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NON-STANDARD HIGHER-ORDER $\mathbb{G}$-STRAND PARTIAL DIFFERENTIAL EQUATIONS ON MATRIX LIE ALGEBRA

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ABSTRACT. Let $\mathbb{G}$ be a Lie group and $g(t,s):\mathbb{R}\times\mathbb{R}\to\mathbb{G}$ be its corresponding map where t and s are independent variables. A $\mathbb{G}$-strand gives rise to dynamical equations for a map $\mathbb{R}\times\mathbb{R}$ into $\mathbb{G}$ that follows from the standard Hamilton's principle. It is believed that a good number of important dynamical equations arising in different fields of sciences can be written as the Euler-Poisson equations on a matrix Lie algebra $\mathfrak{g}$ of $\mathbb{G}$. This picture was extended in literature to the higher-order derivatives case through different contexts in particular when the original configuration space is a configuration manifold $\mathbb{Q}$ on which a Lie group $\mathbb{G}$ acts appropriately. The goal of this paper is to extend $\mathbb{G}$-strand equations on matrix Lie algebra to the higher-order derivatives case through a different approach and more precisely by means of non-standard Lagrangians where higher-order derivatives occur naturally although the Lagrangian holds 1st order derivative terms. Some consequences are discussed accordingly.

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1. INTRODUCTION

In Lagrangian classical mechanics, a well-known result obtained by Henri Poincaré in 1901 concerns the fact that when a Lie algebra $\mathfrak{g}$ acts locally transitively on the configuration space of the system, the Euler-Lagrange equations are equivalent to a new

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system of differential equations known as the Euler-Poincaré equations defined on the product of the configuration space with $\mathfrak{g}$. This result is very important as many important partial differential equations arising in sciences such as the Korteweg-de Vries and the Camassa-Holm equations can be written as the Euler-Poincaré equations on matrix Lie algebra $\mathfrak{g}$ of G [**18-27**].

The extension of this problem for the case of higher-order derivatives was explored in literature through different contexts, e.g. higher-order variational problems that are invariant under Lie group transformations [**16**], higher-order variational problem with a Lagrangian function defined on k^{th}-order tangent bundle $T^{(k)}Q$ and hence depends on the first k^{th}-order time derivatives of the curve [**6,17**] and so on. These higher-order derivatives extensions are very important in sciences mainly in optimal control [**8,28**].

The goal of this paper is to explore G-strand higher-order derivatives equations on matrix Lie algebra through a totally different aspect which is related to the notation of non-standard Lagrangians (NSL) or non-natural Lagrangians as entitled by Arnold [**1**]. In fact NSL which are characterized by a deformed kinetic energy and potential energy terms have proved to have a wide class of applications in the theory of nonlinear differential equations [**3-6**] and dynamical systems [**29-32,34**]. In a recent work [**10**] we have introduced two different types of NSL: exponentially Lagrangians and power-law Lagrangians. It was observed that in such types of Lagrangians, higher-order derivatives arise naturally whereas the Lagrangian functional contains up to a 1^{st}-order derivative term. A good number of interesting equations arising in classical theories were obtained using this formalism [**11-14**] and accordingly represent our basic motivation to deal with NSL. Through this paper, we will deal with exponentially NSL.

The paper is organized as follows: In **Sec. 2**, we setup the basic ingredients of the theory; in particular we derive the G-strand higher-order derivatives equations on matrix Lie algebra, the modified Euler-Poisson equations in terms of momentum and the extended G -strand partial differential equations for $G = \mathrm{Diff}(\mathbb{R})$; in **Sec. 3**, we illustrate by discussing some special examples and finally conclusions are given in **Sec. 4**.

2. BASIC INGREDIENTS: EULER-POINCARE THEORY FOR LEFT G -INVARIANT NON-STANDARD LARANGIANS

2. A: Extended G -strand equations on matrix Lie algebras

We start by the following definition:

Definition 2.1 [**10,26**]: Let G be a Lie group for the map $g(t,s):\mathbb{R}\times\mathbb{R}\to G$ characterized by two types of tangent vectors $\dot{g} := g_t \in TG$ and $g' := g_s \in TG$. We define the left G -invariant exponentially non-standard Lagrangian density function by $e^{\xi L(g,\dot{g},g')}$ and the associated exponentially non-standard Lagrangian $\mathrm{L}:\mathfrak{g}\times\mathfrak{g}\to\mathbb{R}$ by $e^{\xi \mathrm{L}(g^{-1}\dot{g},g^{-1}g')}$ where ξ is a free parameter.

