{Q}$; if $d \leq 5$ these points are described by Tzermias ([11]); if $6 \leq d \leq 10$ these points are given by Theorem 1. Thus, we can assume that $11 \leq d \leq 14$. Let $R_1, \cdots, R_d$ be the Galois conjugates of R. Put $t = [R_1 + \cdots + R_d - dP_\infty] \in J_7(\mathbb{Q})$.
According to the corollary, we can consider the following four cases :

Case 1 : $t = m[\infty - a] + n[\infty - b]$ with $0 \leq m, n \leq 6$.

Then we have $[R_1 + \cdots + R_d - d\infty] = m[\infty - a] + n[\infty - b]$, hence $[R_1 + \cdots + R_d + ma + nb - (d + m + n)\infty] = 0$. Since $d + m + n \leq 28$, Lemma 2 leads to the existence of a quartic polynomial $f(x, y)$ such that

$$\mathrm{div}(f(x, y)/(x + y)^4) = R_1 + \cdots + R_d + ma + nb - (d + m + n)\infty.$$

Thus, by Lemma 1,

$$\begin{aligned}\mathrm{div}(f(x, y)) = R_1 + \cdots + R_d + ma + nb + (28 - d - m - n)\infty \\ - 4(c_0 + \cdots + c_6).\end{aligned}$$

Using the homogenized f^* of f, we have

$$f^*(X, Y, Z) = Z^4 f(\frac{X}{Z}, \frac{Y}{Z}),$$

where $f^*(X, Y, Z)$ defines a curve Γ_4 of degree 4; which shows the existence of a quartic Γ_4 defined over $\mathbb{Q}$. As the curve F_7 is smooth, hence $\mathrm{div}(f(x,y)) = \Gamma_4.F_7 - 4(c_0 + \cdots + c_6)$. Therefore,

$$\Gamma_4.F_7 = R_1 + \cdots + R_d + ma + nb + (28 - d - m - n)\infty.$$

If $m \geq 5$, then, by Lemma 4, Γ_4 is reducible and contains L_a. Moreover, since $m \leq 6$, one of the points R_i is a, which is absurd because R_i and a are not of the same degree. Hence $m \leq 4$. Similarly, we have $n \leq 4$. Therefore $6 \leq 28 - d - m - n \leq 17$. The Lemma 4 also shows that Γ_4 contains L_∞, there exists a cubic Γ_3 such that

$$\Gamma_3.F_7 = R_1 + \cdots + R_d + ma + nb + (21 - d - m - n)\infty. \qquad (*)$$

We must have $0 \leq m, n \leq 3$ and so $1 \leq 21 - d - m - n \leq 10$. The sum of the coefficients of a, b and ∞ equals $21 - d$.

1.1 Suppose that $1 \leq 21 - d - m - n \leq 3$. Then, the sum of the coefficients of a, b and ∞ is ≤ 9, i.e., $21 - d \leq 9$, therefore $d \geq 12$. Let m_1, m_2 and m_3 be the coefficients of a, b and ∞ respectively. We have $0 \leq m_1, m_2 \leq 3$, $1 \leq m_3 \leq 3$ and $m_1 + m_2 + m_3 = 21 - d$.
Thus, we obtain :

1.1.a for $d = 12$, the relation $(*)$ becomes

$$\Gamma_3.F_7 = R_1 + \cdots + R_{12} + 3a + 3b + 3\infty,$$

which shows that algebraic points on F_7 of degree 12 over $\mathbb{Q}$ are obtained as intersection of F_7 with a cubic defined over $\mathbb{Q}$ having a, b and ∞ as contact points of order 3 at each of its points.

1.1.b for $d = 13$, the relation $(*)$ becomes

$$\Gamma_3.F_7 = R_1 + \cdots + R_{13} + m_1 a + m_2 b + m_3 \infty$$

with $m_i \in \{2, 3\}$ and $m_1 + m_2 + m_3 = 8$,
which shows that algebraic points on F_7 of degree 13 over $\mathbb{Q}$ are obtained as intersection of F_7 with a cubic defined over $\mathbb{Q}$ tangent to F_7 at one of the points a, b, ∞ and having a point of contact of order 3 with the other two.

