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SOME ALGEBRAIC PROPERTIES OF GENERALISED CENTRAL LOOPS

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ABSTRACT. Generalised central loops (*GCL*) are loops satisfying the identity
$x(z\cdot z^{\sigma}y)=(xz\cdot z^{\sigma})y$. In this work, three generalised identities corresponding to three of the four left central identities are newly introduced and all of these three generalised identities together with identity $z(z^{\sigma}\cdot xy)=(z\cdot z^{\sigma}x)y$, which was introduced in [14] are shown to be equivalent in any loop. It is shown that every *GCL* $(G,\cdot,\sigma)=(G,\cdot)$ is a $\sigma-$central square loop. Furthermore, it is established that a loop $(G,\cdot,\sigma)=(G,\cdot)$ is a *GCL* if and only if $L_{z^{\sigma}}L_z$ and $R_zR_{z^{\sigma}}$ are crypto-automorphisms of $(G,\cdot,\sigma)=(G,\cdot)$ with companions $c_1=(zz^{\sigma})^{-1}$ and $c_2=e$, and companions $d_1=e$ and $d_2=(zz^{\sigma})^{-1}$ respectively. The necessary and sufficient conditions for a *GCL* to be isomorphic to its principal isotopes are also formulated. Every pseudo-automorphism of a *GCL* $(G,\cdot,\sigma)=(G,\cdot)$ with companion zz^{σ} is shown to be a semi-automorphism. Lastly, a *GCL* was constructed using a group together with an arbitrary subgroup of it.

1. INTRODUCTION

'Central-identity' as named by Ferenc Fenyves [11] and [12] in 1968 and 1969 respectively is one of the 60 identities of Bol-Moufang type. Loops of Bol-Moufang type are variety of loops defined by a single identity that satisfy: (i) involves three distinct variables on both sides, (ii) contains variables in the same order on both sides, (iii) exactly one of the variables appears twice on both sides. Loops satisfying the central identity are called 'central loops'. Closely related to the central identity are the extra, left central (LC) and right central (RC) identities.

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In 2005, Phillips and Vojtechovsky [19] revisiting the work of Fenyves in [11] and [12] obtained four equivalent identities that define LC-loops and four equivalent identities that define RC-loops. Three of these four identities given by Phillips and Vojtechovsky are the same as the three already given by Fenyves. Stated below are the C-loops-, extra loops-, LC-loops- and RC-loops- identities as found in [12].

$$(yx\cdot x)z = y(x\cdot xz) \qquad \textit{central identity} \qquad (1)$$
$$(xy\cdot z)x = x(y\cdot zx) \qquad \textit{extra identity} \qquad (2)$$
$$(xy)\cdot(xz) = x(yx\cdot z) \qquad \textit{extra identity} \qquad (3)$$
$$(yx)\cdot(zx) = (y\cdot xz)x \qquad \textit{extra identity} \qquad (4)$$
$$(xx)\cdot(yz) = (x\cdot xy)z \qquad \textit{left central identity} \qquad (5)$$
$$(x\cdot xy)z = x(x\cdot yz) \qquad \textit{left central identity} \qquad (6)$$
$$(xx\cdot y)z = x(x\cdot yz) \qquad \textit{left central identity} \qquad (7)$$
$$z(x\cdot xy) = (z\cdot xx)y \qquad \textit{left central identity} \qquad (8)$$
$$(yz)\cdot(xx) = y(zx\cdot x) \qquad \textit{right central identity} \qquad (9)$$
$$(yz\cdot x)x = y(zx\cdot x) \qquad \textit{right central identity} \qquad (10)$$
$$(yz\cdot x)x = y(z\cdot xx) \qquad \textit{right central identity} \qquad (11)$$
$$y(xx\cdot z) = (yx\cdot x)z \qquad \textit{right central identity} \qquad (12)$$

In any loop, there is equivalence between any two of the identities corresponding to each of the equation numbers in each of the triples {2, 3, 4}, {5, 6, 7, 8} and {9, 10, 11, 12}. These facts are found in [11] and [12]. Furthermore, for a given loop, any one of (5), (6), (7, 8) and any one of (9), (10), (11, 12) together is equivalent to (1) and vice versa. Although in a loop, any one of (2), (3), (4) implies (1) but the converse is not true. Loops that satisfy (1) are called central loops or C-loops as the short form while loops that satisfy (5) and (9) or their equivalent forms are called left central and right central loops, or LC-loops and RC-loops respectively as the short forms.

Right and left Bol loops, Moufang loops and extra loops are the most studied varieties of loops of Bol-Moufang type. The studies of these loops have also been generalised. Sharma and Sabinin in 1976 introduced the generalised form of left Bol loops which they called half-Bol loops and later in 1979 studied their algebraic properties ([22], [23]). Within this time period, Ajimal [7] introduced the generalised form of right Bol loops which was called generalised Bol loops, its algebraic properties and relationship with generalised Moufang loops were studied. Thereafter, Adeniran [1], Adeniran and Akinleye [2], Adeniran, Jaiyeola and Idowu [4], [5], Adeniran and Solarin [6] studied the algebraic properties of generalised Bol loops.

In 2014, Jaiyeola [14] introduced and studied four generalised identities corresponding to the four right central identities in loop. It was shown that the four new identities are equivalent in a loop. Also, one generalised form each of the central identity and the left central identity were introduced. Furthermore, he investigated the algebraic properties of generalised right central loops $(GRCL)$ $(G,\cdot,\sigma)$ and some equivalent characterizing forms of the selfmap σ were found.

The following definitions were given in [14].

Definition 1.1. $(L,\cdot)$ is called a generalised 1st right central loop (GRC_1L) or σ_{1st} right central loop $(\sigma_{1st}-RCL)$ if it satisfies the identity:

$$x(yz\cdot z^{\sigma}) = (xy\cdot z)z^{\sigma} \tag{13}$$

Definition 1.2. $(L,\cdot)$ is called a generalised 2nd right central loop (GRC_2L) or σ_{2nd} right central loop $(\sigma_{2nd}-RCL)$ if it satisfies the identity:

$$x(yz\cdot z^{\sigma}) = (xy\cdot zz^{\sigma}) \tag{14}$$

Definition 1.3. $(L,\cdot)$ is called a generalised 3rd right central loop (GRC_3L) or σ_{3rd} right central loop $(\sigma_{3rd}-RCL)$ if it satisfies the identity:

$$x(y\cdot zz^{\sigma}) = (xy\cdot z)z^{\sigma} \tag{15}$$

Definition 1.4. $(L,\cdot)$ is called a generalised 4th right central loop (GRC_4L) or σ_{4th} right central loop $(\sigma_{4th}-RCL)$ if it satisfies the identity:

$$x(zz^{\sigma}\cdot y) = (xz\cdot z^{\sigma})y \tag{16}$$

Definition 1.5. $(L,\cdot)$ is called a generalised 1st left central loop (GLC_1L) or σ_{1st} left central loop $(\sigma_{1st}-LCL)$ if it satisfies the identity:

$$z(z^{\sigma}\cdot xy) = (z\cdot z^{\sigma}x)y \tag{17}$$

Definition 1.6. $(L,\cdot)$ is called a generalised 1st central loop (GC_1L) or σ_{1st} central loop $(\sigma_{1st}-CL)$ if it satisfies the identity:

$$x(z\cdot z^{\sigma}y) = (xz\cdot z^{\sigma})y \tag{18}$$

Observe that the identities of the $\sigma_{1st}-, \sigma_{2nd}-, \sigma_{3rd}-$ and $\sigma_{4th}-$right central loop all of which have been proved to be equivalent in any loop are the generalised forms of the right central loop identities while the identities of the $\sigma_{1st}-$left central loop and $\sigma_{1st}-$central loop are the generalised forms of the left central loop and central loop identities respectively. Hence, the $\sigma_{1st}-, \sigma_{2nd}-, \sigma_{3rd}-$ and $\sigma_{4th}-$right central loop are generally referred to as generalised right central loops ($GRCLs$) or $\sigma-$right central loops ($\sigma-RCLs$) while the $\sigma_{1st}-$left central loop is called generalised left central loop ($GLCL$) or $\sigma-$left central loop ($\sigma-LCL$) and the $\sigma_{1st}-$central loop is called generalised central loop (GCL) or $\sigma-$central loop ($\sigma-CL$).

In this work, three generalised identities corresponding to three of the four left central identities are newly introduced and all of these three generalised identities together with identity $z(z^\sigma\cdot xy)=(z\cdot z^\sigma x)y$, which was introduced in [14] are shown to be equivalent in any loop. It is shown that every generalised central loop (GCL) $(G,\cdot,\sigma)$ is a $\sigma-$central square loop. Furthermore, it is established that a loop $(G,\cdot,\sigma)$ is a GCL if and only if $L_{z^\sigma}L_z$ and $R_zR_{z^\sigma}$ are crypto-automorphisms of $(G,\cdot,\sigma)$ with companions $c_1=(zz^\sigma)^{-1}$ and $c_2=e$, and companions $d_1=e$ and $d_2=(zz^\sigma)^{-1}$ respectively. The necessary and sufficient conditions for a GCL to be isomorphic to its principal isotopes are also formulated. Every pseudo-automorphism of a GCL $(G,\cdot,\sigma)$ with companion zz^σ is shown to be a semi-automorphism. Lastly, a GCL was constructed using a group together with an arbitrary subgroup of it.

For definition of concepts in theory of loops readers may consult [9] and [17].

2. PRELIMINARIES

Definition 2.1. Let $(L,\cdot)$ be a loop with a single valued self-map $\sigma:x\to x^\sigma$.
A loop $(L,\cdot)$ is called a *σ-right alternative property loop* ($\sigma-RAPL$) if it satisfies the *σ-right alternative property* ($\sigma-RAP$)

$$xz\cdot z^\sigma=x\cdot zz^\sigma \tag{19}$$

A loop $(L,\cdot)$ is called a *σ-left alternative property loop* ($\sigma-LAPL$) if it satisfies the *σ-left alternative property* ($\sigma-LAP$)

$$zz^\sigma\cdot x=z\cdot z^\sigma x \tag{20}$$

A loop $(L,\cdot)$ is called a *σ-alternative property loop* ($\sigma-APL$) if it is a $\sigma-RAPL$ and a $\sigma-LAPL$.