Definition 2.2 [**26**]: For any $\mathrm{L}(\vec{u},\vec{v}):\mathfrak{g}\times\mathfrak{g}\to\mathbb{R}$ we can also define a left G -invariant function $L:TG\times TG\to\mathbb{R}$ and a map $g(t,s):\mathbb{R}\times\mathbb{R}\to G$ such that $\vec{u}(t,s) = g^{-1}\dot{g}(t,s)$ and $\vec{v}(t,s) = g^{-1}g'(t,s)$.

Due to the fact that $g_{ts} = g_{st}$, it follows that $\vec{v}_t - \vec{u}_s = -\mathrm{ad}_{\vec{u}}\vec{v}$ where $\vec{v}_t := \partial\vec{v}/\partial t$ and $\vec{u}_s := \partial\vec{u}/\partial s$.

Theorem 2.1: For the NSL $e^{\xi L(g(t,s),\dot{g}(t,s),g'(t,s))}$ and $e^{\xi \mathrm{L}(\vec{u}(t,s),\vec{v}(t,s))}$ we associate respectively the actions functionals $\delta S = \int_{t_1}^{t_2} e^{\xi L(g(t,s),\dot{g}(t,s),g'(t,s))} ds dt$ on $TG\times TG$ and $\delta S = \int_{t_1}^{t_2} e^{\xi \mathrm{L}(\vec{u}(t,s),\vec{v}(t,s))} ds dt$ on $\mathfrak{g}\times\mathfrak{g}$. The following statements are equivalent:

i-For $\delta S = 0$, $g(t,s)$ satisfies the following modified Euler-Lagrange equation for t and s explicitly independent Lagrangian [**10**]:

$$\frac{\partial L}{\partial g}-\frac{d}{dt}\left(\frac{\partial L}{\partial g_t}\right)-\frac{d}{ds}\left(\frac{\partial L}{\partial g_s}\right)=\xi\frac{\partial L}{\partial g_t}\left(g_t\frac{\partial L}{\partial g}+g_{tt}\frac{\partial L}{\partial g_t}\right)+\xi\frac{\partial L}{\partial g_s}\left(g_s\frac{\partial L}{\partial g}+g_{ss}\frac{\partial L}{\partial g_s}\right), \quad (1)$$

where $\ddot{g}:=g_{tt}\in TG$ and $g'':=g_{ss}\in TG$ are the higher-order derivatives terms.

j-For $\delta S=0$, the following modified Euler-Poincaré equation or G-strand equations holds on $\mathfrak{g}^*\times\mathfrak{g}^*$

$$\frac{d}{dt}\left(\frac{\delta L}{\delta\vec{u}}\right)-\mathrm{ad}^*_{\vec{u}}\frac{\delta L}{\delta\vec{u}}+\frac{d}{ds}\left(\frac{\delta L}{\delta\vec{v}}\right)-\mathrm{ad}^*_{\vec{v}}\frac{\delta L}{\delta\vec{v}}=-\xi\frac{dL}{dt}\frac{\delta L}{\delta\vec{u}}-\xi\frac{dL}{ds}\frac{\delta L}{\delta\vec{v}}, \quad (2)$$

and $\partial_s\vec{u}-\partial_t\vec{v}=[\vec{u},\vec{v}]=\mathrm{ad}_{\vec{u}}\vec{v}$ where

$$\frac{dL}{dt}=\vec{\dot{u}}\mathrm{ad}^*_{\vec{u}}\frac{\delta L}{\delta\vec{u}}+\vec{u}_t\frac{\delta L}{\delta\vec{u}},$$
$$\frac{dL}{ds}=\vec{v}\mathrm{ad}^*_{\vec{v}}\frac{\delta L}{\delta\vec{v}}+\vec{v}_s\frac{\delta L}{\delta\vec{v}},$$