1.1.c for $d = 14$, the relation $(*)$ becomes

$$\Gamma_3.F_7 = R_1 + \cdots + R_{14} + m_1 a + m_2 b + m_3 \infty$$

with $m_i \in \{1, 2, 3\}$ and $m_1 + m_2 + m_3 = 7$,
which shows that algebraic points on F_7 of degree 14 over $\mathbb{Q}$ are obtained as intersection of F_7 with a cubic defined over $\mathbb{Q}$

- passing through one of the points a, b, ∞ and having a contact point of order 3 with other two,
- tangent to F_7 at two of the points a, b, ∞ and having a contact point of order 3 with the other.

1.2 Suppose that $21 - d - m - n \geq 4$.
Then, by Lemma 4, Γ_3 contains L_∞. Then there exists a conic Γ_2 such that
$\Gamma_2.F_7 = R_1 + \cdots + R_d + ma + nb + (14 - d - m - n)\infty$. $(**)$.
We must have $0 \leq m, n \leq 2$ and $0 \leq 14 - d - m - n \leq 2$. The sum of the coefficients of a, b and ∞ is equal to $14 - d$. Let m_1, m_2 and m_3 be the coefficients of a, b and ∞ respectively. We have $0 \leq m_i \leq 2$ and $m_1 + m_2 + m_3 = 14 - d$. If the $m_i \neq 0$ then, according to, Lemma 5, Γ_2 contains L, which is absurd otherwise one of the R_i would be equal to P or $\overline{P}$. Hence, at least one of m_i is zero. Thus, we obtain :

1.2.a. for $d = 11$, the relation $(**)$ becomes

$$\Gamma_2.F_7 = R_1 + \cdots + R_{11} + m_1 a + m_2 b + m_3 \infty$$

with $m_i \neq m_j \in \{0, 1, 2\}$ and $m_1 + m_2 + m_3 = 3$,
thus, algebraic points on F_7 of degree 11 over $\mathbb{Q}$ are obtained as intersection of F_7 with a conic defined over $\mathbb{Q}$ passing through one of the points a, b, ∞ and tangent to one of the other two.

1.2.b. for $d = 12$, the relation $(**)$ becomes

$$\Gamma_2.F_7 = R_1 + \cdots + R_{12} + m_1 a + m_2 b + m_3 \infty$$

with $m_i \in \{0, 1, 2\}$ and $m_1 + m_2 + m_3 = 2$,
thus, algebraic points on F_7 of degree 12 over $\mathbb{Q}$ are obtained as intersection of F_7 with a conic defined over $\mathbb{Q}$
- passing through two of the points a, b, ∞,
- tangent to F_7 at one of the points a, b, ∞.

1.2.c. for $d = 13$, the relation $(**)$ becomes

$$\Gamma_2.F_7 = R_1 + \cdots + R_{13} + m_1 a + m_2 b + m_3 \infty$$

with $m_i \in \{0, 1\}$ and $m_1 + m_2 + m_3 = 1$,
thus, algebraic points on F_7 of degree 13 over $\mathbb{Q}$ are obtained as intersection of F_7 with a conic defined over $\mathbb{Q}$ passing through one of the points a, b, ∞.

1.2.d. for $d = 14$, the relation $(**)$ becomes

$$\Gamma_2.F_7 = R_1 + \cdots + R_{14},$$

thus, algebraic points on F_7 of degree 14 over $\mathbb{Q}$ are obtained as intersection of F_7 with a conic defined over $\mathbb{Q}$.