Definition 2.2. Let $(G,\cdot)$ be a loop, the group of all permutations on $(G,\cdot)$ is called the *symmetric group* of G and denoted by $SYM(G,\cdot)$ while the group of all automorphisms of $(G,\cdot)$ is denoted by $AUM(G,\cdot)$ where $V\in AUM(G,\cdot)\Leftrightarrow (xy)V=xV\cdot yV\ \forall\ x,y\in G$.
i. $U\in SYM(G,\cdot)$ is called *autotopic* if there exists $(U,V,W)\in AUT(G,\cdot)$;

the set of all such mappings forms a group $\Sigma(G,\cdot)$.
ii. $U\in SYM(G,\cdot)$ is called ρ*-regular* if there exists $(I,U,U)\in AUT(G,\cdot)$; the set of all such mappings forms a group $\rho(G,\cdot)$.
iii. $U\in SYM(G,\cdot)$ is called λ*-regular* if there exists $(U,I,U)\in AUT(G,\cdot)$; the set of all such mappings forms a group $\Lambda(G,\cdot)$.
iv. $U\in SYM(G,\cdot)$ is called μ*-regular* if there exists $U'\in SYM(G,\cdot)$ such that $(U,U'^{-1},I)\in AUT(G,\cdot)$. U' is called the *adjoint* of U. The set of all μ*-regular* mappings forms a group $\Phi(G,\cdot)\leq\Sigma(G,\cdot)$. The set of all adjoint mapping forms a group $\Psi(G,\cdot)$.
v. If there exists $c,\, d\in G$ such that (A,AR_c,AR_c) and (VL_d,V,VL_d) are elements of $AUT(G,\cdot)$, then $A,V\in SYM(G,\cdot)$ are respectively called *right pseudo-automorphism* of $(G,\cdot)$ with *companion c* and *left pseudo-automorphism* of $(G,\cdot)$ with *companion d*. The set of all right pseudo-automorphisms of G forms a group called the *right pseudo-automorphism group* of $(G,\cdot)$ and denoted by $P_\rho AUM(G,\cdot)$ and the set of all left pseudo-automorphisms of G forms a group called the *left pseudo-automorphism group* of $(G,\cdot)$ and denoted by $P_\lambda AUM(G,\cdot)$.
vi. If there exists $c_1,c_2\in G$ such that $(AR_{c_1},AL_{c_2},A)\in AUT(G,\cdot)$, then $A\in SYM(G,\cdot)$ is called a *crypto-automorphism* of $(G,\cdot)$ with *companions* c_1 and c_2. The set of such permutations on G forms a group called the *cryto-automorphism group* of $(G,\cdot)$ and denoted by $CAUM(G,\cdot)$.

Definition 2.3. Let $(G,\cdot)$ be a loop.
The left nucleus of $(G,\cdot)$ denoted by
$$N_\lambda(G,\cdot)=\{a\in G: a\cdot xy=ax\cdot y\quad\forall\quad x,y\in G\}$$
The right nucleus of $(G,\cdot)$ denoted by
$$N_\rho(G,\cdot)=\{a\in G: xy\cdot a=x\cdot ya\quad\forall\quad x,y\in G\}$$
The middle nucleus of $(G,\cdot)$ denoted by
$$N_\mu(G,\cdot)=\{a\in G: xa\cdot y=x\cdot ay\quad\forall\quad x,y\in G\}$$
The nucleus of $(G,\cdot)$ denoted by
$$N(G,\cdot)=N_\lambda(G,\cdot)\bigcap N_\rho(G,\cdot)\bigcap N_\mu(G,\cdot)$$
The centrum of $(G,\cdot)$ denoted by
$$C(G,\cdot)=\{a\in G: ax=xa\quad\forall\quad x\in G\}$$
The center of $(G,\cdot)$ denoted by
$$Z(G,\cdot)=N(G,\cdot)\bigcap C(G,\cdot)$$

Definition 2.4. Let a and b be two elements of a loop L. The *loop commutator* of a and b is the unique element $[a,b]$ of L which satisfies
$$ab=ba\cdot[a,b]$$

Below are some of the results obtained in [14] which will be employed in this work.

Theorem 2.5. Let $(G,\cdot,\sigma)=(G,\cdot)$ be a $\sigma 1st-RCL$. Then
i. $(G,\cdot,\sigma)$ has the $\sigma-RAP$.
ii. a. $R_xR_{x^\sigma}=R_{xx^\sigma}$.
b. $x^\lambda\cdot xx^\sigma=x^\sigma$; σ-self left inverse property($\sigma-SLIP$).
iii. $e^\sigma\in N_\rho(G,\cdot)$.
iv. $(G,\cdot)$ has the RIP.
v. $x^\rho=x^\lambda=x^{-1}$ i.e. $J_\rho=J_\lambda=J$.
vi. $N_\rho(G,\cdot)=N_\mu(G,\cdot)$
vii. $xx^\sigma\in N_\rho(G,\cdot)=N_\mu(G,\cdot)$; σ-right square property.
viii. there exists $n_\sigma\in N_\rho(G,\cdot)$ such that $\sigma=L_{n_\sigma^{-1}}J;x^\sigma=(n_\sigma^{-1}x)^{-1}$.
ix. $\sigma(n_\sigma)=e,\sigma(e)=n_\sigma$.
x. $(yz\cdot x)(n_\sigma^{-1}x)^{-1}=y[zx\cdot(n_\sigma^{-1}x)^{-1}]$.

Theorem 2.6. Let $(G,\cdot,\sigma)=(G,\cdot)$ be a loop. $(G,\cdot,\sigma)$ is a $\sigma_{1st}-RCL$ if and only if any of the following is true.
i. $(I,R_xR_{x^\sigma},R_xR_{x^\sigma})\in AUT(G,\cdot)$.
ii. $R_xR_{x^\sigma}\in\rho(G,\cdot)$.
iii. $R_zR_xR_{x^\sigma}=R_{zx\cdot x^\sigma}$.
iv. $[L_y,R_xR_{x^\sigma}]=I$.
v. $(R_xR_{x^\sigma},L^{-1}_{xx^\sigma},I)\in AUT(G,\cdot)$.
vi. $R_xR_{x^\sigma}\in\Phi(G,\cdot)$ and $L_{xx^\sigma}\in\Psi(G,\cdot)$.

Theorem 2.7. Let $(G,\cdot,\sigma)=(G,\cdot)$ be a $\sigma_{1st}-LCL$. Then
i. $(G,\cdot)$ is an $LIPL$.
ii. $N_\lambda(G,\cdot)=N_\mu(G,\cdot)$.
iii. $(G,\cdot)$ is a $\sigma-LAPL$.
iv. $x^\sigma=x^{-1}n_\sigma,\sigma=JR_{n\sigma}$ and $(xn_\sigma)^{-1}=n_\sigma^{-1}x^{-1}$ for all $x\in G$ and some $n\in N_\mu(G,\cdot)$.

Theorem 2.8. Let $(G,\cdot,\sigma)=(G,\cdot)$ be a $\sigma_{1st}-CL$. Then
i. $(G,\cdot)$ is an IPL.
ii. $N_\rho(G,\cdot)=N_\lambda(G,\cdot)=N_\mu(G,\cdot)=N(G,\cdot)$.
iii. $(G,\cdot)$ is a $\sigma-APL$.
iv. $x^\sigma=x^{-1}n,\sigma=JR_n$ and $(xn)^{-1}=n^{-1}x^{-1}$ for all $x\in G$ and some $n\in N_\mu(G,\cdot)$.
v. $(G,\cdot)$ is a $\sigma_{1st}-RCL$ and a $\sigma_{1st}-LCL$.

Theorem 2.9. Let $(G,\cdot)$ be a loop. The following are equivalent.
i. $(G,\cdot,\sigma)$ is a $\sigma_{1st}-CL$.
ii. $(G,\cdot,\sigma)$ is a $\sigma_{1st}-RCL$ and a $LIPL$.

iii. $(G,\cdot,\sigma)$ is a $\sigma_{1st}-LCL$ and a $RIPL$.
iv. $(G,\cdot,\sigma)$ is a $\sigma_{1st}-RCL$ and a $\sigma_{1st}-LCL$.

Stated below are also some existing results in literatures used in this work:

Theorem 2.10. (Pflugfelder [17]) Let $(G,\cdot)$ be a quasigroup, then the following hold:
i. $L_a^{-1}=L_{a^{-1}}$ for all $a\in N_\mu$
ii. $R_a^{-1}=R_{a^{-1}}$ for all $a\in N_\mu$

Theorem 2.11. (Pflugfelder [17]) If $A=(U,V,W)$ is an autotopism of an inverse property loop (IPL) $(G,\cdot)$, then $A_\rho=(V,U,JWJ), A_\mu=(W,JVJ,U), A_\lambda=(JUJ,W,V)$ are also autotopisms of $(G,\cdot)$.

Remark 2.12. It has been established in [14] that a loop is a $\sigma-CL$ if and only if it is both a $\sigma-LCL$ and a $\sigma-RCL$ (Theorem 2.5). Hence, we shall prove results for $\sigma-CL$ by combining those that are true for $\sigma-LCL$ and $\sigma-RCL$. The identity element is represented with e throughout this work except otherwise stated.