are the total derivative operators, $\delta\vec{u}=\dot{\vec{w}}+\mathrm{ad}_{\vec{u}}\vec{w}$, $\delta\vec{v}=\dot{\vec{w}}+\mathrm{ad}_{\vec{v}}\vec{w}$, $\vec{w}(t,s)=\mathfrak{g}^{-1}\delta\mathfrak{g}^*\in\mathfrak{g}:\vec{w}(t_1,s)=\vec{w}(t_2,s)=0$ and $\mathrm{ad}^*:\mathfrak{g}^*\times\mathfrak{g}^*\to\mathfrak{g}^*$ is defined by means of $\mathrm{ad}:\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ in the dual pairing $<\cdot,\cdot>$ by

$$\left\langle\mathrm{ad}^*_{\vec{u}}\frac{\delta L}{\delta\vec{u}},\vec{v}\right\rangle_g=\left\langle\frac{\delta L}{\delta\vec{u}},\mathrm{ad}^*_{\vec{u}}\vec{v}\right\rangle_g.$$

Proof: The proof is classical [**10,26**].

Corollary 2.1: Let $\vec{\mathsf{m}}:=\delta L/\delta\vec{u}$ and $\vec{\mathsf{n}}:=\delta L/\delta\vec{v}$ in $\mathfrak{g}^*$. The modified G-strand equations for t and s explicitly independent Lagrangians on $\mathfrak{g}^*\times\mathfrak{g}^*$ are:

$$\vec{\mathsf{m}}_t+\vec{\mathsf{n}}_s-\mathrm{ad}^*_{\vec{u}}\vec{\mathsf{m}}\left(1-\xi\vec{\mathsf{m}}\vec{u}\right)-\mathrm{ad}^*_{\vec{v}}\vec{\mathsf{n}}\left(1-\xi\vec{\mathsf{n}}\vec{v}\right)+\xi\left(\vec{\mathsf{m}}\vec{\mathsf{m}}\vec{u}_t+\vec{\mathsf{n}}\vec{\mathsf{n}}\vec{v}_s\right)=0, \quad (3)$$

$$\partial_s\vec{u}-\partial_t\vec{v}-\mathrm{ad}_{\vec{u}}\vec{v}=0. \quad (4)$$

Proof: It follows directly from the standard Euler-Lagrange equation:

$$\frac{d}{dt}\left(\frac{\delta L}{\delta\vec{u}}\right)-\mathrm{ad}^*_{\vec{u}}\frac{\delta L}{\delta\vec{u}}+\frac{d}{ds}\left(\frac{\delta L}{\delta\vec{v}}\right)-\mathrm{ad}^*_{\vec{v}}\frac{\delta L}{\delta\vec{v}}=0,$$

After replacing $\mathrm{L}(u,v) \to e^{\xi \mathrm{L}(u,v)}$. ■

2. B: Lie-Poisson Hamiltonian formulation and the Modified Euler-Poisson Equations

For the case of exponentially NSL, the Legendre transformation of the NSL $\mathrm{L}(\vec{u},\vec{v}) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ yields the non-standard Hamiltonian $\mathsf{H}(\vec{\mathsf{m}},\vec{v}) = < \vec{\mathsf{m}}, \vec{u} > - \mathrm{L}(\vec{u},\vec{v})$. The modified partial derivatives are then given by [**10**]:

$$\vec{\mathsf{m}} = \frac{\delta \mathrm{L}}{\delta \vec{u}}, \tag{5}$$

$$\vec{u} = \frac{\delta \mathsf{H}}{\delta \vec{\mathsf{m}}}, \tag{6}$$

$$\frac{\delta \mathsf{H}}{\delta \vec{v}} = -\frac{\delta \mathrm{L}}{\delta \vec{v}} = \vec{v}\,. \tag{7}$$