Case 2 : $t = m[\infty - a] + n[\infty - b] + x_0$ with $0 \leq m, n \leq 6$.
Then we have

$$[R_1 + \cdots + R_d + ma + nb + P + \overline{P} - (d + m + n + 2)\infty] = 0.$$

Since $d+m+n+2 \leq 28$, Lemmas 2 and 1 ensure the existence of a quartic polynomial $f(x,y)$ such that:

$$\begin{aligned}\operatorname{div}(f(x,y)) = R_1 + \cdots + R_d + ma + nb + P + \overline{P} + (26-d-m-n)\infty \\ - 4(c_0 + \cdots + c_6).\end{aligned}$$

Since the curve F_7 is smooth, there exists a quartic Γ_4 such that $\operatorname{div}(f(x,y)) = \Gamma_4.F_7 - 4(c_0 + \cdots + c_6)$. As a result,

$$\Gamma_4.F_7 = R_1 + \cdots + R_d + ma + nb + P + \overline{P} + (26-d-m-n)\infty.$$

We must have $0 \leq m,n \leq 4$ and so $4 \leq 26-d-m-n \leq 15$.

2.1 If $26-d-m-n=4$, i.e., $d=14$, $m=n=4$, then $\Gamma_4.F_7 = R_1 + \cdots + R_{14} + 4a + 4b + P + \overline{P} + 4\infty$. We see that Γ_4 contains L, which is absurd otherwise one of the R_i's to equal P or $\overline{P}$.

2.2 If $26-d-m-n \geq 5$, then Γ_4 contains L_∞. There exists a cubic Γ_3 such that

$$\Gamma_3.F_7 = R_1 + \cdots + R_d + ma + nb + P + \overline{P} + (19-d-m-n)\infty.$$

We must have $0 \leq m,n \leq 3$ and $19-d-m-n \geq 0$.

2.2.a. If $m=n=0$, then $\Gamma_3.F_7 = R_1 + \cdots + R_d + P + \overline{P} + (19-d)\infty$. We have $19-d \geq 5$, hence Γ_3 contains L_∞. There exists a conic Γ_2 such that

$$\Gamma_2.F_7 = R_1 + \cdots + R_d + P + \overline{P} + (12-d)\infty,$$

which gives $d=12$, hence

$$\Gamma_2.F_7 = R_1 + \cdots + R_{12} + P + \overline{P},$$

thus, algebraic points on F_7 of degree 12 over $\mathbb{Q}$ are obtained as intersection of F_7 with a conic defined over $\mathbb{Q}$ passing through P and $\overline{P}$.

2.2.b. If $m \neq 0$ or $n \neq 0$, then at least two of the coefficients of a, b and ∞ are non-zero. Consequenlty Γ_3 contains L, which is absurd otherwise one of the R_i's to equal P or $\overline{P}$.

Case 3 : $t = m[\infty - a] + n[\infty - b] + kx_0 + x_1$ with $0 \leq m,n \leq 6$ and $0 \leq k \leq 1$.
Then we have $[R_1 + \cdots + R_d - d\infty] = m[\infty - a] + n[\infty - b] + kx_0 + x_1$.
Composing by $f_4^* \circ f_4$ and using (P4) and (P5), we have

$$f_4^* \circ f_4([R_1 + \cdots + R_d - d\infty]) = f_4^* \circ f_4(kx_0) + f_4^* \circ f_4(x_1).$$

Then, combining (P2) and the definition of x_1, we have

$$f_4^* \circ f_4([R_1 + \cdots + R_d - d\infty]) = f_4^* \circ f_4(kx_0) + f_4^* \circ f_4(7x_0).$$

Thus,

$$f_4^* \circ f_4([R_1 + \cdots + R_d - (7+k)x_0 - d\infty]) = 0.$$

From (P6), we obtain

$$[R_1 + \cdots + R_d - (7+k)x_0 - d\infty] = m[\infty - a] + n[\infty - b].$$

What is also writtten

$$[R_1+\cdots+R_d+ma+nb+(7+k)P+(7+k)\overline{P}-(14+d+m+n+2k)\infty] = 0.$$

Since $14 + d + m + n + 2k \leq 42$, Lemmas 2 and 1 ensure the existence of a sextic polynomial $f(x, y)$ such that

$$\begin{aligned}\operatorname{div}(f(x,y)) = {} & R_1 + \cdots + R_d + ma + nb + (7+k)P + (7+k)\overline{P} \\ & + (28 - d - m - n - 2k)\infty - 6(c_0 + \cdots + c_6).\end{aligned}$$