3. Main Results

Theorem 3.1. Let $(G,\cdot)$ be a $\sigma-LCL$, then the following conditions hold in $(G,\cdot)$:
(i) $L_{z^\sigma}L_z=L_{zz^\sigma}$
(ii) $z=zz^\sigma.(z^\sigma)^\rho$
(iii) $e^\sigma\in N_\lambda(G,\cdot)=N_\mu(G,\cdot)$
(iv) $zz^\sigma\in N_\lambda(G,\cdot)=N_\mu(G,\cdot)$ [σ-left nuclear square property]

Proof. (i) $(G,\cdot)$ is a $\sigma-LAPL$, i.e. $(G,\cdot)$ satisfies identity (20). Writing (20) in translation form, we have:
$xL_{z^\sigma}L_z=xL_{zz^\sigma}\Leftrightarrow L_{z^\sigma}L_z=L_{zz^\sigma}$.
(ii) Setting $x=(z^\sigma)^\rho$ in (20), we have:
$z\cdot z^\sigma(z^\sigma)^\rho=zz^\sigma\cdot(z^\sigma)^\rho\Rightarrow z=zz^\sigma\cdot(z^\sigma)^\rho$.
(iii) Setting $z=e$ in (17), we have:
$e(e^\sigma\cdot xy)=(e\cdot e^\sigma x)y$
$\Rightarrow e^\sigma\cdot xy=e^\sigma x\cdot y$ for all $x,y\in(G,\cdot)$
$\Rightarrow e^\sigma\in N_\lambda=N_\mu$.
(iv) Applying the $\sigma-LAP$ to both sides of (17), we have:
$zz^\sigma\cdot xy=z(z^\sigma\cdot xy)=(z\cdot z^\sigma x)y=(zz^\sigma\cdot x)y$
$\Rightarrow zz^\sigma\cdot xy=(zz^\sigma\cdot x)y$ for all $x,y\in(G,\cdot)$
$\Rightarrow zz^\sigma\in N_\lambda=N_\mu$. □

Remark 3.2. The dual of the above results have been obtained in [14] (2.5) for $\sigma-RCL$. Hence, the following results hold for a $\sigma-CL$.

Let $(G,\cdot)$ be a $\sigma-CL$, then the following hold in $(G,\cdot)$:
(i) $L_{z^\sigma}L_z = L_{zz^\sigma}$
(ii) $R_zR_{z^\sigma} = R_{zz^\sigma}$
(iii) $z = zz^\sigma.(z^\sigma)^\rho$
(iv) $z^\sigma = z^\lambda\cdot zz^\sigma$
(v) $e^\sigma \in N(G,\cdot)$
(vi) $zz^\sigma \in N(G,\cdot)$ [σ-nuclear square property]

Proof. By Theorem 2.9, $(G,\cdot)$ is a $\sigma-CL \Leftrightarrow$ it is both a $\sigma-LCL$ and a $\sigma-RCL$. Hence, the proof of (i) - (vi) follows from Theorem 2.5 and Theorem 3.1, and the fact that the three nuclei coincide for an inverse property loop (IPL). □

Remark 3.3. The above result shows that zz^σ is in the intersection of the three nuclei, and in particular lies in the middle nucleus. This property is called the σ-nuclear square property. Hence, it follows from theorem 2.10 that $L^{-1}_{zz^\sigma} = L_{(zz^\sigma)^{-1}}$ and $R^{-1}_{zz^\sigma} = R_{(zz^\sigma)^{-1}}$.

Theorem 3.4. If $(G,\cdot)$ is a $\sigma-LCL$, then $(G,\cdot)$ satisfies the following conditions
(i) $(z\cdot z^\sigma x)^{-1} = (zz^\sigma\cdot x)^{-1} = x^{-1}\cdot z^{\sigma^{-1}}z^{-1}$
(ii) $(zz^\sigma)^{-1} = z^{\sigma^{-1}}z^{-1}$

Proof. (i) $(zz^\sigma\cdot x)(x^{-1}\cdot z^{\sigma^{-1}}z^{-1})$
$= (z\cdot z^\sigma x)(x^{-1}\cdot z^{\sigma^{-1}}z^{-1})$ (By the $\sigma-LAP$)
$= z[z^\sigma\cdot x(x^{-1}\cdot z^{\sigma^{-1}}z^{-1})]$ (By identity (17))
$= z[z^\sigma\cdot z^{\sigma^{-1}}z^{-1}]$ (By the *LIP*)
$= zz^{-1} = e$ (By the *LIP*)
(ii) Set $x = e$ in (i), then the result holds. □

Theorem 3.5. If $(G,\cdot)$ is a $\sigma-RCL$, then $(G,\cdot)$ satisfies the following conditions
(i) $(x\cdot zz^\sigma)^{-1} = (xz\cdot z^\sigma)^{-1} = z^{\sigma^{-1}}z^{-1}\cdot x^{-1}$
(ii) $(zz^\sigma)^{-1} = z^{\sigma^{-1}}z^{-1}$

Proof. (i) $(z^{\sigma^{-1}}z^{-1}\cdot x^{-1})(x\cdot zz^\sigma)$
$= (z^{\sigma^{-1}}z^{-1}\cdot x^{-1})(xz\cdot z^\sigma)$ (By the $\sigma-RAP$)
$= [[[(z^{\sigma^{-1}}z^{-1}\cdot x^{-1})x]z]z^\sigma]$ (By identity (13))
$= (z^{\sigma^{-1}}z^{-1}\cdot z)z^\sigma$ (By the *RIP*)
$= z^{\sigma^{-1}}z^\sigma = e$ (By the *RIP*)
(ii) Set $x = e$ in (i), then the result holds. □

If $(G,\cdot)$ is a $\sigma-CL$, then the following conditions hold in $(G,\cdot)$
(i) $(z\cdot z^\sigma x)^{-1} = (zz^\sigma\cdot x)^{-1} = x^{-1}\cdot z^{\sigma^{-1}}z^{-1}$

(ii) $(x\cdot zz^{\sigma})^{-1}=(xz\cdot z^{\sigma})^{-1}=z^{\sigma^{-1}}z^{-1}\cdot x^{-1}$
(iii) $(zz^{\sigma})^{-1}=z^{\sigma^{-1}}z^{-1}$

Proof. By Theorem 2.9, $(G,\cdot)$ is a $\sigma-CL \Leftrightarrow$ it is both a $\sigma-LCL$ and a $\sigma-RCL$. The proof of (i) - (iii) then follows by combining Theorem 3.4 and Theorem 3.5. □

Theorem 3.6. For a loop $(G,\cdot)$ the following identities are equivalent:

$$(L_1): \qquad (z\cdot z^{\sigma}y)x=z(z^{\sigma}\cdot yx) \tag{21}$$

$$(L_2): \qquad (z\cdot z^{\sigma}y)x=zz^{\sigma}\cdot yx \tag{22}$$

$$(L_3): \qquad (zz^{\sigma}\cdot y)x=z(z^{\sigma}\cdot yx) \tag{23}$$

$$(L_4): \qquad y(z\cdot z^{\sigma}x)=(y\cdot zz^{\sigma})x \tag{24}$$

Proof. $(L_1)\Rightarrow(L_2)$: Assume (L_1) holds in $(G,\cdot)$. In [14], it has been shown that a loop $(G,\cdot)$ satisfying (L_1) is both an $LIPL$ and a $\sigma-LAPL$. So, applying the $\sigma-LAP$ to the right hand side (RHS) of (L_1) we have:

$$(z\cdot z^{\sigma}y)x=zz^{\sigma}\cdot yx$$

$(L_2)\Rightarrow(L_3)$: Assume (L_2) holds in $(G,\cdot)$, setting $x=e$ in (L_2) we have:

$$z\cdot z^{\sigma}y=zz^{\sigma}\cdot y$$

which is the $\sigma-LAP$. Now, applying the $\sigma-LAP$ to both sides of (L_2) we have:

$$(zz^{\sigma}\cdot y)x=z(z^{\sigma}\cdot yx)$$

$(L_3)\Rightarrow(L_1)$: Assume (L_3) holds in $(G,\cdot)$, setting $x=e$ in (L_3) we have:

$$zz^{\sigma}\cdot y=z\cdot z^{\sigma}y$$

which is the $\sigma-LAP$. Now, applying the $\sigma-LAP$ to the left hand side LHS of (L_3) we have:

$$(z\cdot z^{\sigma}y)x=z(z^{\sigma}\cdot yx)$$

$(L_1)\Leftrightarrow(L_4)$: Suppose (L_1) holds in $(G,\cdot)$, then writing (L_1) in autotopic form we have:

$$(L_{z^{\sigma}}L_z,I,L_{z^{\sigma}}L_z)=(L_{zz^{\sigma}},I,L_{z^{\sigma}}L_z)\in AUT(G,\cdot)\text{ (By Corollary 3(i))}$$

Since $(G,\cdot)$ is an $LIPL$, then the last autotopism implies that:

$$(JL_{zz^{\sigma}}J,L_{z^{\sigma}}L_z,I)\in AUT(G,\cdot)$$

For every $x \in (G,\cdot)$ we have that:
$xJL_{zz^\sigma}J = (zz^\sigma \cdot x^{-1})^{-1} = x \cdot (z^\sigma)^{-1}z^{-1}$ (By Theorem 3.4(i))
$= xR_{(z^\sigma)^{-1}z^{-1}} = xR_{(zz^\sigma)^{-1}}$ (By Theorem 3.4(ii))
Hence, $JL_{zz^\sigma}J = R_{(zz^\sigma)^{-1}}$ and the last autotopism can therefore be re-written as

$$(R_{(zz^\sigma)^{-1}}, L_{z^\sigma}L_z, I) \in AUT(G,\cdot)$$

By Corollary 3(vi) and Theorem 2.10(ii), the last autotopism is equivalent to

$$(R^{-1}_{(zz^\sigma)}, L_{z^\sigma}L_z, I) \in AUT(G,\cdot)$$

Applying the last autotopism to the product ty, we have:
$tR^{-1}_{(zz^\sigma)} \cdot yL_{z^\sigma}L_z = (ty)I = ty$
$\Rightarrow tR^{-1}_{(zz^\sigma)}(z \cdot z^\sigma y) = ty$
Setting $t = xR_{zz^\sigma}$, we have:

$$x(z \cdot z^\sigma y) = xR_{zz^\sigma} \cdot y$$
$$\Rightarrow x(z \cdot z^\sigma y) = (x \cdot zz^\sigma)y$$

which is identity (L_4).
Conversely, suppose that (L_4) holds in $(G,\cdot)$, then setting $y = e$ in (L_4) we have:

$$z \cdot z^\sigma x = zz^\sigma \cdot x$$

which is the $\sigma - LAP$. Now, applying the $\sigma - LAP$ to the LHS of (L_4), we have:

$$y(zz^\sigma \cdot x) = (y \cdot zz^\sigma)x$$

which implies that $zz^\sigma \in N_\mu$.By applying (L_4), we have that:
$e = x^\sigma(x^\sigma)^\rho = [(x^\sigma/xx^\sigma)xx^\sigma](x^\sigma)^\rho = (x^\sigma/xx^\sigma)[x \cdot x^\sigma(x^\sigma)^\rho] = (x^\sigma/xx^\sigma)x$
Hence $x^\sigma/xx^\sigma = x^\lambda \Rightarrow x^\sigma = x^\lambda \cdot xx^\sigma$.
Again, applying (L_4), we have that:
$x^\lambda \cdot xy = x^\lambda \cdot x[x^\sigma(x^\sigma\backslash y)] = (x^\lambda \cdot xx^\sigma)(x^\sigma\backslash y) = x^\sigma(x^\sigma\backslash y) = y.$
This implies that $(G,\cdot)$ has the LIP. Hence, if identity (L_4) holds in $(G,\cdot)$ then it implies that $(G,\cdot)$ is both a $\sigma - LAPL$ and an $LIPL$, and $zz^\sigma \in N_\mu = N_\lambda$. Now, writing (L_4) in autotopic form we have:

$$(R^{-1}_{(zz^\sigma)}, L_{z^\sigma}L_z, I) \in AUT(G,\cdot)$$

By Corollary 3(iv) and Theorem 2.10(ii), the last autotopism can be re-written as

$$(R_{(zz^\sigma)^{-1}}, L_{z^\sigma}L_z, I) \in AUT(G,\cdot)$$

Since $(G,\cdot)$ is an *LIPL*, then we have that:

$$(JR_{(zz^{\sigma})^{-1}}J, I, L_{z^{\sigma}}L_{z}) \in AUT(G,\cdot)$$

Recall that $JL_{zz^{\sigma}}J = R^{-1}_{zz^{\sigma}}$, then $JR_{(zz^{\sigma})^{-1}}J = JR^{-1}_{zz^{\sigma}}J = L_{zz^{\sigma}}$
So, the last autotopism implies that

$$(L_{zz^{\sigma}}, I, L_{z^{\sigma}}L_{z}) \in AUT(G,\cdot)$$

By the $\sigma - LAP$, the above autotopism is equivalent to

$$\Rightarrow (L_{z^{\sigma}}L_{z}, I, L_{z^{\sigma}}L_{z}) \in AUT(G,\cdot)$$

But the last autotopism is the autotopic form of (L_1). Hence, this completes the proof. □

Remark 3.7. Henceforth, we call a loop $(G,\cdot)$ satisfying any of the (equivalent) identities (21), (22), (23), (24) a generalised left central loop $(GLCL)$ or $\sigma-$left central loop ($\sigma - LCL$).

Theorem 3.8. A loop $(G,\cdot)$ is a $\sigma - LCL$ if and only if, for all $z \in G, (G,\cdot)$ satisfies any of the following (equivalent) conditions:

$$(i) \qquad (L_{z^{\sigma}}L_{z}, I, L_{z^{\sigma}}L_{z}) \in AUT(G,\cdot) \tag{25}$$

$$(ii) \qquad (L_{z^{\sigma}}L_{z}, I, L_{zz^{\sigma}}) \in AUT(G,\cdot) \tag{26}$$

$$(iii) \qquad (L_{zz^{\sigma}}, I, L_{z^{\sigma}}L_{z}) \in AUT(G,\cdot) \tag{27}$$

$$(iv) \qquad (R^{-1}_{zz^{\sigma}}, L_{z^{\sigma}}L_{z}, I) \in AUT(G,\cdot) \tag{28}$$

Proof. The proof of (i) - (iv) follows by writing each of the $\sigma - LCL$ identities (21), (22), (23), (24) in autotopic form. □

Theorem 3.9. A loop $(G,\cdot)$ is a $\sigma - RCL$ if and only if, for all $z \in G, (G,\cdot)$ satisfies any of the following (equivalent) conditions:

$$(i) \qquad (I, R_{z}R_{z^{\sigma}}, R_{z}R_{z^{\sigma}}) \in AUT(G,\cdot) \tag{29}$$

$$(ii) \qquad (I, R_{z}R_{z^{\sigma}}, R_{zz^{\sigma}}) \in AUT(G,\cdot) \tag{30}$$

$$(iii) \qquad (I, R_{zz^{\sigma}}, R_{z}R_{z^{\sigma}}) \in AUT(G,\cdot) \tag{31}$$

$$(iv) \qquad (R^{-1}_{z^{\sigma}}R^{-1}_{z}, L_{zz^{\sigma}}, I) \in AUT(G,\cdot) \tag{32}$$

Proof. The proof of (i) - (iv) follows by writing each of the $\sigma - RCL$ identities (13), (14), (15), (16) in autotopic form. □

Theorem 3.10. A loop $(G,\cdot)$ is a $\sigma - CL$ if and only if, for all $z \in G, (R^{-1}_{z^{\sigma}}R^{-1}_{z}, L_{z^{\sigma}}L_{z}, I) \in AUT(G,\cdot)$

Proof. The proof of this follows by writing the $\sigma - CL$ identity (18) in autotopic form. □

Theorem 3.11. Every $\sigma-$central loop ($\sigma - CL$) $(G,\cdot)$ is a $\sigma-$central square loop ($\sigma - CSL$) (i.e. $zz^{\sigma} \in Z(G,\cdot)$ for all $z \in (G,\cdot)$).

Proof. Suppose $(G,\cdot)$ is a $\sigma-$central loop ($\sigma - CL$), then by Theorem 3.10 $(R_{z^{\sigma}}^{-1}R_z^{-1}, L_{z^{\sigma}}L_z, I) \in AUT(G,\cdot)$ for all $z \in G$. Applying 2.11, we have that:

$$A_{\rho} = (L_{z^{\sigma}}L_z, R_{z^{\sigma}}^{-1}R_z^{-1}, JIJ) = (L_{z^{\sigma}}L_z, R_{z^{\sigma}}^{-1}R_z^{-1}, I) \in AUT(G,\cdot) \text{ for all } z \in G$$

Applying the last autotopism to the product xy, we have:

$$xL_{z^{\sigma}}L_z \cdot yR_{z^{\sigma}}^{-1}R_z^{-1} = (xy)I = xy$$

By setting $y = yR_zR_{z^{\sigma}}$ in the last equation we have:
$xL_{z^{\sigma}}L_z \cdot y = x \cdot yR_zR_{z^{\sigma}}$
$\Rightarrow (z \cdot z^{\sigma}x)y = x(yz \cdot z^{\sigma})$
Applying the $\sigma-$left alternative property and the $\sigma-$right alternative property to the LHS and RHS respectively of the last equation we have:
$(zz^{\sigma} \cdot x)y = x(y \cdot zz^{\sigma})$
Setting $y = e$ in the last equation we have that:
$zz^{\sigma} \cdot x = x \cdot zz^{\sigma}$ for all $x \in (G,\cdot)$
$\Rightarrow zz^{\sigma} \in C(G,\cdot)$ for all $z \in (G,\cdot)$.
But by Corollary 3(vi), $zz^{\sigma} \in N(G,\cdot)$ for all $z \in (G,\cdot)$. Hence, this implies that $zz^{\sigma} \in Z(G,\cdot) = N(G,\cdot) \bigcap C(G,\cdot)$ for all $z \in (G,\cdot)$. Thus, $(G,\cdot)$ is a $\sigma-$central square loop. □

If $(G,\cdot)$ is a $\sigma-$central loop, then the following conditions hold in $(G,\cdot)$:
(i) $e^{\sigma} \in Z(G,\cdot)$
(ii) $R_{zz^{\sigma}} = L_{zz^{\sigma}}$ for all $z \in (G,\cdot)$.

Proof. (i) By Theorem 3.11 $zz^{\sigma} \in Z(G,\cdot)$, setting $z = e$ in this result, we have that $e^{\sigma} \in Z(G,\cdot)$.
(ii)By Theorem 3.11, $zz^{\sigma} \in Z(G,\cdot) \Rightarrow zz^{\sigma} \in C(G,\cdot) \Rightarrow x \cdot zz^{\sigma} = zz^{\sigma} \cdot x \Rightarrow R_{zz^{\sigma}} = L_{zz^{\sigma}}$ for all $z \in (G,\cdot)$. □

Theorem 3.12. A loop $(G,\cdot)$ is a $\sigma - LCL$ if and only if any of the following is true:
(i) $L_{z^{\sigma}}L_z \in \bigwedge(G,\cdot)$
(ii) $L_{z \cdot z^{\sigma}y} = L_yL_{z^{\sigma}}L_z$
(iii) $[R_x, L_{z^{\sigma}}L_z] = I$
(iv)$R_{zz^{\sigma}} \in \Phi(G,\cdot)$ and $L_{z^{\sigma}}L_z \in \Psi(G,\cdot)$

Proof. (i) By Theorem 3.8, we have that $(G,\cdot)$ is a $\sigma-LCL$ if and only if

$$(L_{z^\sigma}L_z, I, L_{z^\sigma}L_z) \in AUT(G,\cdot)$$

Hence, by Definition 2.2(iii), it follows from the last autotopism that $L_{z^\sigma}L_z \in \bigwedge(G,\cdot)$.
(ii) The result follows by writing identity (21) in left translation form.
(iii) Writing identity (21) in left and right translation forms, we have:
$yL_{z^\sigma}L_zR_x = yR_xL_{z^\sigma}L_z \Leftrightarrow L_{z^\sigma}L_zR_x = R_xL_{z^\sigma}L_z \Leftrightarrow [R_x, L_{z^\sigma}L_z] = I$
(iv) From Theorem 3.8, we have that $(G,\cdot)$ is a $\sigma-LCL$ if and only if

$$(R^{-1}_{zz^\sigma}, L_{z^\sigma}L_z, I) \in AUT(G,\cdot)$$

Taking the inverse of this autotopism, we have:

$$(R^{-1}_{zz^\sigma}, L_{z^\sigma}L_z, I)^{-1} = (R_{zz^\sigma}, (L_{z^\sigma}L_z)^{-1}, I) \in AUT(G,\cdot)$$

Hence, by Definition 2.2(iv) the last autotopism holds if and only if $R_{zz^\sigma} \in \Phi(G,\cdot)$ and $L_{z^\sigma}L_z \in \Psi(G,\cdot)$. □

Remark 3.13. The dual of the above results have been obtained for $\sigma-RCL$ in [14] Theorem (2.6). Hence, the following results hold for a $\sigma-CL$.