We can now rewrite equations (3) and (4) as:

$$\partial_t \vec{\mathsf{m}} - \partial_s \frac{\delta \mathsf{H}}{\delta \vec{v}} - \mathrm{ad}^*_{\frac{\delta \mathsf{H}}{\delta \vec{\mathsf{m}}}} \vec{\mathsf{m}} \left(1 - \xi \vec{\mathsf{m}} \frac{\delta \mathsf{H}}{\delta \vec{\mathsf{m}}} \right) + \mathrm{ad}^*_{\vec{v}} \frac{\delta \mathsf{H}}{\delta \vec{v}} \left(1 - \xi \vec{\mathsf{n}} \frac{\delta \mathsf{H}}{\delta \vec{v}} \right) + \xi \left(\vec{\mathsf{m}} \vec{\mathsf{m}} \partial_t \frac{\delta \mathsf{H}}{\delta \vec{\mathsf{m}}} + \vec{\mathsf{n}} \vec{\mathsf{n}} \partial_s \frac{\delta \mathsf{H}}{\delta \vec{v}} \right) = 0 \tag{8}$$

$$\partial_t \vec{v} = \partial_s \frac{\delta \mathsf{H}}{\delta \vec{\mathsf{m}}} - \mathrm{ad}_{\frac{\delta \mathsf{H}}{\delta \vec{\mathsf{m}}}} \vec{v}\,. \tag{9}$$

or in the following matrix form:

$$\frac{\partial}{\partial t} \begin{pmatrix} \vec{\mathsf{m}} \\ \vec{v} \end{pmatrix} + \xi \vec{\mathsf{m}} \vec{\mathsf{m}} \frac{\partial}{\partial t} \begin{pmatrix} \frac{\delta \mathsf{H}}{\delta \vec{\mathsf{m}}} \\ 0 \end{pmatrix} + \xi \vec{\mathsf{n}} \vec{\mathsf{n}} \frac{\partial}{\partial s} \begin{pmatrix} \frac{\delta \mathsf{H}}{\delta \vec{v}} \\ 0 \end{pmatrix} = \begin{pmatrix} \left(1 - \xi \vec{\mathsf{m}} \frac{\delta \mathsf{H}}{\delta \vec{\mathsf{m}}} \right) \mathrm{ad}^*(\cdot) \vec{\mathsf{m}} & \left(\partial_s - \mathrm{ad}^*_{\vec{v}} \left(1 - \xi \vec{\mathsf{n}} \frac{\delta \mathsf{H}}{\delta \vec{v}} \right) \right) \\ \partial_s + \mathrm{ad}_{\vec{v}} & 0 \end{pmatrix} \begin{pmatrix} \frac{\delta \mathsf{H}}{\delta \vec{\mathsf{m}}} \\ \frac{\delta \mathsf{H}}{\delta \vec{v}} \end{pmatrix} \tag{10}$$

where $(\mathrm{ad}^*(\cdot)\vec{\mathsf{m}})\vec{v} = \mathrm{ad}^*_{\vec{v}} \vec{\mathsf{m}}$ for $\vec{v} \in \mathfrak{g}$ and $\vec{\mathsf{m}} \in \mathfrak{g}^*$ [**10**]. It is obvious that for negligible value of the parameter ξ, equation (10) is reduced to the standard Lie-Poisson Hamiltonian matrix.

2. C: Extended G-strand partial differential equations for $G = \mathrm{Diff}(\mathbb{R})$

In this subsection we derive the extended G-strands partial differential equations for $G = \mathrm{Diff}(\mathbb{R})$ in particular for the case of a two-parametric group with two tangent vectors $\vec{u}(t,s,x)$ and $\vec{v}(t,s,x)$ defined through the composition of functions operator $\circ$ by: $\partial_t g = u \circ g$ and $\partial_s g = v \circ g$ [**10**]. In this right-invariant case, the G-strands partial differential equations for $L(u,v) \to e^{\xi L(u,v)}$ are:

$$\frac{d}{dt}\left(\frac{\delta L}{\delta \vec{u}}\right) + \mathrm{ad}^*_{\vec{u}} \frac{\delta L}{\delta \vec{u}} + \frac{d}{ds}\left(\frac{\delta L}{\delta \vec{v}}\right) + \mathrm{ad}^*_{\vec{v}} \frac{\delta L}{\delta \vec{v}} = -\xi \frac{dL}{dt}\frac{\delta L}{\delta \vec{u}} - \xi \frac{dL}{ds}\frac{\delta L}{\delta \vec{v}}, \tag{11}$$

and $\partial_t \vec{v} - \partial_s \vec{u} = \mathrm{ad}_{\vec{u}} \vec{v}$. We can now use equation (3) and mainly for maps $\mathbb{R} \times \mathbb{R} \to G = \mathrm{Diff}(\mathbb{R})$ in one spatial dimension to obtain directly the following system of partial differential equations in (t,s,x):