As the plane curve F_7 is smooth, there exists a sextic Γ_6 such that $\operatorname{div}(f(x,y)) = \Gamma_6.F_7 - 6(c_0 + \cdots + c_6)$. Therefore,

$$\begin{aligned}\Gamma_6.F_7 = {} & R_1 + \cdots + R_d + ma + nb + (7+k)P + (7+k)\overline{P} \\ & + (28 - d - m - n - 2k)\infty.\end{aligned}$$

3.1. $m = 0$ or $n = 0$

3.1.a. If $m = n = 0$, then

$$\Gamma_6.F_7 = R_1 + \cdots + R_d + (7+k)P + (7+k)\overline{P} + (28 - d - 2k)\infty$$

with $12 \leq 28 - d - 2k \leq 17$. The curve Γ_6 contains L_∞, there exists a quintic Γ_5 such that

$$\Gamma_5.F_7 = R_1 + \cdots + R_d + (7+k)P + (7+k)\overline{P} + (21 - d - 2k)\infty$$

with $5 \leq 21 - d - 2k \leq 10$.

3.1.a.i. If $21 - d - 2k = 5$, i.e., $d = 14$ and $k = 1$, then

$$\Gamma_5.F_7 = R_1 + \cdots + R_{14} + 8P + 8\overline{P} + 5\infty,$$

that is, algebraic points on F_7 of degree 14 over $\mathbb{Q}$ are obtained as intersection of F_7 with a quintic defined over $\mathbb{Q}$ having a contact point of order 5 at ∞ and of order 8 at each of its points P and $\overline{P}$.

3.1.a.ii. If $21 - d - 2k \geq 6$, then Γ_5 contains L_∞, there exists a quartic Γ_4 such that

$$\Gamma_4.F_7 = R_1 + \cdots + R_d + (7+k)P + (7+k)\overline{P} + (14 - d - 2k)\infty$$

with $0 \leq 14 - d - 2k \leq 3$. We see that the coefficient of ∞ must be zero otherwise one of R_i should be equal to a or b. Thus $14 - d - 2k = 0$, i.e., we have ($d = 14$ and $k = 0$) or ($d = 12$ and $k = 1$). As a result,

$$\Gamma_4.F_7 = R_1 + \cdots + R_{14} + 7P + 7\overline{P},$$

in other words, algebraic points on F_7 of degree 14 over $\mathbb{Q}$ are obtained as intersection of F_7 with a quartic defined over Q having P and $\overline{P}$ as contact points of order 7 at each of its points; or

$$\Gamma_4.F_7 = R_1 + \cdots + R_{12} + 8P + 8\overline{P},$$

in other words, algebraic points on F_7 of degree 12 over $\mathbb{Q}$ are obtained as intersection of F_7 with a quartic defined over Q having P and $\overline{P}$ as contact points of order 8 at each of its points.

3.1.b. If $m = 0$ and $n \geq 1$ (resp. $m \geq 1$ and $n = 0$), then

$$\Gamma_6.F_7 = R_1 + \cdots + R_d + nb + (7+k)P + (7+k)\overline{P} + (28-d-n-2k)\infty$$

with $6 \leq 28 - d - n - 2k \leq 16$.

3.1.b.i. If $28 - d - n - 2k = 6$, i.e., $d = 14$, $n = 6$ and $k = 1$, then

$$\Gamma_6.F_7 = R_1 + \cdots + R_{14} + 6b + 8P + 8\overline{P} + 6\infty,$$

which proves that algebraic points on F_7 of degree 14 over $\mathbb{Q}$ are obtained as intersection of F_7 with a sextic defined over $\mathbb{Q}$ having a contact point of order 6 at b and ∞ and of order 8 at each of its points P and $\overline{P}$.