A loop $(G,\cdot)$ is a $\sigma-CL$ if and only if any of the following is true:
(i) $L_{z^\sigma}L_z \in \bigwedge(G,\cdot)$ and $R_zR_{z^\sigma} \in \rho(G,\cdot)$
(ii) $L_{z\cdot z^\sigma y} = L_yL_{z^\sigma}L_z$ and $R_{yz\cdot z^\sigma} = R_yR_zR_{z^\sigma}$
(iii) $[R_x, L_{z^\sigma}L_z] = I$ and $[L_x, R_zR_{z^\sigma}] = I$
(iv) $R_{zz^\sigma} = R_zR_{z^\sigma} \in \Phi(G,\cdot)$ and $L_{zz^\sigma} = L_{z^\sigma}L_z \in \Psi(G,\cdot)$.

Proof. By Theorem 2.9, $(G,\cdot)$ is a $\sigma-CL \Leftrightarrow$ it is both a $\sigma-LCL$ and a $\sigma-RCL$. The proof of (i) - (iv) then follows by combining Theorem 2.6 and Theorem 3.9. □

Theorem 3.14. A loop $(G,\cdot)$ is a $\sigma-LCL$ if and only if $L_{z^\sigma}L_z$ is a *crypto-automorphism* of $(G,\cdot)$ with companions $c_1 = (zz^\sigma)^{-1}$ and $c_2 = e$.

Proof. Let $(G,\cdot)$ be a $\sigma-LCL$, then by Theorem 3.8, we have that:
$(L_{z^\sigma}L_z, I, L_{z^\sigma}L_z)$ and $(R^{-1}_{zz^\sigma}, L_{z^\sigma}L_z, I)$ are both autotopisms of $(G,\cdot)$. But by Corollary 3(vi) and Theorem 2.10(ii),
$(R^{-1}_{zz^\sigma}, L_{z^\sigma}L_z, I) = (R_{(zz^\sigma)^{-1}}, L_{z^\sigma}L_z, I)$
Hence, $(G,\cdot)$ is a $\sigma-LCL$ if and only if the product

$$A = (L_{z^\sigma}L_z, I, L_{z^\sigma}L_z)(R_{(zz^\sigma)^{-1}}, L_{z^\sigma}L_z, I) = (L_{z^\sigma}L_zR_{(zz^\sigma)^{-1}}, L_{z^\sigma}L_z, L_{z^\sigma}L_z)$$

is an autotopism of $(G,\cdot)$. But the product

$$A=(L_{z^\sigma}L_zR_{(zz^\sigma)^{-1}},L_{z^\sigma}L_z,L_{z^\sigma}L_z)=(L_{z^\sigma}L_zR_{(zz^\sigma)^{-1}},L_{z^\sigma}L_zL_e,L_{z^\sigma}L_z)$$

Hence, by Definition 2.2(vi), it follows that the product A is an autotopism of $(G,\cdot)$ if and only if $L_{z^\sigma}L_z$ is a *crypto-automorphism* of $(G,\cdot)$ with companions $c_1=(zz^\sigma)^{-1}$ and $c_2=e$. □

Theorem 3.15. A loop $(G,\cdot)$ is a $\sigma-RCL$ if and only if $R_zR_{z^\sigma}$ is a *crypto-automorphism* of $(G,\cdot)$ with companions $c_1=e$ and $c_2=(zz^\sigma)^{-1}$.

Proof. Let $(G,\cdot)$ be a $\sigma-RCL$, then by Theorem 3.9, we have that: $(I,R_zR_{z^\sigma},R_zR_{z^\sigma})$ and $(R_{z^\sigma}^{-1}R_z^{-1},L_{zz^\sigma},I)$ are both autotopisms of $(G,\cdot)$. Taking the inverse of the last autotopism, and applying Corollary 3(vi) and Theorem 2.10(i), we have that:

$$(R_{z^\sigma}^{-1}R_z^{-1},L_{zz^\sigma},I)^{-1}=(R_zR_{z^\sigma},L_{zz^\sigma}^{-1},I)=(R_zR_{z^\sigma},L_{(zz^\sigma)^{-1}},I)\in AUT(G,\cdot)$$

Then $(G,\cdot)$ is a $\sigma-RCL$ if and only if the product

$$B=(I,R_zR_{z^\sigma},R_zR_{z^\sigma})(R_zR_{z^\sigma},L_{(zz^\sigma)^{-1}},I)=(R_zR_{z^\sigma},R_zR_{z^\sigma}L_{(zz^\sigma)^{-1}},R_zR_{z^\sigma})$$

is an autotopism of $(G,\cdot)$. But the product

$$B=(R_zR_{z^\sigma},R_zR_{z^\sigma}L_{(zz^\sigma)^{-1}},R_zR_{z^\sigma})=(R_zR_{z^\sigma}R_e,R_zR_{z^\sigma}L_{(zz^\sigma)^{-1}},R_zR_{z^\sigma})$$

Hence, by Definition 2.2(vi), it follows that the product B is an autotopism of $(G,\cdot)$ if and only if $R_zR_{z^\sigma}$ is a *crypto-automorphism* of $(G,\cdot)$ with companions $c_1=e$ and $c_2=(zz^\sigma)^{-1}$. □

A loop $(G,\cdot)$ is a $\sigma-CL$ if and only if $L_{z^\sigma}L_z$ and $R_zR_{z^\sigma}$ are *crypto-automorphisms* of $(G,\cdot)$ with companions $c_1=(zz^\sigma)^{-1}$ and $c_2=e$, and companions $d_1=e$ and $d_2=(zz^\sigma)^{-1}$ respectively.

Proof. By Theorem 2.9, $(G,\cdot)$ is a $\sigma-CL \Leftrightarrow$ it is both a $\sigma-LCL$ and a $\sigma-RCL$, then the rest of the proof follows by combining Theorem 3.14 and Theorem 3.15. □

Theorem 3.16. If $(G,\cdot)$ is a $\sigma-LCL$, then $z\in N_\lambda$ if and only if $z^\sigma\in N_\lambda$.

Proof. Let $(G,\cdot)$ be a $\sigma-LCL$. Now, suppose $z\in N_\lambda$ and applying this condition to the *LHS* of (21), we have:
$z(z^\sigma y\cdot x)=(z\cdot z^\sigma y)x=z(z^\sigma\cdot yx)$
$\Rightarrow z^\sigma y\cdot x=z^\sigma\cdot yx \quad \forall \quad x,y\in(G,\cdot)\Rightarrow z^\sigma\in N_\lambda$
Conversely, suppose $z^\sigma\in N_\lambda$ and applying this condition to the *RHS* of (21), we have:
$(z\cdot z^\sigma y)x=z(z^\sigma\cdot yx)=z(z^\sigma y\cdot x)$
Setting $z^\sigma y=q$, we have:
$zq\cdot x=z\cdot qx \quad \forall \quad q,x\in(G,\cdot)\Rightarrow z\in N_\lambda$. □

If $(G,\cdot)$ is a $\sigma-CL$, then $z\in N_\lambda$ if and only if $z^\sigma\in N_\lambda$.

Proof. By Theorem 2.9, $(G,\cdot)$ is a $\sigma - CL \Rightarrow$ it is a $\sigma - LCL$. Then the result follows from Theorem 3.16. □

Theorem 3.17. If $(G,\cdot)$ is a $\sigma - RCL$, then $z \in N_\rho$ if and only if $z^\sigma \in N_\rho$.

Proof. Let $(G,\cdot)$ be a $\sigma - RCL$ and suppose $z \in N_\rho$. Now, applying this condition to the *RHS* of (13), we have:
$x(yz \cdot z^\sigma) = (xy \cdot z)z^\sigma = (x \cdot yz)z^\sigma$
Setting $yz = q$, we have:
$x \cdot qz^\sigma = xq \cdot z^\sigma \quad \forall \quad q,x \in (G,\cdot) \Rightarrow z \in N_\rho$
Conversely, suppose $z^\sigma \in N_\rho$ and applying this condition to the *LHS* of (13), we have:
$(x \cdot yz)z^\sigma = x(yz \cdot z^\sigma) = (xy \cdot z)z^\sigma$
$\Rightarrow x \cdot yz = xy \cdot z \quad \forall \quad x,y \in (G,\cdot) \Rightarrow z \in N_\rho$. □

If $(G,\cdot)$ is a $\sigma - CL$, then $z \in N_\rho$ if and only if $z^\sigma \in N_\rho$.

Proof. By Theorem 2.9, $(G,\cdot)$ is a $\sigma - CL \Rightarrow$ it is a $\sigma - RCL$. Then the result follows from Theorem 3.17. □

Theorem 3.18. Let $(G,\cdot)$ be a $\sigma - LCL$, and $z \in G$ be such that

$$u \circ v = uR^{-1}_{(zz^\sigma)} \cdot VL_{zz^\sigma} \quad \forall \quad u,v \in G,$$

that is $(G,\circ)$ is a principal isotope of $(G,\cdot)$. Then $(G,\cdot)$ and $(G,\circ)$ are isomorphic if and only if $\exists$ a left pseudo-automorphism of $(G,\cdot)$ with companion $(zz^\sigma)^{-1}$.