$$m_t + n_s = -\mathrm{ad}^*_u m(1-\xi m u) - \mathrm{ad}^*_v n(1-\xi n v) - \xi(m^2 u_t + n^2 v_s)$$

$$= -((um)_x + m u_x)(1-\xi m u) - ((vn)_x + n v_x)(1-\xi n v) - \xi(m^2 u_t + n^2 v_s), \tag{12}$$

and

$$v_t - u_s = -\mathrm{ad}_v u = -u v_x + v u_x. \tag{13}$$

In matrix form, this system of equations is written as:

$$\frac{\partial}{\partial t}\begin{pmatrix} m \\ v \end{pmatrix} + \xi m^2 \frac{\partial}{\partial t}\begin{pmatrix} u \\ 0 \end{pmatrix} + \xi n^2 \frac{\partial}{\partial s}\begin{pmatrix} v \\ 0 \end{pmatrix} = \begin{pmatrix} -(1-\xi m u)\mathrm{ad}^*(\cdot)m & \partial_s + (1-\xi n v)\mathrm{ad}^*_v \\ \partial_s - \mathrm{ad}_v & 0 \end{pmatrix} \begin{pmatrix} \frac{\delta H}{\delta m} = u \\ \frac{\delta H}{\delta v} = -n \end{pmatrix} \tag{14}$$

The matrix that appears in equation (14) is the extended Lie-Poisson bracket. In the next section, we will discuss some applications of equations obtained.

Remark 2.1: For $s-$ independent solutions, equations (12) and (13) are reduced to:

$$m_t = -((um)_x + m u_x)(1-\xi m u) - ((vn)_x + n v_x)(1-\xi n v) - \xi m^2 u_t,$$

and

$$v_t = -u v_x + v u_x.$$

This case corresponds for a NSL defined on $T(\mathrm{Diff}(\mathbb{R}) \times \Xi(\mathbb{R}))/\mathrm{Diff}_{\Xi_0}(\mathbb{R}) : \mathrm{Diff}_{\Xi_0}(\mathbb{R}) \subset \mathrm{Diff}(\mathbb{R})$ where $\mathrm{Diff}_{\Xi_0}(\mathbb{R})$ is the isotropy subgroup of the vector field parameter $\Xi_0(\mathrm{R})$ [**10**]. Equation (14) is then reduced to:

$$\frac{\partial}{\partial t}\begin{pmatrix} m \\ v \end{pmatrix} + \xi m^2 \frac{\partial}{\partial t}\begin{pmatrix} u \\ 0 \end{pmatrix} = \begin{pmatrix} -(1-\xi m u)\mathrm{ad}^*(\cdot)m & (1-\xi n v)\mathrm{ad}_v^* \\ -\mathrm{ad}_v & 0 \end{pmatrix}\begin{pmatrix} \frac{\delta \mathrm{H}}{\delta m} = u \\ \frac{\delta \mathrm{H}}{\delta v} = -n \end{pmatrix}.$$

This is the modified Lie-Poisson bracket dual to the action of the semidirect product $\otimes$ Lie algebra $\mathrm{g} = \Xi(\mathrm{R}) \otimes \Lambda^1(\mathrm{Den})(\mathrm{R})$ where $\Xi(\mathrm{R})$ is the space of vector fields and $\Lambda^1(\mathrm{Den})(\mathrm{R})$ is the space of 1-form densities on the real line R.

3-ILLUSTRATIONS

3.1: As a starting illustration we consider the case $\mathrm{G} = \mathrm{SO}(3)$ and $e^{\xi \mathrm{L}} \equiv e^{\xi \mathrm{L}(\vec{u},\vec{v})} : \mathrm{R}^3 \times \mathrm{R}^3 \to \mathrm{R}$ in spacetime dimension 1+1. The modified Euler-Poincaré equations (2) and (4) become respectively in that case:

$$\frac{d}{dt}\left(\frac{\delta \mathrm{L}}{\delta \vec{u}}\right) + \vec{u} \times \frac{\delta \mathrm{L}}{\delta \vec{u}} + \frac{d}{ds}\left(\frac{\delta \mathrm{L}}{\delta \vec{v}}\right) + \vec{v} \times \frac{\delta \mathrm{L}}{\delta \vec{v}}$$