3.1.b.ii. If $28-d-n-2k \geq 7$, then Γ_6 contains L_∞, there exists a quintic Γ_5 such that

$$\Gamma_5.F_7 = R_1 + \cdots + R_d + nb + (7+k)P + (7+k)\overline{P} + (21-d-n-2k)\infty$$

with $0 \leq 21 - d - n - 2k \leq 9$. Since $n \neq 0$, the coefficient of ∞ must be zero, i.e., $21 - d - n - 2k = 0$, in this case, we have ($d = 13$, $n = 6$ and $k = 1$) or ($d = 14$, $n = 5$ and $k = 1$). Thus, $\Gamma_5.F_7 = R_1 + \cdots + R_{13} + 6b + 8P + 8\overline{P}$, this case is absurd otherwise one of the R_i's to equal b or

$$\Gamma_5.F_7 = R_1 + \cdots + R_{14} + 5b + 8P + 8\overline{P},$$

that is, algebraic points on F_7 of degree 14 over $\mathbb{Q}$ are obtained as intersection of F_7 with a quintic defined over $\mathbb{Q}$ having a contact point of order 5 at b and of order 8 at each of its points P and $\overline{P}$.

3.2. $m = 1$ or $n = 1$

3.2.a. If $m = n = 1$, then

$$\Gamma_6.F_7 = R_1 + \cdots + R_d + a + b + (7+k)P + (7+k)\overline{P} + (26-d-2k)\infty.$$

The curve Γ_6 contains L, there exists a quintic Γ_5 such that

$$\Gamma_5.F_7 = R_1 + \cdots + R_d + (5+k)P + (5+k)\overline{P} + (25-d-2k)\infty$$

with $9 \leq 25 - d - 2k \leq 14$. The curve Γ_5 contains L_∞, there exists a quartic Γ_4 such that

$$\Gamma_4.F_7 = R_1 + \cdots + R_d + (5+k)P + (5+k)\overline{P} + (18-d-2k)\infty$$

with $2 \leq 18 - d - 2k \leq 7$. As the coefficient of ∞ is non-zero, then Γ_4 contains L which is absurd otherwise one of the R_i is a or b.

3.2.b If $m = 1$ and $n \geq 2$ (resp. $m \geq 2$ and $n = 1$), then

$$\Gamma_6.F_7 = R_1+\cdots+R_d+a+nb+(7+k)P+(7+k)\overline{P}+(27-d-n-2k)\infty.$$

We see that Γ_6 contains L, there exists a quintic Γ_5 such that

$$\Gamma_5.F_7 = R_1+\cdots+R_d+(n-1)b+(7+k)P+(7+k)\overline{P}+(26-d-n-2k)\infty.$$

Γ_5 contains L which is absurd otherwise one of the R_i is a.

3.3. $2 \leq m, n \leq 6$

3.3.a. If $28 - d - m - n - 2k = 0$, i.e., $d = 14$, $m = n = 6$ and $k = 1$, then

$$\Gamma_6.F_7 = R_1 + \cdots + R_{14} + 6a + 6b + 8P + 8\overline{P},$$

which proves that algebraic points on F_7 of degree 14 over $\mathbb{Q}$ are obtained as intersection of F_7 with a sextic defined over $\mathbb{Q}$ having a contact point of order 6 at a and b and of order 8 at each of its points P and $\overline{P}$.

3.3.b If $28 - d - m - n - 2k \geq 1$, then Γ_6 contains L, there exists a quintic Γ_5 such that

$$\Gamma_5.F_7 = R_1 + \cdots + R_d + (m-1)a(n-1)b + (5+k)P$$
$$+ (5+k)\overline{P} + (27 - d - m - n - 2k)\infty.$$

We see that Γ_5 contains L, there exists a quartic Γ_4 such that

$$\Gamma_4.F_7 = R_1 + \cdots + R_d + (m-2)a + (n-2)b + (3+k)P+$$
$$(3+k)\overline{P} + (26 - d - m - n - 2k)\infty.$$

Since, the coefficients of a, b and ∞ are not simultaneously zero, then Γ_4 contains L, there exists a cubic Γ_3 such that $\Gamma_3.F_7 = R_1 + \cdots + R_d + (m-3)a + (n-3)b + (1+k)P + (1+k)\overline{P} + (25 - d - m - n - 2k)\infty$.