Proof. Let $(G,\cdot)$ be a $\sigma - LCL$ and $(G,\circ)$ its principal isotope. Suppose $(G,\cdot)$ and $(G,\circ)$ are isomorphic, then $\exists$ a permutation T of G such that
$(u \cdot v)T = uT \circ vT \quad \forall \quad u,v \in G$
$\Leftrightarrow (u \cdot v)T = UTR^{-1}_{(zz^\sigma)} \cdot vTL_{zz^\sigma} \quad \forall \quad u,v \in G$
$\Leftrightarrow A = (TR^{-1}_{(zz^\sigma)}, TL_{zz^\sigma}, T) \in AUT(G,\cdot)$
By Theorem 3.8, we have that

$$(L_{z^\sigma}L_z, I, L_{z^\sigma}L_z) = (L_{zz^\sigma}, I, L_{zz^\sigma}) \text{ and } (R^{-1}_{zz^\sigma}, L_{z^\sigma}L_z, I) = (R^{-1}_{zz^\sigma}, L_{zz^\sigma}, I)$$

are autotopisms of $(G,\cdot)$. The product of these two autotopisms gives:

$$B = (L_{zz^\sigma}R^{-1}_{zz^\sigma}, L_{zz^\sigma}, L_{zz^\sigma}) \in AUT(G,\cdot).$$

Taking the inverse of B, we have:

$$B^{-1} = (R_{zz^\sigma}L^{-1}_{(zz^\sigma)}, L^{-1}_{(zz^\sigma)}, L^{-1}_{(zz^\sigma)}) \in AUT(G,\cdot).$$

Hence, A is an autotopism of $(G,\cdot)$ if and only if the product

$$AB^{-1} = (TL^{-1}_{(zz^\sigma)}, T, TL^{-1}_{(zz^\sigma)})$$

is also an autotopism of $(G,\cdot)$. By Theorem 2.10(i) and 3(vi), AB^{-1} is an autotopism of $(G,\cdot)$ if and only if

$$C = (TL_{(zz^\sigma)^{-1}}, T, TL_{(zz^\sigma)^{-1}})$$

is also an autotopism of $(G,\cdot)$. Hence, this completes the proof. $\square$

Theorem 3.19. Let $(G,\cdot)$ be a $\sigma - RCL$, and $z \in G$ be such that

$$u \circ v = uR^{-1}_{(zz^\sigma)} \cdot VL_{zz^\sigma} \quad \forall \quad u, v \in G,$$

that is $(G,\circ)$ is a principal isotope of $(G,\cdot)$. Then $(G,\cdot)$ and $(G,\circ)$ are isomorphic if and only if $\exists$ a right pseudo-automorphism of $(G,\cdot)$ with companion zz^σ.

Proof. Let $(G,\cdot)$ be a $\sigma - RCL$ and $(G,\circ)$ its principal isotope. Suppose $(G,\cdot)$ and $(G,\circ)$ are isomorphic, then $\exists$ a permutation T of G such that
$(u \cdot v)T = uT \circ vT \quad \forall \quad u, v \in G$
$\Leftrightarrow (u \cdot v)T = UTR^{-1}_{(zz^\sigma)} \cdot vTL_{zz^\sigma} \quad \forall \quad u, v \in G$
$\Leftrightarrow A = (TR^{-1}_{(zz^\sigma)}, TL_{zz^\sigma}, T) \in AUT(G,\cdot)$
By Theorem 3.9, we have that

$$B = (I, R_z R_{z^\sigma}, R_z R_{z^\sigma}) = (I, R_{zz^\sigma}, R_{zz^\sigma})$$

and

$$C = (R^{-1}_{z^\sigma} R^{-1}_z, L_{zz^\sigma}, I)^{-1} = (R_z R_{z^\sigma}, L^{-1}_{zz^\sigma}, I) = (R_{zz^\sigma}, L^{-1}_{zz^\sigma}, I)$$

are autotopisms of $(G,\cdot)$. Taking The product of B and C gives:

$$D = (R_{zz^\sigma}, L^{-1}_{zz^\sigma} R_{zz^\sigma}, R_{zz^\sigma}) \in AUT(G,\cdot).$$

Hence, A is an autotopism of $(G,\cdot)$ if and only if the product

$$AD = (T, TR_{zz^\sigma}, TR_{zz^\sigma})$$

is also an autotopism of $(G,\cdot)$. This therefore completes the proof. $\square$

Let $(G,\cdot)$ be a $\sigma - CL$, and $z \in G$ be such that

$$u \circ v = uR^{-1}_{(zz^\sigma)} \cdot VL_{zz^\sigma} \quad \forall \quad u, v \in G,$$

that is $(G,\circ)$ is a principal isotope of $(G,\cdot)$. Then $(G,\cdot)$ and $(G,\circ)$ are isomorphic if and only if $\exists$ a left pseudo-automorphism of $(G,\cdot)$ with companion $(zz^\sigma)^{-1}$(or if and only if $\exists$ a right pseudo-automorphism of $(G,\cdot)$ with companion zz^σ).

Proof. By Theorem 2.9, $(G,\cdot)$ is a $\sigma - CL \Leftrightarrow$ it is both a $\sigma - LCL$ and a $\sigma - RCL$, then the rest of the proof follows by combining Theorem 3.18 and Theorem 3.19. $\square$

Remark 3.20. Theorem 3.18 and Theorem 3.19 above gave the necessary and sufficient conditions for a $\sigma-$LCL and a $\sigma-$RCL respectively to be isomorphic to every of its principal isotopes. Hence, from these results necessary and sufficient for a $\sigma-$CL to be isomorphic to every of its principal isotopes are deduced.

Theorem 3.21. If (U,V,W) is an autotopism of a $\sigma - LCL$ $(G,\cdot)$ and if $eU = u$ and $eV = (vv^{\sigma})^{-1}$, then $Y = VL_{vv^{\sigma}}$ is a left pseudo-automorphism of $(G,\cdot)$ with companion $c = (vv^{\sigma})u \cdot (vv^{\sigma})^{-1}$.

Proof. Let (U,V,W) be an autotopism of a $\sigma - LCL$ $(G,\cdot)$ with identity element e such that $eU = u$ and $eV = (vv^{\sigma})^{-1}$. By Theorem 3.8 and Theorem 2.10(ii), we have that

$$(L_{v^{\sigma}}L_v, I, L_{v^{\sigma}}L_v) = (L_{vv^{\sigma}}, I, L_{vv^{\sigma}}) \text{ and}$$
$$(R^{-1}_{vv^{\sigma}}, L_{v^{\sigma}}L_v, I) = (R_{(vv^{\sigma})^{-1}}, L_{vv^{\sigma}}, I)$$

are autotopisms of $(G,\cdot)$. The product of these two autotopisms:

$$B = (L_{vv^{\sigma}}R_{(vv^{\sigma})^{-1}}, L_{vv^{\sigma}}, L_{vv^{\sigma}}).$$

is also an autotopism of $(G,\cdot)$ and so is the product:

$$(X,Y,Z) = (U,V,W)(L_{vv^{\sigma}}R_{(vv^{\sigma})^{-1}}, L_{vv^{\sigma}}, L_{vv^{\sigma}})$$

Hence, we have that: $X = UL_{vv^{\sigma}}R_{(vv^{\sigma})^{-1}}, Y = VL_{vv^{\sigma}}$ and $Z = WL_{vv^{\sigma}}$. Applying (X,Y,Z) to the product xy for any $x,y \in G$, we have:

$$xX \cdot yY = (xy)Z \tag{33}$$

Setting $y = e$ in (33), we have: $xX \cdot eY = xZ$, where $eY = eVL_{vv^{\sigma}} = vv^{\sigma} \cdot (vv^{\sigma})^{-1} = e$. Hence, substituting for eY gives: $xX \cdot e = xZ \Rightarrow xX = xZ \Rightarrow Z = X = UL_{vv^{\sigma}}R_{(vv^{\sigma})^{-1}}$. Now, set $x = e$ in (33), we have:
$eX \cdot yY = yZ$, where $eX = eUL_{vv^{\sigma}}R_{(vv^{\sigma})^{-1}} = (vv^{\sigma})u(vv^{\sigma})^{-1}$. Hence, substituting for eX gives: $yZ = eX \cdot yY = (vv^{\sigma})u(vv^{\sigma})^{-1} \cdot yY = yYL_{(vv^{\sigma})u(vv^{\sigma})^{-1}}$ $\Rightarrow YL_{(vv^{\sigma})u(vv^{\sigma})^{-1}} = Z = X$. Hence, the autotopism (X,Y,Z) is now of the form $(X,Y,Z) = (YL_{(vv^{\sigma})u(vv^{\sigma})^{-1}}, Y, YL_{(vv^{\sigma})u(vv^{\sigma})^{-1}})$, which implies that $Y = VL_{vv^{\sigma}}$ is a left pseudo-automorphism of $(G,\cdot)$ with companion $c = (vv^{\sigma})u(vv^{\sigma})^{-1}$. □

If (U,V,W) is an autotopism of a $\sigma - CL$ $(G,\cdot)$ and if $eU = u$ and $eV = (vv^{\sigma})^{-1}$, then $Y = VL_{vv^{\sigma}}$ is a left pseudo-automorphism of $(G,\cdot)$ with companion $c = (vv^{\sigma})u \cdot (vv^{\sigma})^{-1}$.

Proof. By Theorem 2.9, $(G,\cdot)$ is a $\sigma - CL \Rightarrow$ it is a $\sigma - LCL$, and the result then follows. □

Theorem 3.22. If (U,V,W) is an autotopism of a $\sigma-RCL$ $(G,\cdot)$ and if $eU=(uu^{\sigma})^{-1}$ and $eV=v$, then $X=UR_{uu^{\sigma}}$ is a right pseudo-automorphism of $(G,\cdot)$ with companion $c=(uu^{\sigma})^{-1}v\cdot(uu^{\sigma})$.