$$= -\xi\left(-\vec{u}\left(\vec{u} \times \frac{\delta \mathrm{L}}{\delta \vec{u}}\right)\frac{\delta \mathrm{L}}{\delta \vec{u}} + \frac{\partial \vec{u}}{\partial t}\left(\frac{\delta \mathrm{L}}{\delta \vec{u}}\right)^2\right) - \xi\left(-\vec{v}\left(\vec{v} \times \frac{\delta \mathrm{L}}{\delta \vec{v}}\right)\frac{\delta \mathrm{L}}{\delta \vec{v}} + \frac{\partial \vec{v}}{\partial s}\left(\frac{\delta \mathrm{L}}{\delta \vec{v}}\right)^2\right), \quad (15)$$

and

$$\frac{\partial \vec{v}}{\partial t} = \frac{\partial \vec{u}}{\partial s} + \vec{v} \times \vec{u}. \quad (16)$$

We choose at the beginning the singular Lagrangian $\mathrm{L} = \frac{1}{2}\int (C\vec{u} + D\vec{v})ds, (C, D) \in \mathrm{R}$. Accordingly, we find:

$$C^2 \frac{\partial \vec{u}}{\partial t} + D^2 \frac{\partial \vec{v}}{\partial s} = 0, \quad (17)$$

and

$$\frac{\partial \vec{v}}{\partial t} - \frac{\partial \vec{u}}{\partial s} + \vec{u} \times \vec{v} = 0. \quad (18)$$

When $C^2 = -D^2 = 1$, i.e. complexified singular Lagrangian, we find a similar solution of the $SO(3)$ spin chain model derived in [**22**]. It is thus amazing to obtain the same solutions from two different types of Lagrangians and in particular form a totally singular Lagrangian which is affine in velocities. It is notable that complexified Lagrangian mechanics is not new and it was discussed in literature through different aspects [**2,15,33**]. This gives us a hope to obtain integrable systems from non-standard singular Lagrangians.

If for instance we choose the NSL $L = \frac{1}{2}\int(\vec{u}\cdot\vec{u} - \vec{v}\cdot\vec{v})ds + \frac{\zeta}{2}\int(\vec{u}\cdot A\vec{v} + \vec{v}\cdot B\vec{u})ds, (A, B, \zeta) \in \mathbb{R}$, we find:

$$\frac{d}{dt}\left(\vec{u} + \frac{\zeta}{2}A\vec{v}\right) + \vec{u}\times\left(\vec{u} + \frac{\zeta}{2}A\vec{v}\right) + \frac{d}{ds}\left(-\vec{v} + \frac{\zeta}{2}B\vec{u}\right) + \vec{v}\times\left(-\vec{v} + \frac{\zeta}{2}B\vec{u}\right)$$

$$= \xi\left(\vec{u}\left(\vec{u}\times\left(\vec{u} + \frac{\zeta}{2}A\vec{v}\right)\right)\left(\vec{u} + \frac{\zeta}{2}A\vec{v}\right)\right) + \xi\left(\vec{v}\left(\vec{v}\times\left(-\vec{v} + \frac{\zeta}{2}B\vec{u}\right)\right)\left(-\vec{v} + \frac{\zeta}{2}B\vec{u}\right)\right)$$

$$-\xi\frac{\partial\vec{u}}{\partial t}\left(\vec{u} + \frac{\zeta}{2}A\vec{v}\right)^2 - \xi\frac{\partial\vec{v}}{\partial s}\left(-\vec{v} + \frac{\zeta}{2}B\vec{u}\right)^2, \tag{19}$$

and

$$\frac{\partial\vec{v}}{\partial t} - \frac{\partial\vec{u}}{\partial s} + \vec{u}\times\vec{v} = 0. \tag{20}$$

After arrangement and in particular for $A = B = 2$, we can simplify equation (19) to:

$$\vec{u}_t\left(1 + \xi\left(\vec{u} + \zeta\vec{v}\right)^2\right) + \zeta\vec{v}_t + \zeta\vec{u}_s - \vec{v}_s\left(1 - \xi\left(-\vec{v} + \zeta\vec{u}\right)^2\right) = \xi\zeta\left(\vec{u}\times\vec{v}\right)\left(u^2 + v^2\right). \tag{21}$$