We must have $3 \leq m, n \leq 6$ and $25 - d - m - n - 2k \geq 0$.

3.3.b.i. If $m = n = 3$, then

$$\Gamma_3.F_7 = R_1 + \cdots + R_d + (1+k)P + (1+k)\overline{P} + (19 - d - 2k)\infty$$

with $3 \leq 19 - d - 2k \leq 8$.

- If $19 - d - 2k = 3$, i.e., $d = 14$ and $k = 1$, then

$$\Gamma_3.F_7 = R_1 + \cdots + R_{14} + 2P + 2\overline{P} + 3\infty.$$

  We see that Γ_3 contains L, which is absurd.
- If $19-d-2k \geq 4$, then Γ_3 contains L_∞, there exists a conic Γ_2 such that

$$\Gamma_2.F_7 = R_1 + \cdots + R_d + (1+k)P + (1+k)\overline{P} + (12 - d - 2k)\infty.$$

  We must have $12 - d - 2k = 0$, i.e., $d = 12$ and $k = 0$ so $\Gamma_2.F_7 = R_1 + \cdots + R_{12} + P + \overline{P}$. Thus, algebraic points on F_7 of degree 12 over $\mathbb{Q}$ are obtained as intersection of F_7 with a conic defined over $\mathbb{Q}$ passing through P and $\overline{P}$.

3.3.b.ii. If $m \neq 3$ or $n \neq 3$, then Γ_3 contains L, there exists a conic Γ_2 such that

$$\Gamma_2.F_7 = R_1 + \cdots + R_d + (m-4)a + (n-4)b + (-1+k)P + (-1+k)\overline{P} + (24-d-m-n-2k)\infty.$$

We must have
$4 \leq m, n \leq 6$, $k = 1$ and $0 \leq 24 - d - m - n - 2k \leq 2$.
The sum of the coefficients of a, b and ∞ is equal to $14-d$. We have

- $\Gamma_2.F_7 = R_1 + \cdots + R_{11} + m_1 a + m_2 b + m_3 \infty$ with $m_i \neq m_j \in \{0,1,2\}$ and $m_1+m_2+m_3 = 3$, thus, algebraic points on F_7 of degree 11 over $\mathbb{Q}$ are obtained as intersection of F_7 with a conic defined over $\mathbb{Q}$ passing through one of the points a, b, ∞ and tangent to one of the other two.
- $\Gamma_2.F_7 = R_1 + \cdots + R_{12} + m_1 a + m_2 b + m_3 \infty$ with $m_i \in \{0,1,2\}$ and $m_1 + m_2 + m_3 = 2$, thus, algebraic points on F_7 of degree 12 over $\mathbb{Q}$ are obtained as intersection of F_7 with a conic defined over $\mathbb{Q}$
    * passing through two of the points a, b, ∞,
    * tangent to F_7 at one of the points a, b, ∞.
- $\Gamma_2.F_7 = R_1 + \cdots + R_{13} + m_1 a + m_2 b + m_3 \infty$ with $m_i \in \{0,1\}$ and $m_1 + m_2 + m_3 = 1$, thus, algebraic points on F_7 of degree 13 over $\mathbb{Q}$ are obtained as intersection of F_7 with a conic defined over $\mathbb{Q}$ passing through one of the points a, b, ∞.
- $\Gamma_2.F_7 = R_1 + \cdots + R_{14}$, thus, algebraic points on F_7 of degree 14 over $\mathbb{Q}$ are obtained as intersection of F_7 with a conic defined over $\mathbb{Q}$.

Case 4 : $t = m[\infty - a] + n[\infty - b] + kx_0 + x_2$ with $0 \leq m, n \leq 6$ and $0 \leq k \leq 1$.
Composing by $f_2^* \circ f_2$ and using the properties (P2), (P4), (P5) and (P6), we find exactly the same expression as in the Case 3 and therefore we obtain the same results.

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