Proof. Let (U,V,W) be an autotopism of a $\sigma-RCL$ $(G,\cdot)$ with identity element e such that $eU=(uu^{\sigma})^{-1}$ and $eV=v$. By Theorem 2.10(i) and Theorem 3.9, we have that

$$A=(I,R_uR_{u^{\sigma}},R_uR_{u^{\sigma}})=(I,R_{uu^{\sigma}},R_{uu^{\sigma}})$$

and

$$B=(R_{u^{\sigma}}^{-1}R_u^{-1},L_{uu^{\sigma}},I)^{-1}=(R_uR_{u^{\sigma}},L_{uu^{\sigma}}^{-1},I)=(R_{uu^{\sigma}},L_{(uu^{\sigma})^{-1}},I)$$

are autotopisms of $(G,\cdot)$. The product of these two autotopisms:

$$BA=(R_{uu^{\sigma}},L_{(uu^{\sigma})^{-1}}R_{uu^{\sigma}},R_{uu^{\sigma}})$$

is an autotopism of $(G,\cdot)$ and so is the product

$$(X,Y,Z)=(U,V,W)(R_{uu^{\sigma}},L_{(uu^{\sigma})^{-1}}R_{uu^{\sigma}},R_{uu^{\sigma}})$$

Hence, we have that: $X=UR_{uu^{\sigma}},Y=VL_{(uu^{\sigma})^{-1}}R_{uu^{\sigma}}$ and $Z=WR_{uu^{\sigma}}$. Now, applying (X,Y,Z) to the product xy for any $x,y\in G$, we have:

$$xX\cdot yY=(xy)Z \tag{34}$$

Setting $x=e$ in (34), we have: $eX\cdot yY=yZ$, where $eX=eUR_{uu^{\sigma}}=(uu^{\sigma})^{-1}\cdot uu^{\sigma}=e$. Substituting for eX gives: $e\cdot yY=yZ\Rightarrow yY=yZ\Rightarrow Z=Y=VL_{(uu^{\sigma})^{-1}}R_{uu^{\sigma}}$. Now, set $y=e$ in (34) to give: $xX\cdot eY=xZ$, where $eY=eVL_{(uu^{\sigma})^{-1}}R_{uu^{\sigma}}=(uu^{\sigma})^{-1}v\cdot(uu^{\sigma})$. Substituting for eY gives: $xZ=xX\cdot eY=xX\cdot(uu^{\sigma})^{-1}v\cdot(uu^{\sigma})=xXR_{(uu^{\sigma})^{-1}v\cdot(uu^{\sigma})}\Rightarrow XR_{(uu^{\sigma})^{-1}v\cdot(uu^{\sigma})}=Z=Y$. So the autotopism (X,Y,Z) is now of the form $(X,Y,Z)=(X,XR_{(uu^{\sigma})^{-1}v\cdot(uu^{\sigma})},XR_{(uu^{\sigma})^{-1}v\cdot(uu^{\sigma})})$, which implies that $X=UR_{uu^{\sigma}}$ is a right pseudo-automorphism of $(G,\cdot)$ with companion $c=(uu^{\sigma})^{-1}v\cdot(uu^{\sigma})$, and this completes the proof. □

If (U,V,W) is an autotopism of a $\sigma-CL$ $(G,\cdot)$ and if $eU=(uu^{\sigma})^{-1}$ and $eV=v$, then $X=UR_{uu^{\sigma}}$ is a right pseudo-automorphism of $(G,\cdot)$ with companion $c=(uu^{\sigma})^{-1}v\cdot(uu^{\sigma})$.

Proof. By Theorem 2.9, $(G,\cdot)$ is a $\sigma-CL\Rightarrow$ it is a $\sigma-RCL$, and the rest of the result then follows. □

Theorem 3.23. If $(G,\cdot)$ is a $\sigma-CL$, then a bijection U is a right pseudo-automorphism of $(G,\cdot)$ with companion zz^{σ} if and only if U is a left pseudo-automorphism of $(G,\cdot)$ with companion zz^{σ}.

Proof. Let $(G,\cdot)$ be a $\sigma-CL$, then a bijection U is a right pseudo-automorphism of $(G,\cdot)$ with companion zz^{σ} if and only if $A=(U,UR_{zz^{\sigma}},UR_{zz^{\sigma}})\in AUT(G,\cdot)$. Applying the autotopism A to the product xy, then $A\in AUT(G,\cdot)$ if and only if $xU\cdot yUR_{zz^{\sigma}}=(xy)UR_{zz^{\sigma}}\Leftrightarrow xU(yU\cdot zz^{\sigma})=(xy)U\cdot zz^{\sigma}\Leftrightarrow(xU\cdot yU)zz^{\sigma}=(xy)U\cdot zz^{\sigma}$ (By $zz^{\sigma}\in N_{\rho}$) $\Leftrightarrow zz^{\sigma}(xU\cdot yU)=zz^{\sigma}\cdot(xy)U$ (By $zz^{\sigma}\in C(G)$) $\Leftrightarrow(zz^{\sigma}\cdot xU)yU=zz^{\sigma}\cdot(xy)U$ (By $zz^{\sigma}\in N_{\lambda}$) $\Leftrightarrow xUL_{zz^{\sigma}}\cdot yU=(xy)UL_{zz^{\sigma}}\Leftrightarrow(UL_{zz^{\sigma}},U,UL_{zz^{\sigma}})\in AUT(G,\cdot)\Leftrightarrow$ U is a left pseudo-automorphism of $(G,\cdot)$ with companion zz^{σ}. □

Theorem 3.24. Let $(G,\cdot)$ be a $\sigma-LCL$ and V a bijection on $(G,\cdot)$, then V is an automorphism of $(G,\cdot)$ if and only if it is a left pseudo-automorphism of $(G,\cdot)$ with companion zz^{σ}.

Proof. Let $(G,\cdot)$ be a $\sigma-LCL$, then a bijection V on $(G,\cdot)$ is an automorphism of $(G,\cdot)$ if and only if $xV\cdot yV=(xy)V\ \forall\, x,y\in G\Leftrightarrow zz^{\sigma}(xV\cdot yV)=zz^{\sigma}((xy)V)$ (pre-multiplying both sides with zz^{σ}) $\Leftrightarrow(zz^{\sigma}\cdot xV)yV=zz^{\sigma}((xy)V)$ (By $zz^{\sigma}\in N_{\lambda}(G)$)$\Leftrightarrow xVL_{zz^{\sigma}}\cdot yV=(xy)VL_{zz^{\sigma}}\Leftrightarrow(VL_{zz^{\sigma}},V,VL_{zz^{\sigma}})\in AUT(G,\cdot)\Leftrightarrow V$ is a left pseudo-automorphism of $(G,\cdot)$ with companion zz^{σ}. □

Theorem 3.25. Let $(G,\cdot)$ be a $\sigma-RCL$ and V a bijection on $(G,\cdot)$, then V is an automorphism of $(G,\cdot)$ if and only if it is a right pseudo-automorphism of $(G,\cdot)$ with companion zz^{σ}.

Proof. Let $(G,\cdot)$ be a $\sigma-RCL$, then a bijection V on $(G,\cdot)$ is an automorphism of $(G,\cdot)$ if and only if $xV\cdot yV=(xy)V\ \forall\, x,y\in G\Leftrightarrow(xV\cdot yV)zz^{\sigma}=((xy)V)zz^{\sigma}$ (post-multiplying both sides with zz^{σ}) $\Leftrightarrow xV(yV\cdot zz^{\sigma})=((xy)V)zz^{\sigma}$ (By $zz^{\sigma}\in N_{\rho}(G)$)$\Leftrightarrow xV\cdot yVR_{zz^{\sigma}}=(xy)VR_{zz^{\sigma}}\Leftrightarrow(V,VR_{zz^{\sigma}},VR_{zz^{\sigma}})\in AUT(G,\cdot)\Leftrightarrow V$ is a right pseudo-automorphism of $(G,\cdot)$ with companion zz^{σ}. □

Let $(G,\cdot)$ be a $\sigma-CL$ and V a bijection on $(G,\cdot)$, then V is an automorphism of $(G,\cdot)$ if and only if it is both a left and a right pseudo-automorphism of $(G,\cdot)$ with companion zz^{σ}.

Proof. By Theorem 2.9, $(G,\cdot)$ is a $\sigma-CL\Leftrightarrow$ it is both a $\sigma-LCL$ and a $\sigma-RCL$, then the result follows from Theorem 3.23 and by applying Theorem 3.24 and Theorem 3.25. □

Theorem 3.26. Let $(G,\cdot)$ be a $\sigma-LCL$, then every left pseudo-automorphism V of $(G,\cdot)$ with companion zz^{σ} is a semi-automorphism of $(G,\cdot)$.

Proof. Let $(G,\cdot)$ be a $\sigma-LCL$ and suppose V is a left pseudo-automorphism of $(G,\cdot)$ with companion zz^{σ}, then this implies that
$(VL_{zz^{\sigma}},V,VL_{zz^{\sigma}})\in AUT(G,\cdot)\Rightarrow(zz^{\sigma}\cdot xV)yV=zz^{\sigma}\cdot(xy)V\ \forall\, x,y\in G$.
Now, applying the last autotopism to the product $xy\cdot x$, we have:

$zz^\sigma \cdot (xy \cdot x)V = (zz^\sigma \cdot (xy)V)xV = (zz^\sigma \cdot xVyV)xV$ (since V is an automorphism of $(G,\cdot)$ by Theorem 3.24 $\Rightarrow zz^\sigma \cdot (xy \cdot x)V = zz^\sigma(xVyV \cdot xV)$ (By $zz^\sigma \in N_\lambda$) $\Rightarrow (xy \cdot x)V = (xV \cdot yV)xV \ \forall \ x,y \in G$. Also, by Theorem 3.24, V is an automorphism of $(G,\cdot) \Rightarrow eV = e$. Hence, V is a semi-automorphism of $(G,\cdot)$. □

Theorem 3.27. Let $(G,\cdot)$ be a $\sigma - RCL$, then every right pseudo-automorphism V of $(G,\cdot)$ with companion zz^σ is a semi-automorphism of $(G,\cdot)$.