Making use of equation (20), we can rewrite the system of equations as:

$$\vec{u}_t\left(1 + \xi\left(\vec{u} + \zeta\vec{v}\right)^2\right) + \zeta(1 + \xi(u^2 + v^2))\vec{v}_t + \zeta(1 - \xi(u^2 + v^2))\vec{u}_s - \vec{v}_s\left(1 - \xi\left(-\vec{v} + \zeta\vec{u}\right)^2\right) = 0 \tag{22}$$

$$\vec{v}_t - \vec{u}_s + \vec{u}\times\vec{v} = 0. \tag{23}$$

This system of equations belongs to the class of non-integrable Hamiltonian systems which play an important role in chaos since non-integrable systems can exhibit chaos [**9**]. In case $\xi << 1$ and $\zeta << 1$, we find the $SO(3)$ spin chain model obtained in [**22**]. Hence equations (22) and (23) correspond to a perturbed $SO(3)$ spin chain model.

3.2: We consider now the case $G = \mathrm{Diff}(\mathbb{R})$ where $L = \int_{-\infty}^{+\infty} u_x dx$ with vanishing boundary conditions at infinity ($|x| \to \infty$) and we set $\xi = 1$ for convenience. From the variational principle, we have:

$$\delta \int e^{L} dt = -e^{L} \iint u_{xx} \delta u dx dt := e^{L} \iint m \delta u dx dt\ , \tag{24}$$

and hence $m = -u_{xx}$. From equation (12) we find:

$$m_t = -((um)_x + m u_x)(1 - mu) - m^2 u_t\ . \tag{25}$$

and hence the equation of motion is:

$$u_{xxt} + u_{xx}^2 u_t + (1 + u u_{xx})(u u_{xxx} + 2 u_{xx} u_x) = 0\ . \tag{26}$$

We consider now $L = \frac{1}{2}\int_{-\infty}^{+\infty} u^2 dx$ with vanishing boundary conditions at infinity. In such a case, we have

$$\delta \int e^{L} dt = e^{L} \iint u \delta u dx dt := e^{L} \iint m \delta u dx dt\ , \tag{27}$$

and hence $m = u$ and hence using equation (12) we find the PDE:

$$(1 + u^2) u_t + 3(1 - u^2) u u_x = 0\ . \tag{28}$$

For $L = \int_{-\infty}^{+\infty} u^{\alpha} dx, \alpha \in \mathbb{R} - \{-1\}$ with vanishing boundary conditions at infinity, we find $m = \alpha u^{\alpha - 1}$ and the corresponding PDE

$$\left(\alpha(\alpha - 1) u^{\alpha - 2} + \alpha^2 u^{2(\alpha - 1)}\right) u_t = -\alpha(1 - \alpha u^{\alpha})(\alpha + 1) u^{\alpha - 1} u_x\ . \tag{29}$$

In particular when $\alpha = 1$, we find $m = 1$ and the following 1st-order PDE:

$$2 u_x (1 - u) + u_t = 0\ . \tag{30}$$

The partial differential equations (28) and (30) require certain initial conditions and are similar to the Cauchy problem discussed in nonlinear 1^{st} order PDE. Numerical simulations are required, yet our main aim was to prove that a number of interesting PDE may be obtained from NSL.

4. CONCLUDING REMARKS

In this paper, we have introduced the basic concepts of higher-order G-strand equations on matrix Lie algebra based on the notion of non-standard Lagrangians or Arnold's non-natural Lagrangians. We have set up the basic modified dynamical equations mainly the modified Euler-Poincaré and the modified Euler-Poisson equations in terms of momentum through the Hamiltonian formulation. We have illustrated our results by discussing some specific examples. It was observed that for complex singular Lagrangians which are affine in velocities, we find the same integrable solution for the $SO(3)$ spin chain model derived from the standard approach. This gives us the optimism that for matrix Lie groups based on NSL, we can find on G-strand integrable systems. For the case of $G = \mathrm{Diff}(\mathbb{R})$, a number of interesting PDE were obtained as well similar to the Cauchy problem found in 1^{st}-order nonlinear PDE. The next step is to solve numerically some of the equations obtained in this work and to compare the results obtained with the solution behavior for the PDE obtained in the standard formalism.

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