Proof. Let $(G,\cdot)$ be a $\sigma - RCL$ and suppose V is a right pseudo-automorphism of $(G,\cdot)$ with companion zz^σ, then this implies that
$(V, VR_{zz^\sigma}, VR_{zz^\sigma}) \in AUT(G,\cdot) \Rightarrow xV(yV \cdot zz^\sigma) = (xy)V \cdot zz^\sigma \ \forall \ x,y \in G$.
Now, applying the last autotopism to the product $xy \cdot x$, we have:
$(xy \cdot x)V \cdot zz^\sigma = (xy)V(xV \cdot zz^\sigma) = (xV \cdot yV)(xV \cdot zz^\sigma)$ (since V is an automorphism of $(G,\cdot)$ by Theorem 3.25) $\Rightarrow (xy \cdot x)V \cdot zz^\sigma = (xVyV \cdot xV)zz^\sigma$ (By $zz^\sigma \in N_\rho$) $\Rightarrow (xy \cdot x)V = (xV \cdot yV)xV \ \forall \ x,y \in G$. Also, by Theorem 3.25, V is an automorphism of $(G,\cdot) \Rightarrow eV = e$. Hence, V is a semi-automorphism of $(G,\cdot)$. □

Let $(G,\cdot)$ be a $\sigma - CL$, then every left (right) pseudo-automorphism V of $(G,\cdot)$ with companion zz^σ is a semi-automorphism of $(G,\cdot)$.

Proof. By Theorem 2.9, $(G,\cdot)$ is a $\sigma - CL \Leftrightarrow$ it is both a $\sigma - LCL$ and a $\sigma - RCL$, then the result follows by combining Theorem 3.26 and Theorem 3.27. □

The following theorem shows how a $\sigma - CL$ can be constructed from a group G with a subgroup H.

Theorem 3.28. Let H be a subgroup of a group G and let $g_1^{g_2} = g_2^{-1}g_1g_2$ denote the conjugate of g_1 by g_2. Define $'\circ'$ on $A = H \times G$ such that for all $x,y \in A, x = (h_1,g_1)$ and $y = (h_2,g_2)$,

$$x \circ y = (h_1,g_1) \circ (h_2,g_2) = (h_1h_2, h_2g_1h_2^{-1}g_2) \tag{35}$$

Let $\sigma : A \to A \uparrow \sigma(h,g) = (\delta_1 h, \delta_2 g)$ where $\delta_1, \delta_2 : G \to G$ are singled valued mappings and $\delta_1 h \in Z(G)$ for all $h \in H$. Then $(A,\circ,\sigma)$ is a $\sigma - CL$ if and only if $g^h = g^{h'^{-1}hh'}$ for all $g \in G$ and $h,h' \in H$.

Proof. First, we show that $(A,\circ,\sigma)$ is a loop.
(i) Closure Property: Clearly, the operation $'\circ'$ is closed on A since for all $x,y \in A, x \circ y = (h_1,g_1) \circ (h_2,g_2) = (h_1h_2, h_2g_1h_2^{-1}g_2)$, we have that $h_1h_2 = h \in H$ and $h_2g_1h_2^{-1}g_2 = g \in G$. Hence, $x \circ y = (h,g) \in A$.
(ii) Existence of a unique identity element: The element $(e,e) \in A$ is the identity element of A where e is the identity element of the group G, since $(h,g) \circ (e,e) = (e,e) \circ (h,g) = (h,g)$ for all $(h,g) \in A$.

(iii) Existence of a unique inverse for each element in A : Let $x=(h,g)\in A$ and let (h',g') denote the two sided inverse of (h,g), we will show that $(h',g')\in A$. If (h',g') is the two sided inverse of (h,g), then this implies that
$(h,g)\circ(h',g')=(e,e)=(h',g')\circ(h,g)\Rightarrow(hh',h'gh'^{-1}g')=(e,e)=(h'h,hg'h^{-1}g)$
$\Rightarrow hh'=e$ or $h'h=e\Rightarrow h'=h^{-1}\in H$ and $h'gh'^{-1}g'=e$ or $hg'h^{-1}g=e$
$\Rightarrow h^{-1}ghg'=e$ or $hg'h^{-1}g=e\Rightarrow g'=h^{-1}g^{-1}h\in G$. Hence, $(h',g')=(h^{-1},h^{-1}g^{-1}h)\in A$ is the inverse of $(h,g)\in A$.
Next, we show that the $\sigma-CL$ identity holds in $(A,\circ,\sigma)$. Let $x=(h_1,g_1),y=(h_2,g_2)$ and $z=(h_3,g_3)$ be elements of A, and given that $\sigma(h,g)=(\delta_1h,\delta_2g)$ where $\delta_1,\delta_2:G\longrightarrow G$ are single valued mappings such that $\delta_1h\in Z(G)$ for all $h\in H$, then:

$$y\circ x=(h_2,g_2)\circ(h_1,g_1)=(h_2h_1,h_1g_2h_1^{-1}g_1)$$
$$(y\circ x)\circ x^{\sigma}=(h_2h_1,h_1g_2h_1^{-1}g_1)\circ(h_1,g_1)^{\sigma}=(h_2h_1,h_1g_2h_1^{-1}g_1)\circ(\delta_1h_1,\delta_2g_1)$$
$$=(h_2h_1\delta_1h_1,(\delta_1h_1)h_1g_2h_1^{-1}g_1(\delta_1h_1)^{-1}\delta_2g_1)$$
$$[(y\circ x)\circ x^{\sigma}]\circ z=(h_2h_1\delta_1h_1,(\delta_1h_1)h_1g_2h_1^{-1}g_1(\delta_1h_1)^{-1}\delta_2g_1)\circ(h_3,g_3)$$
$$=(h_2h_1(\delta_1h_1)h_3,h_3(\delta_1h_1)h_1g_2h_1^{-1}g_1(\delta_1h_1)^{-1}(\delta_2g_1)h_3^{-1}g_3)\quad(36)$$

Similarly,

$$x^{\sigma}\circ z=(h_1,g_1)^{\sigma}\circ(h_3,g_3)=(\delta_1h_1,\delta_2g_1)\circ(h_3,g_3)=((\delta_1h_1)h_3,h_3(\delta_2g_1)h_3^{-1}g_3)$$
$$[x\circ(x^{\sigma}\circ z)]=(h_1,g_1)\circ((\delta_1h_1)h_3,h_3(\delta_2g_1)h_3^{-1}g_3)$$
$$=(h_1(\delta_1h_1)h_3,(\delta_1h_1)h_3g_1((\delta_1h_1)h_3)^{-1}h_3(\delta_2g_1)h_3^{-1}g_3)$$
$$y\circ[x\circ(x^{\sigma}\circ z)]=(h_2,g_2)\circ(h_1(\delta_1h_1)h_3,(\delta_1h_1)h_3g_1((\delta_1h_1)h_3)^{-1}h_3(\delta_2g_1)h_3^{-1}g_3)$$
$$=(h_2h_1(\delta_1h_1)h_3,h_1(\delta_1h_1)h_3g_2(h_1(\delta_1h_1)h_3)^{-1}(\delta_1h_1)h_3g_1((\delta_1h_1)h_3)^{-1}h_3(\delta_2g_1)h_3^{-1}g_3)$$
$$=(h_2h_1(\delta_1h_1)h_3,h_1(\delta_1h_1)h_3g_2h_3^{-1}(\delta_1h_1)^{-1}h_1^{-1}(\delta_1h_1)h_3g_1h_3^{-1}(\delta_1h_1)^{-1}h_3(\delta_2g_1)h_3^{-1}g_3)\quad(37)$$

$(A,\circ,\sigma)$ is a $\sigma-CL$ if and only if (3.36) and (3.37) are equal, and this is true if and only if

$$h_3(\delta_1h_1)h_1g_2h_1^{-1}g_1(\delta_1h_1)^{-1}(\delta_2g_1)h_3^{-1}g_3$$
$$=h_1(\delta_1h_1)h_3g_2h_3^{-1}(\delta_1h_1)^{-1}h_1^{-1}(\delta_1h_1)h_3g_1h_3^{-1}(\delta_1h_1)^{-1}h_3(\delta_2g_1)h_3^{-1}g_3$$
$$\Leftrightarrow h_3(\delta_1h_1)h_1g_2h_1^{-1}g_1(\delta_1h_1)^{-1}$$
$$=h_1(\delta_1h_1)h_3g_2h_3^{-1}(\delta_1h_1)^{-1}h_1^{-1}(\delta_1h_1)h_3g_1h_3^{-1}(\delta_1h_1)^{-1}h_3$$
$$\Leftrightarrow h_3(\delta_1h_1)h_1g_2h_1^{-1}g_1(\delta_1h_1)^{-1}$$
$$=h_1(\delta_1h_1)h_3g_2h_3^{-1}(\delta_1h_1)^{-1}(\delta_1h_1)h_1^{-1}h_3g_1h_3^{-1}h_3(\delta_1h_1)^{-1}$$
$$\Leftrightarrow h_3(\delta_1h_1)h_1g_2h_1^{-1}=h_1(\delta_1h_1)h_3g_2h_3^{-1}h_1^{-1}h_3$$
$$\Leftrightarrow h_1g_2h_1^{-1}=(\delta_1h_1)^{-1}h_3^{-1}h_1(\delta_1h_1)h_3g_2h_3^{-1}h_1^{-1}h_3$$
$$\Leftrightarrow h_1g_2h_1^{-1}=(\delta_1h_1)^{-1}(\delta_1h_1)h_3^{-1}h_1h_3g_2h_3^{-1}h_1^{-1}h_3$$
$$\Leftrightarrow h_1g_2h_1^{-1}=h_3^{-1}h_1h_3g_2h_3^{-1}h_1^{-1}h_3$$

$\Leftrightarrow h_1g_2h_1^{-1} = (h_3^{-1}h_1^{-1}h_3)^{-1}g_2h_3^{-1}h_1^{-1}h_3$
$\Leftrightarrow g_2^{h_1^{-1}} = g_2^{h_3^{-1}h_1^{-1}h_3} \Leftrightarrow g_2^{h} = g_2^{h_3^{-1}hh_3}$ (where $h = h_1^{-1}$). □

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